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Quantitative unique continuation for hyperbolic and hypoelliptic equations

Camille Laurent* and Matthieu Léautaud†

Abstract

We review recent results of the authors concerning quantitative unique continuation estimates for operators with coefficients that are analytic in some (or all the) variables. We describe several applications for wave-like equations, but also equations based on hypoelliptic operators. These proceedings are a survey of the general results in [LL19] together with applications to wave equations [LL16] and to hypoelliptic equations [LL17, LL20b].

Keywords

Unique continuation, control theory, observability, approximate controllability, wave equation, hypoelliptic operators.

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1 Introduction

In this note, we are interested in the quantification of *global unique continuation* results of the following form: given a differential operator P on an open set $\Omega \subset \mathbb{R}^n$, and given a small subset U of Ω , having

$$Pu = 0 \text{ in } \Omega, \quad u|_U = 0 \implies u = 0 \text{ on } \Omega. \quad (1.1)$$

More generally, in cases where (1.1) is known to hold, we are interested in proving a quantitative version of

$$Pu \text{ small in } \Omega, \quad u \text{ small in } U \implies u \text{ small in } \Omega.$$

This can sometimes be expressed by a stability estimate of the form

$$\|u\|_\Omega \leq \varphi(\|u\|_U, \|Pu\|_{\tilde{\Omega}}, \|u\|_{\tilde{\Omega}}), \quad \text{with } \varphi(a, b, c) \rightarrow 0 \text{ when } (a, b) \rightarrow 0 \text{ with } c \text{ bounded,} \quad (1.2)$$

where $U \subset \Omega \subset \tilde{\Omega}$ are nonempty, and for appropriate norms. As we will see, both qualitative and quantitative unique continuation properties have several applications in control theory.

A more tractable problem than (1.1) is the so called *local unique continuation* problem: given $x^0 \in \mathbb{R}^n$ and S an oriented local hypersurface containing x^0 , do we have the following implication:

$$\text{There is a neighborhood } \Omega \text{ of } x^0, \text{ such that } Pu = 0 \text{ in } \Omega, u|_{\Omega \cap S^-} = 0 \implies x^0 \notin \text{supp}(u), \quad (1.3)$$

where S^- denotes one side of S . It turns out that proving (1.3) for a suitable class of hypersurfaces (with regard to the operator P) is in general a key step in the proof of properties of the type (1.1). The first general unique continuation result of the form (1.3) is the Holmgren-John Theorem [Hol01, Joh49], stating that, for operators with analytic coefficients, unique continuation holds across any noncharacteristic hypersurface S (see e.g. [Hör90, Theorem 8.6.5] for a precise statement).

When focusing on operators with (only) smooth (C^∞) coefficients, the most general result was proved by Hörmander [Hör94, Chapter XXVIII]. Uniqueness across a hypersurface holds assuming a strong pseudoconvexity condition (see e.g. Definition 2.1 below). This result uses as a key tool Carleman estimates, which were introduced in [Car39] and developed at first for elliptic operators in [Cal58].

Starting from the 70's, and motivated by applications to the unique continuation of the wave equation with non analytic coefficients, people tried to lower the geometric conditions (with respect to “strong pseudoconvexity”) without imposing analyticity of all coefficients of the operator involved. The first results of this kind were [RT73], which provides with unique continuation for the wave operator with (only) C^∞ coefficients from subsets of the form $(-\infty, +\infty) \times \omega$. This work was then followed by the seminal [Rob91] proving a similar result but from a set $(-T, T) \times \omega$ with T finite yet non optimal. Then, it was understood in [Tat95] that the right framework was that of **partial analyticity**, that is analyticity with only a subset of variables (e.g. the time variable when dealing with wave equation). This led to successive improvements [Hör97, RZ98, Tat99b] that interpolate in a satisfying way between Holmgren and Hörmander Theorem. So far, the local unique continuation results are summarized in the first two lines of the Table 1. The specific setting of partial analyticity and the related pseudoconvexity property is described more precisely in the Section 2.

Roughly speaking, our work [LL19] (see [BKL16] for a related result with loss for the wave operator, obtained simultaneously) is concerned with the last line of Table 1. It consists in giving *local and global* quantification (i.e. stability estimates) related to this unique continuation result under partial analyticity. Logarithmic or Hölder type dependence refers to the form of the function φ

Theorem	Holmgren, John [Hol01, Joh49]	Tataru, Robbiano-Zuily, Hörmander [Tat95, Hör97, RZ98, Tat99b]	Hörmander [Hör94, Chapter XXVIII]
Regularity of the coefficients needed	analytic coefficients	partially analytic coefficients in some variable x_a	C^∞ (even C^1) coefficients
Geometrical assumption of the hypersurface	Φ non characteristic for P : $p(x, \nabla\Phi) \neq 0$	Φ pseudoconvex in $\{\xi_a = 0\}$	Φ pseudoconvex ($\{p, \{p, \Phi\}\} > 0$ sufficient if real order 2)
Stability estimates	Logarithmic type, see John [Joh60] and [LL19] (reviewed in Section 2)	Logarithmic type, see [LL19] (reviewed in Section 2)	Hölder type, see [Bah87, LR95], see also [LL19] (reviewed in Section 2)

Table 1: Panorama of the different unique continuation results for a differential operator P with principal symbol p .

in (1.2). This allowed us to apply these results to the classical wave equation [LL19, LL16] and equations involving hypoelliptic (sum of squares) operators [LL17, LL20b]. To summarize, the results we obtain are as follows:

1. A general global quantification of the unique continuation property under partial analyticity assumptions. The main result we present is Theorem 2.5, which gives a global quantification of the unique continuation along a foliation of hypersurfaces satisfying the appropriate conditions. The general estimates obtained in this setting are described in Section 2 and were obtained in [LL19].
2. A logarithmic stability estimate for the observation of the waves from any non empty open subset (that is, without geometric assumptions). This is mainly Theorem 3.1 below. The results are described in Section 3.1 and were obtained in [LL19].
3. A constructive proof of the Bardos-Lebeau-Rauch observability result of the wave equation under the Geometric Control Condition (GCC) [BLR92]. We describe how the estimate described in Item 2 is useful to obtain a constructive proof of the observability of the wave equation under the GCC. This is useful when one wants to have estimates of the observability constants in some regimes. These results are described in Section 3.2 and rely on [LL16].
4. Some stability estimates for the observation of hypoelliptic (sum of squares) operators and their evolution counterparts. Namely we obtain quantitative unique continuation for eigenfunctions, wave-type, heat-type equations related to a sum of squares operator. As usual, such estimates can be transferred to (approximate) control results, but also to stabilization of damped equations. The results and some ideas of the proof are presented in Section 4. They were obtained in [LL17, LL20b].

2 Quantitative unique continuation under partial analyticity

2.1 General results of quantitative unique continuation

In this section, we describe the setting of the general stability result we obtained and present the class of partial differential operators we deal with. We consider domains $\Omega \subset \mathbb{R}^n = \mathbb{R}^{n_a} \times \mathbb{R}^{n_b}$, where $n_a + n_b = n$. We denote by $x = (x_a, x_b)$ the global variables and $\xi = (\xi_a, \xi_b)$ the associated dual variables. The variables x_a will denote the set of variables in which the considered operator is analytic. In the examples studied below, this will be the time variable t when we consider the classical (Riemannian) wave equation, while it will be the full set of variables when hypoelliptic operators are considered.

Given a bounded domain $\Omega \subset \mathbb{R}^n = \mathbb{R}^{n_a} \times \mathbb{R}^{n_b}$, we say that a smooth function $f : \Omega \rightarrow \mathbb{C}$ is analytic with respect to x_a if, for any point $x^0 = (x_a^0, x_b^0) \in \Omega$, there is $\varepsilon > 0$ such that f extends as a holomorphic function in the variable x_a for $x = (x_a, x_b) \in (B(x_a^0, \varepsilon) + iB(0, \varepsilon)) \times B(x_b^0, \varepsilon)$.

The unique continuation and the associated quantification is proved true under some very general assumptions for the operators, named analytically principally normal operators by Tataru [Tat99b, Definition 2.2]. Yet, for simplification, we will give a particular case of this condition that will be sufficient for most of the applications we have in mind.

Assumption 2.1. Let P be a differential operator on an open set $\Omega \subset \mathbb{R}^{n_a} \times \mathbb{R}^{n_b}$ of order $m \in \mathbb{N}^*$ with smooth coefficients and **real-valued** principal symbol $p(x_a, x_b, \xi_a, \xi_b)$. We will assume that all coefficients of P are **real-analytic in the variable** x_a and that

$$\partial_{x_a} p(x_a, x_b, 0, \xi_b) = 0, \quad \text{for } (x_a, x_b) \in \Omega, \xi_b \in \mathbb{R}^{n_b}. \quad (2.1)$$

We now formulate the definition of strongly pseudoconvex surfaces for an operator P .

Definition 2.1 (Strongly pseudoconvex oriented surface). Let $\Omega \subset \mathbb{R}^n$, Γ be a closed conic subset of $T^*\Omega$, and let P be a differential operator with principal symbol p . Let S be a C^2 oriented hypersurface of Ω and $x^0 \in S \cap \Omega$. We say that S is strongly pseudoconvex in Γ at x^0 for P if there exists $\phi \in C^2(\Omega; \mathbb{R})$ such that $S = \{\phi = 0\}$, $\nabla\phi(x^0) \neq 0$, satisfying:

$$\operatorname{Re} \{\bar{p}, \{p, \phi\}\}(x^0, \xi) > 0, \quad \text{if } p(x^0, \xi) = \{p, \phi\}(x^0, \xi) = 0 \text{ and } \xi \in \Gamma_{x^0}, \xi \neq 0; \quad (2.2)$$

$$\frac{1}{i\tau} \{\bar{p}_\phi, p_\phi\}(x^0, \xi) > 0, \quad \text{if } p_\phi(x^0, \xi) = \{p_\phi, \phi\}(x^0, \xi) = 0 \text{ and } \xi \in \Gamma_{x^0}, \tau > 0, \quad (2.3)$$

where $p_\phi(x, \xi) = p(x, \xi + i\tau\nabla\phi)$.

Note that this is a property of the *oriented* surface S solely, and not of the defining function ϕ (see [Hör94], beginning of Section 28.3). If $\Gamma = T^*\Omega$, it is the usual condition of the Hörmander Theorem (see [Hör94, Section 28.3]), that is, under which uniqueness holds for P at x^0 across the hypersurface S , i.e. from $\phi > 0$ to $\phi < 0$.

Below, this condition will always be used for $\Gamma = \{\xi_a = 0\}$. In this case, and using the homogeneity of p in ξ , Assumption (2.3) may be rephrased as:

$$\frac{1}{i} \{\bar{p}(x, \xi - i\nabla\phi), p(x, \xi + i\nabla\phi)\}(x^0, 0, \xi_b) > 0, \quad \text{if } p(\zeta) = \{p, \phi\}(\zeta) = 0, \quad \xi_b \in \mathbb{R}^{n_b},$$

where $\zeta = (x^0, i\nabla_a\phi(x^0), \xi_b + i\nabla_b\phi(x^0))$. An important feature of this definition is that it is invariant by changes of coordinates.

Before stating our main result, let us discuss some cases of operators of particular interest.

Remark 2.2 (Hörmander case). If $n_a = 0$, there is no analytic variable. In this case, Definition 2.1 coincides with the definition of principally normal operators [Hör94, Chapter XXVIII] and Definition 2.1 with $\Gamma = T^*\Omega$ that of strictly pseudoconvex functions. The unique continuation result under consideration is the classical Hörmander theorem [Hör94, Chapter XXVIII].

Remark 2.3 (Holmgren-John case). If $n_a = n$, that is the operator is analytic in all the variables, we have $x_a = x, \xi_a = \xi$, and hence $\Gamma = \Omega \times \{\xi_a = 0\} = \Omega \times \{\xi = 0\}$. In this situation, Condition (2.1) is empty ($\partial_{x_a} p(x_a, \xi_a)$ is a homogeneous polynomial of degree $m \geq 1$ in ξ_a , where m is the order of P ; hence it vanishes at $\xi_a = 0$).

Next, concerning the conditions on the surface $\{\phi = 0\}$, notice that (2.2) is also empty since $\Gamma_{x^0} \cap \{\xi \neq 0\} = \emptyset$. For (2.3), if $\xi \in \Gamma_{x^0}$, that is $\xi = 0$, we have $p_\phi(x^0, \xi) = p(x^0, i\tau \nabla \phi(x^0)) = (i\tau)^m p(x^0, \nabla \phi(x^0))$: any noncharacteristic surface at x^0 (i.e. satisfying $p(x^0, \nabla \phi(x^0)) \neq 0$) is a strongly pseudoconvex oriented surface. The unique continuation result under consideration is the classical Holmgren-John theorem.

Note that, in the case $n_a = n$, the results presented here hold under Condition (2.3), namely:

$$p(x^0, \nabla \phi(x^0)) = \{p, \phi\}(x^0, \nabla \phi(x^0)) = 0 \implies \frac{1}{i} \{\bar{p}(x, \xi - i\nabla \phi), p(x, \xi + i\nabla \phi)\}(x^0, 0) > 0,$$

which is weaker than the noncharactericity condition $p(x^0, \nabla \phi(x^0)) \neq 0$ of the Holmgren-John theorem.

Remark 2.4 (Wave type and Schrödinger type operators). Let us now consider the case of operators P of principal symbol of the form $p_2(x, \xi) = Q_x(\xi)$, where Q_x is a smooth x -family of real quadratic forms in ξ , such that $Q_x(0, \xi_b)$ is positive (or negative) definite on \mathbb{R}^{n_b} . This is the case of the wave operator or Schrödinger type operators when x_a is the time variable. Then, Assumption (2.2) holds (uniformly with respect to $x \in \Omega$) again according to the positive definiteness of $Q_x(0, \xi_b)$. It is indeed empty since $p_2(x, (0, \xi_b))$ does not vanish for $\xi_b \neq 0$. Moreover, we have $\{p_2, \phi\}(x, \xi) = 2\tilde{Q}_x(\xi, \nabla \phi)$, where \tilde{Q}_x is the polar form of Q_x , and

$$\{p_2, \phi\}(x, \xi + i\nabla \phi) = 2\tilde{Q}_x(\xi, \nabla \phi) + 2iQ_x(\nabla \phi).$$

As a consequence (Q being real), $\text{Im}\{p_2, \phi\}(x, \xi + i\nabla \phi) = 2Q_x(\nabla \phi)$ so that (2.3) is also empty (and thus satisfied) for any noncharacteristic hypersurface.

In conclusion, for real quadratic forms which are positive (or negative) definite on \mathbb{R}^{n_b} at $\xi_a = 0$, any noncharacteristic hypersurface is strongly pseudoconvex in the sense of Definition 2.1. In the case $n_a = 1$, this includes the following operators of particular interest:

- $P = D_{x_a}^2 - \sum_{i,j=1}^{n-1} \alpha_{ij}(x) D_{x_b^j} D_{x_b^i} + \ell.o.t.$ (wave operator) with $p = \xi_a^2 - \sum_{i,j=1}^{n-1} \alpha_{ij}(x) \xi_b^j \xi_b^i$,
- $P = D_{x_a} - \sum_{i,j=1}^{n-1} \alpha_{ij}(x) D_{x_b^j} D_{x_b^i} + \ell.o.t.$ (Schrödinger operator) with $p = -\sum_{i,j=1}^{n-1} \alpha_{ij}(x) \xi_b^j \xi_b^i$,

where the quadratic form with coefficients $\alpha_{i,j}$ is positive definite.

We are now prepared to formulate our main result in the general framework. We first describe the geometric context and then state the Theorem.

Geometric setting: (see Figure 1) We first fix two splittings of \mathbb{R}^n as $\mathbb{R}^n = \mathbb{R}_{x'}^{n-1} \times \mathbb{R}_{x_n}$ and $\mathbb{R}^n = \mathbb{R}_{x_a}^{n_a} \times \mathbb{R}_{x_b}^{n_b}$, possibly in two different bases. We let D be a bounded open subset of \mathbb{R}^{n-1} with smooth boundary and $G = G(x', \varepsilon)$ a C^2 function defined in a neighborhood of $\bar{D} \times [0, 1]$, such that

- For all $\varepsilon \in (0, 1]$, we have $\{x' \in \mathbb{R}^{n-1}, G(x', \varepsilon) \geq 0\} = \overline{D}$;
- for all $x' \in D$, the function $\varepsilon \mapsto G(x', \varepsilon)$ is strictly increasing;
- for all $\varepsilon \in (0, 1]$, we have $\{x' \in \mathbb{R}^{n-1}, G(x', \varepsilon) = 0\} = \partial D$.

We set $G(x', 0) = 0$, $S_0 = \overline{D} \times \{0\}$ and, for $\varepsilon \in (0, 1]$,

$$\begin{aligned} S_\varepsilon &= \{(x', x_n) \in \mathbb{R}^n, x_n \geq 0 \text{ and } G(x', \varepsilon) = x_n\} \\ &= (\overline{D} \times \mathbb{R}) \cap \{(x', x_n) \in \mathbb{R}^n, G(x', \varepsilon) = x_n\}; \\ K &= \{x \in \mathbb{R}^n, 0 \leq x_n \leq G(x', 1)\}. \end{aligned}$$

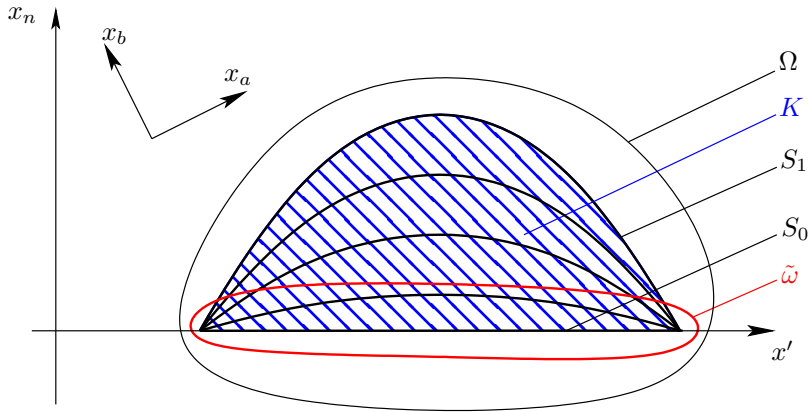


Figure 1: Geometric setting of Theorem 2.5

Theorem 2.5. *In the above geometric setting, we moreover let Ω be a bounded open neighborhood of K , and P be a differential operator of order m , satisfying Assumption 2.1 on Ω in $\{\xi_a = 0\}$.*

Assume also that, for any $\varepsilon \in [0, 1 + \eta)$, the oriented surfaces $S_\varepsilon = \{\phi_\varepsilon = 0\}$ with $\phi_\varepsilon(x', x_n) := G(x', \varepsilon) - x_n$ are strictly pseudoconvex in $\{\xi_a = 0\}$ for P on the whole S_ε , in the sense of Definition 2.1.

Then, for any open neighborhood $\tilde{\omega} \subset \Omega$ of S_0 , there exists a neighborhood U of K , and constants $\kappa, C, \mu_0 > 0$ such that for all $\mu \geq \mu_0$ and $u \in C_0^\infty(\mathbb{R}^n)$, we have

$$\|u\|_{L^2(U)} \leq C e^{\kappa\mu} \left(\|u\|_{H_b^{m-1}(\tilde{\omega})} + \|Pu\|_{L^2(\Omega)} \right) + \frac{C}{\mu^{m-1}} \|u\|_{H^{m-1}(\Omega)},$$

where we have denoted $\|u\|_{H_b^{m-1}(\tilde{\omega})} = \sum_{|\beta| \leq m-1} \|D_b^\beta u\|_{L^2(\tilde{\omega})}$.

Note that in the framework of the Hörmander theorem ($n_a = 0$), we can obtain the stronger polynomial-type dependence :

$$\|u\|_{H^{m-1}(U)} \leq C \left(\|u\|_{H^{m-1}(\tilde{\omega})} + \|Pu\|_{L^2(\Omega)} \right)^\delta \|u\|_{H^{m-1}(\Omega)}^{1-\delta}$$

for some $\delta \in (0, 1)$. This result was more or less already known even if not written explicitly in this geometric framework for general operators (see [Bah87, Rob95, LR95, LRL12]).

The formulation of the above result using a foliation by hypersurfaces is inspired by that of [Joh49, Theorem p. 224] in the context of the Holmgren-John theorem. Most of the global Theorems for the wave equations and hypoelliptic equations presented below are proved in the setting of Theorem 2.5, after some suitable change of coordinates.

2.2 Idea of the proof

As already mentioned, unique continuation theorems (e.g. the Hörmander theorem) are often proved with Carleman estimates, that is, weighted L^2 estimates of the form

$$\left\| e^{\tau\psi} u \right\|_{L^2} \leq C \left\| e^{\tau\psi} Pu \right\|_{L^2}, \quad (2.4)$$

where τ is a large parameter and ψ a weight function having levelsets appropriately situated with respect to the surface S . Such inequalities are already quantitative, and hence furnish a good starting point towards local quantitative unique continuation results. This strategy has already been followed in [Rob95, LR95] in the case of elliptic operators, see also [Bah87]. Starting from the Carleman inequality (2.4), the idea is to apply the estimates to some function $\chi(x)u$ where χ is a cutoff function according to the levelsets of ψ . The exponential weight $e^{\tau\psi(x)}$ in (2.4) (giving an exponentially large/small strength to the large/small values of ψ) naturally leads to inequalities of the form

$$\|u\|_{V_2} \leq e^{\kappa\mu} (\|u\|_{V_1} + \|Pu\|_{V_3}) + e^{-\kappa\mu} \|u\|_{V_3}, \quad (2.5)$$

uniformly for $\mu \geq \mu_0$ and for small open sets $V_1 \subset V_2 \subset V_3$ depending on the local geometry (namely, on the cutoff function χ , the support of $[P, \chi]$, and hence on the levelsets of ψ). Optimizing in μ (see [Rob95] or [LRL12, Lemma 5.2]) this can then be written as an interpolation estimate

$$\|u\|_{V_2} \leq (\|u\|_{V_1} + \|Pu\|_{V_3})^\delta \|u\|_{V_3}^{1-\delta},$$

for some $\delta \in (0, 1)$. The interest of these interpolation estimates (or directly of estimates like (2.5)) is that they can be easily iterated, leading to some global ones. This procedure ends up with a Hölder type dependence. We refer for instance to the survey article [LRL12] for a description of these estimates in the elliptic case, with application to spectral estimates and control results for the heat equation.

Yet, in the context of the unique continuation theorem for partially analytic operators, the Carleman estimates proved in [Tat95, RZ98, Hör97, Tat99b] contain a “microlocal” weight of the form $e^{-\frac{\varepsilon}{2\tau}|D_a|^2} e^{\tau\psi(x)}$ instead of $e^{\tau\psi(x)}$. Whereas the usual $e^{\tau\psi}$ is still here to give strength to the levelsets of ψ , the additional term $e^{-\frac{\varepsilon}{2\tau}|D_a|^2}$ is now aimed at localizing in the low frequencies in the variable x_a . In this context, the proof of unique continuation proceeds with a (qualitative) complex analytic argument (maximum principle). Here, this additional argument in the proof of unique continuation also requires to be quantified. As in [Rob95], this procedure naturally leads to local logarithmic (instead of Hölder) stability estimates. The main issue one then has to face when quantifying unique continuation is that such estimate cannot be iterated (or would yield dependence estimates of the type (1.2) with a function φ being a composition of as many “log” as steps needed in the iteration).

One idea to overcome this difficulty, proposed by Tataru in his unpublished lecture notes [Tat99a], was to propagate some low frequency estimates of the form

$$\begin{cases} \|u\|_{H^{m-1}} = 1 \\ \|m(D_a/\mu) \sigma(x/R) Pu\|_{L^2} \leq e^{-\mu^\alpha} \end{cases} \implies \|m(D_a/\tau) \sigma(x/r) u\|_{H^{m-1}} \leq e^{-\tau}, \quad \forall \tau < c \mu^\alpha$$

for functions u supported in $\{\phi < \phi(x_0)\}$, for some appropriate compactly supported cutoff functions σ and $m(\xi)$ in the Gevrey class $1/\alpha$, $\alpha < 1$, and for some $r < R$. Such estimates could be propagated and would lead to some global stability estimates of the form (1.2) with $\varphi_\varepsilon(a, b, c) = c(\log(1 + c/(a + b)))^{-(1-\varepsilon)}$.

The loss $1 - \varepsilon$ in the power of log is due to the use of functions of class Gevrey α with compact support. The optimal case $\alpha = 1$ would correspond to analytic functions. Yet, analytic functions cannot have compact support, which is a key ingredient in the usual application of Carleman estimates.

Let us now explain our strategy to solve this problem.

2.2.1 Obtaining local information at low frequency

Part of the proof of the present paper is inspired by this idea of propagating only low frequency (in the analytic variable x_a) estimates. However, we replace the Gevrey cutoff functions by some analytic “almost cutoff” functions of the form

$$\chi_\lambda := e^{-|D_a|^2/\lambda} \chi, \quad (2.6)$$

where χ is smooth with the expected compact support, being convolved/regularized with a heat kernel in the variable x_a , hence analytic in this variable. It turns out that the right choice of the regularization parameter λ is $\lambda = C\mu$ where μ is the frequency where we want to measure our solution. That such functions are not compactly supported makes all commutator estimates (e.g. when applying the Carleman estimate to functions like $\chi_\lambda u$ instead of χu , as explained above) much more intricate and requires a careful study of the dependence with respect the regularisation parameter λ , the local frequency μ and the parameter τ in the Carleman inequality. All estimates are carried out up to an exponentially small remainder (in terms of these parameters).

Following this procedure, the local estimate we prove (which we are in addition able to propagate) is a generalization of (2.5), but truncated at low frequencies in the analytic variable x_a . In a neighborhood of a point x^0 , it is of the form

$$\begin{aligned} & \|m_\mu(D_a/\beta\mu) \chi_{2,\mu} u\|_{H^{m-1}} \\ & \leq C e^{\kappa\mu} \left(\|m_\mu(D_a/\mu) \chi_{1,\mu} u\|_{H^{m-1}} + \|Pu\|_{L^2(B(x^0,R))} \right) + C e^{-\kappa'\mu} \|u\|_{H^{m-1}}, \end{aligned} \quad (2.7)$$

uniformly for $\mu \geq \mu_0$. Here, χ_1 and χ_2 are some cutoff functions in the physical space that localize respectively to the place where the information is taken (locally in $\{\phi > 0\}$) and to where it is propagated (a small neighborhood of x^0). These functions respectively correspond to $\mathbb{1}_{V_1}$ and $\mathbb{1}_{V_2}$ in (2.5). The Fourier multipliers m_μ cuts off (analytically) the ξ_a frequencies (m has to be thought of as $\mathbb{1}_{B_{\mathbb{R}^{n_a}}(0,1)}$). All these cutoff functions are used only with their analytic regularization according to (2.6) with $\lambda = \mu$. They never localize exactly. Using such regularized cutoff functions and Fourier multipliers follows the spirit of analytic semiclassical analysis. However, we do not make use of that theory and rather construct by hand the appropriate mollifiers, making the proof selfcontained in this respect.

The proof of estimates like (2.7) mainly proceeds in three steps.

First, as in the usual proofs of unique continuation results, starting from the hypersurface $\{\phi = 0\}$, one needs to construct a weight function ψ with both properties:

- to satisfy the assumptions required to apply the Carleman estimate (ψ should be a strictly pseudoconvex *function*);
- to have level sets appropriately located with respect to those of ϕ (so that propagating uniqueness across levelsets of ψ still corresponds to propagating zero locally from $\phi > 0$ to $\phi < 0$).

This corresponds to the so called “convexification process”, see [Hör94, Chapter XXVIII].

Second, we apply as a black box the Carleman estimates of [Tat95, RZ98, Hör97, Tat99b] (or some similar ones that we prove in the presence of boundary) to χu , where χ is a particular cutoff function (localizing near the point of interest, and according to levelsets of ψ), containing both rough cutoffs and mollified ones. We then need to estimate all terms arising from the commutator $e^{-\varepsilon/2\tau|D_a|^2} e^{\tau\psi} [P, \chi]$, that are either well localized or yield an exponentially small contribution.

Finally, we need to transfer the information given by the Carleman inequality to some estimate like (2.7) on the low frequencies of the function. This is done through a complex analysis argument,

the Carleman parameter τ playing the role of complex variable, as in [Tat95]. If ζ is the complex variable, the Carleman estimates corresponds to an estimate on $\zeta = i\tau \in i\mathbb{R}_+$. Combined with *a priori* estimates, a Phragmén-Lindelöf type theorem allows to extend this estimate to part of the real domain, where it corresponds to estimating $\|m(D_a/\beta\mu)\chi u\|$. To obtain estimates that are uniform with respect to the frequency (and regularization) parameter μ , we also need, following [Tat99a], to use a scaling argument, replacing τ by τ/μ .

2.2.2 Propagating local informations to global ones

Once the local estimates are proved, we need to iterate them to obtain a global estimate. At first, we define some tools that will allow later in an abstract way to propagate easily our local estimate (2.7). Estimate (2.7) says essentially that, for a solution of $Pu = 0$, information can be transferred from the support of χ_1 to the support of χ_2 . We formalize that with the notion of *zone of dependence*. Roughly speaking, we say that on open set O_2 depends on O_1 if (2.7) holds for every χ_1 equals to 1 on O_1 and any χ_2 supported in O_2 . This part allows to formulate the proof of Theorem 2.5 as a complete geometric one. Even if quite different in definition, it is close in spirit to the interpolation theory developped in Lebeau [Leb92] to propagate globally the local information obtained by the Cauchy-Kowalevski theorem. Moreover, it should adapt to some more general kinds of foliations. Note that at each step of this propagation argument, we have a loss in the range of frequency: from an information on frequencies $\leq \mu$, we obtain from (2.7) an information on frequencies $\leq \beta\mu$, with β small. This is overcome by the fact that we only have a finite number of steps in this iterative procedure.

Once this propagation result is done, we are left with a low frequency information of the solution u . Since we have no information about the high frequency part, the only thing to do is to use some trivial bound of the type

$$\|(1 - m(D_a/\mu))u\|_{L^2} \leq \frac{C}{\mu^{m-1}} \|u\|_{m-1}.$$

This is actually much worse than the negative exponential that we already had. But it turns out to be the best we can do without any more information. It gives the final estimate of Theorem 2.5.

3 Applications to the observability and control of the wave equation

3.1 Logarithmic stability without geometric assumption

In this section, we describe the motivating applications of the results presented in Section 2, i.e. to the wave equation with Dirichlet boundary conditions. In this very particular setting, we are also able to tackle the boundary value problem.

When dealing with a manifold with boundary \mathcal{M} , we will always assume that the manifold, the boundary and the metric are smooth. Moreover, $\text{Int}(\mathcal{M})$ will denote the set of points in \mathcal{M} which have a neighborhood homeomorphic to an open subset of \mathbb{R}^n . The boundary of \mathcal{M} , denoted by $\partial\mathcal{M}$, is the complement of $\text{Int}(\mathcal{M})$ in \mathcal{M} . All manifolds considered will be assumed to be connected. For a subset $\omega \subset \mathcal{M}$, we will define $\mathcal{L}(\mathcal{M}, \omega) := \sup_{x \in \mathcal{M}} \text{dist}(x, \omega)$ which is finite since \mathcal{M} is compact and connected.

Theorem 3.1 (Quantitative unique continuation for waves). *Let \mathcal{M} be a compact Riemannian manifold with (or without) boundary. For any nonempty open subset ω of \mathcal{M} and any $T > 2\mathcal{L}(\mathcal{M}, \omega)$,*

there exist $C, \kappa, \mu_0 > 0$ such that for any $(u_0, u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$ and associated solution u of

$$\begin{cases} \partial_t^2 u - \Delta_g u = 0 & \text{in } (0, T) \times \text{Int}(\mathcal{M}), \\ u|_{\partial\mathcal{M}} = 0 & \text{in } (0, T) \times \partial\mathcal{M}, \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) & \text{in } \text{Int}(\mathcal{M}), \end{cases} \quad (3.1)$$

we have, for any $\mu \geq \mu_0$,

$$\|(u_0, u_1)\|_{L^2 \times H^{-1}} \leq C e^{\kappa\mu} \|u\|_{L^2((0, T); H^1(\omega))} + \frac{1}{\mu} \|(u_0, u_1)\|_{H^1 \times L^2}. \quad (3.2)$$

If $\partial\mathcal{M} \neq \emptyset$ and Γ is a non empty open subset of $\partial\mathcal{M}$, for any $T > 2\mathcal{L}(\mathcal{M}, \Gamma)$, there exist $C, \kappa, \mu_0 > 0$ such that for any $(u_0, u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$ and associated solution u of (3.1), we have

$$\|(u_0, u_1)\|_{L^2 \times H^{-1}} \leq C e^{\kappa\mu} \|\partial_\nu u\|_{L^2((0, T) \times \Gamma)} + \frac{1}{\mu} \|(u_0, u_1)\|_{H^1 \times L^2}.$$

Theorem 3.1 remains valid if Δ_g is perturbed by lower order terms that are analytic in time but may have low regularity in space. In the special case where they are time independent, the constants in the previous estimates may be chosen uniformly with respect to these perturbations (in the appropriate norms). Note that in (3.2), the $L^2(0, T; H^1(\omega))$ norm can actually be replaced by a $L^2(0, T; L^2(\omega))$ norm according to [LL17, Section 5.3]. This result can also be formulated in an equivalent way, that looks more like a stability estimate. We only give the boundary observation case, the internal observation case being similar.

Corollary 3.2. *Assume $\partial\mathcal{M} \neq \emptyset$ and Γ is a non empty open subset of $\partial\mathcal{M}$. Then, for any $T > 2\mathcal{L}(\mathcal{M}, \Gamma)$, there exists $C > 0$ such that for any $(u_0, u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M}) \setminus \{(0, 0)\}$ and associated solution u of (3.1), we have*

$$\begin{aligned} \|(u_0, u_1)\|_{L^2 \times H^{-1}} &\leq C \frac{\|(u_0, u_1)\|_{H^1 \times L^2}}{\log\left(1 + \|(u_0, u_1)\|_{H^1 \times L^2} / \|\partial_\nu u\|_{L^2(0, T \times \Gamma)}\right)}, \\ \|(u_0, u_1)\|_{H^1 \times L^2} &\leq C e^{C\Lambda} \|\partial_\nu u\|_{L^2((0, T) \times \Gamma)}, \quad \text{with } \Lambda = \frac{\|(u_0, u_1)\|_{H^1 \times L^2}}{\|(u_0, u_1)\|_{L^2 \times H^{-1}}}. \end{aligned}$$

In the first estimate, the function on the right hand-side is to be understood as being $(\log(1 + 1/x))^{-1}$ for $x > 0$ and 0 for $x = 0$.

In the second estimate, Λ has to be considered as the typical frequency of the initial data. So, the estimate states a cost of observability of the order of an exponential of the typical frequency. As an illustration, taking for initial data $(u_0, u_1) = (\psi_\lambda, 0)$ with ψ_λ a normalized eigenfunction of the Laplace-Dirichlet operator on \mathcal{M} , associated to the eigenvalue λ , one has $\Lambda \sim \sqrt{\lambda}$ and Corollary 3.2 recovers the tunneling estimate $\|\partial_\nu \psi_\lambda\|_{L^2(\Gamma)} \geq C^{-1} e^{-C\sqrt{\lambda}}$ (see [LR95]).

The minimal time $2\mathcal{L}(\mathcal{M}, \omega)$ or $2\mathcal{L}(\mathcal{M}, \Gamma)$ in these two theorems is optimal (even for qualitative unique continuation) in view of the finite speed of propagation for the wave equation. Moreover, as proved by Lebeau [Leb92] in the analytic context, this exponential dependence is sharp in general. More precisely, the form of the estimates of Theorem 3.1 and Corollary 3.2 is optimal as soon as there is a ray of geometric optics (travelling at speed one) which does not intersect the region $\bar{\Gamma}$ (resp. $\bar{\omega}$ in the internal observation case) in the time interval $[0, T]$ (and only has transverse intersection with the boundary). See [Leb92, Section 2, pages 5 and 6].

The proof of Theorem 3.1 (and the variant Theorem 3.4) relies on several applications of our main quantitative unique continuation Theorem 2.5 with some well chosen foliation of non characteristic hypersurfaces.

As a consequence of Theorem 3.1, we obtain the following approximate controllability results, with (optimal in general) estimate on the cost. For the sake of brevity, we only state the case of a boundary control.

Theorem 3.3 (Cost of boundary approximate control). *For any $T > 2\mathcal{L}(\mathcal{M}, \Gamma)$, there exist $C, c > 0$ such that for any $\varepsilon > 0$ and any $(u_0, u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$, there exists $g \in L^2((0, T) \times \Gamma)$ with*

$$\|g\|_{L^2((0, T) \times \Gamma)} \leq C e^{\frac{c}{\varepsilon}} \|(u_0, u_1)\|_{H_0^1(\mathcal{M}) \times L^2(\mathcal{M})},$$

such that the solution of

$$\begin{cases} (\partial_t^2 - \Delta)u = 0 & \text{in } (0, T) \times \text{Int}(\mathcal{M}), \\ u|_{\partial\mathcal{M}} = \mathbf{1}_\Gamma g & \text{in } (0, T) \times \partial\mathcal{M}, \\ (u, \partial_t u)|_{t=0} = (u_0, u_1), & \text{in } \text{Int}(\mathcal{M}), \end{cases}$$

satisfies $\|(u, \partial_t u)|_{t=T}\|_{L^2(\mathcal{M}) \times H^{-1}(\mathcal{M})} \leq \varepsilon \|(u_0, u_1)\|_{H_0^1(\mathcal{M}) \times L^2(\mathcal{M})}$.

That this result is a consequence of Theorem 3.1 is proved in [Rob95, Proof of Theorem 2, Section 3]. The solution of the nonhomogeneous boundary value problem is defined in the sense of transposition, see [Lio88].

The estimates of Theorem 3.1 and Corollary 3.2 can actually be stated more locally, and interpreted in a different physical context (motivated by [RT73]). The following Theorem shows that they are independent on the global geometry, and, in particular, do not require that \mathcal{M} is compact if one only wants to recover data supported in a given compact set.

Theorem 3.4 (Penetration into shadow for waves). *Let \mathcal{M} be a complete Riemannian manifold with (possibly empty) compact boundary $\partial\mathcal{M}$. Let ω_0 be an open set of \mathcal{M} and ω_1 a compact set of \mathcal{M} . Then, for any*

$$T > \mathcal{L}(\omega_1, \omega_0) := \sup_{x \in \omega_1} \text{dist}(x, \omega_0),$$

there exist $C > 0$ such that for any $(u_0, u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M}) \setminus \{(0, 0)\}$ supported in ω_1 and associated solution u of (3.1) (taken on the time interval $(-T, T)$ instead of $(0, T)$), we have,

$$\|(u_0, u_1)\|_{H^1 \times L^2} \leq C e^{C\Lambda} \|u\|_{L^2((-T, T); H^1(\omega_0))}, \quad \text{with } \Lambda = \frac{\|(u_0, u_1)\|_{H^1 \times L^2}}{\|(u_0, u_1)\|_{L^2 \times H^{-1}}}.$$

Roughly speaking, the theorem describes the following physical situation: take a noise creating an initial data compactly supported in ω_1 , and take an observer located in a zone ω_0 . Then, by observing during the time interval $(-\mathcal{L}(\omega_1, \omega_0) - \varepsilon, \mathcal{L}(\omega_1, \omega_0) + \varepsilon)$, $\varepsilon > 0$, the observer will be able to recover at least a proportion of the initial energy of the order $e^{-C\Lambda}$ where Λ is the typical frequency of the data. This result is particularly interesting if the zone ω_1 is in the “shadow” of an obstacle when seen from ω_0 , that is if no rays of geometric optic starting from ω_1 ever reach ω_0 . In that case, the classical geometric optic approximation would predict that the observer does not receive any information. We refer to [RT73] for a qualitative result in infinite time; here, Theorem 3.4 provides a quantitative result in finite time, which is optimal with respect to the time and the form of the estimate if ω_1 is indeed in the “shadow” region when observed from ω_0 . More precisely, [Leb92, Section 2] implies that the $e^{C\Lambda}$ is optimal as soon as there is a ray of geometric optics (having only transverse intersections with $\partial\mathcal{M}$) starting from the interior of ω_1 at time zero and not intersecting $\overline{\omega_0}$ during the time interval $[-T, T]$. Such an estimate in the shadow region is reminiscent to the tunneling effect for waves and in semiclassical analysis.

We also obtain related results for the Schrödinger equation, see [LL19, Theorem 1.5]. The latter formulate in a similar way but hold in arbitrary small time.

Idea of proof Theorem 3.1 is proved using several applications of Theorem 2.5. According to Remark 2.4, it is applicable under the (quite weak) assumption that the hypersurfaces are non characteristic. Then, for each point $x \in \mathcal{M}$, we construct a path that links x to a point $x_1 \in \omega$ of length less than $T > \mathcal{L}(\mathcal{M}, \omega)$. Then, it is possible to construct a foliation of non characteristic hypersurfaces along which the application of Theorem 2.5 is possible. The construction of such hypersurfaces was inspired by that of [Leb92]. This allows then to transfer the information from a neighborhood of x_1 included in ω to a neighborhood of x . Since we can do that for any $x \in \mathcal{M}$, a compactness argument gives a global estimate quite close to the expected one.

$$\|u\|_{L^2(\cdot)_{-\varepsilon, \varepsilon}(\times \mathcal{M})} \leq C e^{\kappa \mu} \|u\|_{L^2(\cdot)_{-T, T}(\times \omega)} + \frac{C}{\mu} \|u\|_{H^1(\cdot)_{-T, T}(\times \mathcal{M})}.$$

Energy estimates finally allow to estimate by below and above the global space time norms with related norms of the initial data.

Note also that in the case of manifold with boundary, we needed to write new Carleman estimates with boundary for operators of order two with real valued principal symbol.

3.2 A constructive proof of the Bardos-Lebeau-Rauch theorem

Another application of Theorem 3.1, given in [LL16] and which was at the origin of the present work, is concerned with the *exact* observability/controllability problem. This property was completely characterized (with optimal geometric conditions) in the seminal paper [BLR92].

The purpose is to prove that if (ω, T) satisfies the Geometric Control Condition (that is, any ray of geometric optics travelling in $(0, T) \times \mathcal{M}$ with unit speed intersects $(0, T) \times \omega$), then, any solution of (3.1) satisfies the observability estimate

$$\|(u_0, u_1)\|_{H^1 \times L^2} \leq C_0 \|u\|_{L^2(0, T; H^1(\omega))}. \quad (3.3)$$

The proof in [BLR92] is non constructive, with the drawback that it does not give any information about the constant C_0 involved. The latter is of primary importance in applications since it may be interpreted as the cost of exact controls for the controlled wave equation (via the Hilbert Uniqueness Method, see [Lio88]). It is made in two steps, both of them being non constructive in the previously known literature:

1. high frequency part: proof of a weaker estimate

$$\|(u_0, u_1)\|_{H^1 \times L^2} \leq C \|u\|_{L^2((0, T); H^1(\omega))} + C \|(u_0, u_1)\|_{L^2 \times H^{-1}} \quad (3.4)$$

2. low frequency part: getting rid of the lower order term by reducing to a unique continuation type argument.

Step 1 concerning high frequency is proved with microlocal analysis and is therefore always performed “up to lower order terms”, which explains the presence of the remainder term in $L^2 \times H^{-1}$ -norm. It uses techniques like propagation of regularity, microlocal defect measures or Egorov theorem that are out of the scope of this review article. We implement in [LL16] a constructive proof inspired by [DL09] and relying on the Egorov Theorem, in case $\partial \mathcal{M} = \emptyset$.

Step 2 was usually performed in the literature with an argument by contradiction combined with a unique continuation theorem. Theorem 3.1 allows to give a completely constructive and direct proof of this step as follows. Combining the high frequency estimate (3.4) with our quantitative unique continuation result (3.2) gives directly uniformly for $\mu \geq \mu_0$,

$$\|(u_0, u_1)\|_{H^1 \times L^2} \leq C \|u\|_{L^2((0, T); H^1(\omega))} + C e^{\kappa \mu} \|u\|_{L^2((0, T); H^1(\omega))} + \frac{1}{\mu} \|(u_0, u_1)\|_{H^1 \times L^2}.$$

It implies (3.3) when μ is taken large enough so that to absorb the last term of the right hand side.

This methods allows us to obtain estimates of the observability constant (and therefore the cost of the control) in two regimes. We obtain

- the dependence of the control cost (the constant C_0 in (3.3)) with respect to the the addition of a potential $V(x)$ in the wave operator
- the dependence of control cost when the the observation time T approaches the critical time of the Geometric Control Condition.

We refer to [LL16] for precise statements.

4 Applications to the observability and control of hypoelliptic equations

The general result stated in Theorem 2.5 actually contains a quantitative version of the classical Holmgren-John theorem, see Remark 2.3. A classical result of Bony [Bon69], relying on the Holmgren-John Theorem, proves unique continuation for solutions to $\mathcal{L}u = Vu$ where \mathcal{L} is a hypoelliptic operators with analytic coefficients (and V and analytic potential). In this Section, we propose a quantitative version of this unique continuation result, together with generalizations to eigenfunctions, solutions to wave and heat equations associated to such operators \mathcal{L} . Most of the results are taken from [LL17], and a few of them from [LL20b].

In Section 4.1, we first present generalities about hypoelliptic operators and their analysis.

In Section 4.2, we detail our main results concerning eigenfunctions, wave-type operators, heat-type operators and damped equations.

In Section 4.3 we give some ideas of the proofs. The main technical part is the proof for the sub-Riemannian wave operator in Section 4.3.1, where we apply our quantitative version of the Holmgren-John theorem (Theorem 2.5) combined with hypoelliptic estimates. Sections 4.3.2, 4.3.3 and 4.3.4 describe the abstract functional analytic framework to deduce the results from the wave equation to eigenfunctions, heat-like and damped equations.

4.1 Generalities about sub-Riemannian geometry and analysis in this context

Let \mathcal{M} be a smooth compact connected manifold without boundary. We denote by \mathcal{X}^∞ the space of smooth vector fields on \mathcal{M} (with real coefficients), which we identify to derivations on \mathcal{M} . We assume \mathcal{M} is endowed with a smooth positive density measure ds , so that we may integrate functions on \mathcal{M} . We may then define the space $L^2(\mathcal{M}) = L^2(\mathcal{M}, ds)$ of square integrable functions with respect to this measure. For $X \in \mathcal{X}^\infty$, we define by X^* its formal dual operator for the duality of $L^2(\mathcal{M})$, that is,

$$\int_{\mathcal{M}} X^*(u)(x)v(x)ds(x) = \int_{\mathcal{M}} u(x)X(v)(x)ds(x), \quad \text{for any } u, v \in C^\infty(\mathcal{M}).$$

Given $m \in \mathbb{N}$ and m vector fields. $X_1, \dots, X_m \in \mathcal{X}^\infty$, we are interested in properties of the following (non-positive) second order operator, associated to the X_i 's (namely the so-called type I Hörmander operator).

$$\mathcal{L} = \sum_{i=1}^m X_i^* X_i. \tag{4.1}$$

Note that this operator is formally symmetric nonnegative, when defined on functions in $C^\infty(\mathcal{M})$, since we have

$$(\mathcal{L}u, u)_{L^2(\mathcal{M})} = \sum_{i=1}^m \|X_i u\|_{L^2(\mathcal{M})}^2.$$

Both from the geometric control and the operator theoretic points of view, it is in this context natural to consider iterated Lie brackets of the vector fields X_i . We refer for instance to [ABB20].

Definition 4.1. For any family \mathcal{F} of smooth vector fields on \mathcal{M} and $\ell \in \mathbb{N}^*$, we define the subspaces $\text{Lie}^\ell(\mathcal{F})$ of \mathcal{X}^∞ by iteration as follows:

- $\text{Lie}^1(\mathcal{F})$ is the space spanned by \mathcal{F} in \mathcal{X}^∞ ,
- $\text{Lie}^{\ell+1}(\mathcal{F}) = \text{span}(\text{Lie}^\ell(\mathcal{F}) \cup \{[X, Y]; X \in \mathcal{F}, Y \in \text{Lie}^\ell(\mathcal{F})\})$.

For any point $x \in \mathcal{M}$, $\ell \in \mathbb{N}^*$, we denote by $\text{Lie}^\ell(\mathcal{F})(x)$ the set of all tangent vectors $X(x)$ with $X \in \text{Lie}^\ell(\mathcal{F})$.

We shall always assume that the family (X_i) satisfies the Chow-Rashevski-Hörmander condition (or is “bracket generating”).

Assumption 4.1. There exists $\ell \geq 1$ so that for any $x \in \mathcal{M}$, $\text{Lie}^\ell(X_1, \dots, X_m)(x) = T_x \mathcal{M}$. Denote then by $k \in \mathbb{N}^*$ the minimal ℓ for which this holds.

The integer k will sometimes be referred to as the *hypoellipticity index* of \mathcal{L} . Assumption 4.1 is central in control theory and operator theory, for it characterizes both the controllability of the controlled ODE driven by the vector fields (X_i) and the Hypoellipticity of the operator \mathcal{L} . Let us now recall these two seminal results, namely the Chow-Rashevski theorem and the Hörmander theorem, which we both use in the sequel.

Theorem 4.2 (Chow-Rashevski). *Under Assumption 4.1, the following statement holds: for any $x_0, x_1 \in \mathcal{M}$, any $T > 0$, there exist $u_i \in L^1(0, T)$ for $i \in \{1, \dots, m\}$ such that the unique solution of*

$$\dot{\gamma}(t) = \sum_{i=1}^m u_i(t) X_i(\gamma(t)), \quad \gamma(0) = x_0$$

satisfies $\gamma(T) = x_1$.

This theorem motivates the following definition.

Definition 4.3 (Horizontal path). We say that an absolutely continuous function $\gamma : [0, T] \rightarrow \mathcal{M}$ is a horizontal path if there exist $u_i \in L^1(0, T; \mathbb{R})$ for $i = 1, \dots, m$ such that for almost every $t \in [0, T]$, we have $\dot{\gamma}(t) = \sum_{i=1}^m u_i(t) X_i(\gamma(t))$.

Such a trajectory is in particular absolutely continuous and almost everywhere tangent to the so-called horizontal distribution $\text{span}(X_1, \dots, X_m)$. The second key role played by Assumption 4.1 in analysis is summarized in the following result.

Theorem 4.4 (Hörmander [Hör67], Rothschild-Stein [RS76]). *Under Assumption 4.1, the operator \mathcal{L} in (4.1) is hypoelliptic, that is, for all $u \in \mathcal{D}'(\mathcal{M})$ and $x_0 \in \mathcal{M}$, if $\mathcal{L}u \in C^\infty$ near x_0 then $u \in C^\infty$ near x_0 .*

Moreover, it is subelliptic of order $1/k$, that is, the following estimates hold: there is $C > 0$ such that for any $u \in C^\infty(\mathcal{M})$, we have

$$\|u\|_{H^{1/k}(\mathcal{M})}^2 \leq C \sum_{i=1}^m \|X_i u\|_{L^2(\mathcal{M})}^2 + C \|u\|_{L^2(\mathcal{M})}^2, \quad (4.2)$$

$$\|u\|_{H^{2/k}(\mathcal{M})}^2 \leq C \|\mathcal{L}u\|_{L^2(\mathcal{M})}^2 + C \|u\|_{L^2(\mathcal{M})}^2. \quad (4.3)$$

The hypoellipticity was shown by Hörmander [Hör67], who also provided with a subelliptic estimate with loss. The optimal subelliptic estimate (4.2) with gain of $1/k$ derivatives is proved by [RS76].

Since the operator \mathcal{L} is symmetric non-negative, the hypoellipticity of $\mathcal{L} + 1$ and the compactness of \mathcal{M} directly imply that \mathcal{L} is essentially selfadjoint (see e.g. Reed-Simon [RS80, Theorem X.26]). Hence, it extends uniquely as a selfadjoint operator (its Friedrich extension)

$$\mathcal{L} : D(\mathcal{L}) \subset L^2(\mathcal{M}) \rightarrow L^2(\mathcal{M}),$$

with, according to (4.3), $H^2(\mathcal{M}) \subset D(\mathcal{L}) \subset H^{2/k}(\mathcal{M})$ (still under Assumption 4.1). The operator \mathcal{L} is hence selfadjoint on $L^2(\mathcal{M})$, with compact resolvent: it admits a Hilbert basis of eigenfunctions $(\varphi_j)_{j \in \mathbb{N}}$, associated with the real eigenvalues $(\lambda_j)_{j \in \mathbb{N}}$, sorted increasingly, that is

$$\mathcal{L}\varphi_i = \lambda_i \varphi_i, \quad (\varphi_i, \varphi_j)_{L^2(\mathcal{M})} = \delta_{ij}, \quad 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_j \rightarrow +\infty.$$

Note that a bootstrap argument in (4.3) shows that $\varphi_j \in C^\infty(\mathcal{M})$. Also, the spectral decomposition allows to define solutions of the hypoelliptic wave and heat equations (respectively $(\partial_t^2 + \mathcal{L})v = f$ and $(\partial_t + \mathcal{L})u = f$), which we shall consider in this paper.

In addition to Assumption 4.1, we will also assume in the main part of the article that everything is real-analytic.

Assumption 4.2. The manifold \mathcal{M} , the density ds , and the vector fields X_i are **real-analytic**.

In particular, it implies that the operator \mathcal{L} has analytic coefficients in any analytic coordinate set compatible with the manifold \mathcal{M} .

This assumption can be lowered (and we give some examples in [LL17]), but seems hard to avoid totally, due to some counterexamples to unique continuation [Bah86].

Finally, let us mention that hypoelliptic operators appear naturally in several physical and mathematical contexts such as stochastic processes and the theory of functions of several complex variables. We refer to [Bra14, Chapter 2] for a presentation of some of these applications. Classical examples of operators \mathcal{L} encompassed by this frameworks is also provided in [LL17, Section 1.1]: elliptic operators ($k = 1$, Grushin operators ($k \in N^*$), Heisenberg ($k = 2$), Lie Groups...

4.2 Main results for hypoelliptic equations

Our main results under Assumptions 4.1 and 4.2 are of four different types:

1. Tunneling estimates for eigenfunctions φ_j of \mathcal{L} (Section 4.2.1);
2. Quantitative approximate observability (and associated controllability) of the hypoelliptic wave equation $(\partial_t^2 + \mathcal{L})v = 0$ from a subset $\omega \subset \mathcal{M}$ (Section 4.2.2);

3. Quantitative approximate observability (and associated controllability) of the hypoelliptic heat equation $(\partial_t + \mathcal{L})u = 0$ from ω (Section 4.2.3).
4. Decay for damped hypoelliptic wave equations $(\partial_t^2 + \mathcal{L} + 1_\omega \partial_t)v = 0$, Schrödinger $(i\partial_t + \mathcal{L} + i1_\omega \partial_t)v = 0$ or damped plates $(\partial_t^2 + \mathcal{L}^2 + 1_\omega \partial_t)v = 0$ (Section 4.2.4)

All of these results depend explicitly on the hypoellipticity index k of the operator considered, i.e. the minimal number of iterated brackets necessary to span the whole tangent space, given by Assumption 4.1. We finally prove with an example that the results are optimal in general.

4.2.1 Eigenfunction tunneling

Our first result is the following.

Theorem 4.5. *Let ω be a nonempty open subset of \mathcal{M} . Then, there is $C, c > 0$ such that for all $(\lambda, \varphi) \in \mathbb{R}_+ \times L^2(\mathcal{M})$ satisfying $\mathcal{L}\varphi = \lambda\varphi$, we have*

$$\|\varphi\|_{L^2(\mathcal{M})} \leq C e^{c\lambda^{k/2}} \|\varphi\|_{L^2(\omega)}. \quad (4.4)$$

This estimate may be read as $\|\varphi\|_{L^2(\omega)} \geq \frac{1}{C} e^{-c\lambda^{k/2}}$ for all normalized eigenfunctions φ , and hence quantizes the possible vanishing rate of eigenfunctions on any subdomain ω .

In the case $k = 1$, i.e. when \mathcal{L} is an elliptic operator, the analyticity assumption 4.2 is not needed and the result follows from the Donnelly-Fefferman paper [DF88]. In this situation, it also holds on a manifold with boundary for Dirichlet eigenfunctions [DF90, LR95] (see also [LR97] for other boundary conditions).

We shall also deduce from estimates of [BCG14, Section 2.3] that the tunneling estimate (4.4) is optimal in the following particular setting.

Example 4.6 (Higher order Grushin operators on the rectangle). Consider the manifold *with boundary* $\mathcal{M} = [-1, 1] \times [0, 1]$ or $\mathcal{M} = [-1, 1] \times (\mathbb{R}/\mathbb{Z})$, endowed with the Lebesgue measure dx , and for $\gamma > 0$, define the operator $\mathcal{L}_\gamma = -(\partial_{x_1}^2 + x_1^{2\gamma} \partial_{x_2}^2)$ with Dirichlet conditions on $\partial\mathcal{M}$. If $\gamma \in \mathbb{N}$, then the operator \mathcal{L}_γ is hypoelliptic of order $k = \gamma + 1$ (i.e. Assumption 4.1 is fulfilled with $k = \gamma + 1$).

Proposition 4.7. *Consider, for $\gamma > 0$ the situation of Example 4.6. Assume that $\bar{\omega} \cap \{x_1 = 0\} = \emptyset$. Then there exists $C, c_0 > 0$ and a sequence (λ_j, φ_j) of eigenvalues and associated eigenfunctions of \mathcal{L}_γ with $\lambda_j \rightarrow +\infty$ such that*

$$\|\varphi_j\|_{L^2(\omega)} \leq C e^{-c_0 \lambda_j^{(\gamma+1)/2}} \|\varphi_j\|_{L^2(\mathcal{M})}.$$

We recall that if $\gamma \in \mathbb{N}^*$, then \mathcal{L}_γ is hypoelliptic of order $k = \gamma + 1$, so that Proposition 4.7 shows that, in general, one cannot expect a better estimate than that of Theorem 4.5.

Note that in the analytic context, the *qualitative* uniqueness:

$$\left(\mathcal{L}\varphi = \lambda\varphi \text{ on } \mathcal{M}, \quad \varphi = 0 \text{ on } \omega \right) \implies \varphi \equiv 0 \text{ on } \mathcal{M},$$

was proved by Bony [Bon69], as a consequence of the Holmgren-John theorem. Removing the analyticity assumption, even for such a qualitative unique continuation property, remains a very subtle issue (see [Bah86]).

4.2.2 Quantitative approximate observability of the hypoelliptic wave equation

We will need to define the Sobolev spaces related to the operators \mathcal{L} .

$$\mathcal{H}_{\mathcal{L}}^s = \{u \in \mathcal{D}'(\mathcal{M}), (1 + \mathcal{L})^{s/2}u \in L^2(\mathcal{M})\}, \quad s \in \mathbb{R},$$

and associated norms

$$\|u\|_{\mathcal{H}_{\mathcal{L}}^s} = \left\| (1 + \mathcal{L})^{s/2}u \right\|_{L^2(\mathcal{M})}, \quad s \in \mathbb{R}.$$

Let us now also introduce basic notions of sub-Riemannian geometry needed to formulate our main result. We refer to [ABB20] for a comprehensive introduction to sub-Riemannian geometry, as well as for further developments. The so-called sub-Riemannian metric associated to the vector fields (X_1, \dots, X_m) is defined, for $x \in \mathcal{M}$ and $v \in T_x\mathcal{M}$, by

$$g(x, v) := \begin{cases} \inf \left\{ \sum_{i=1}^m u_i^2 \mid (u_1, \dots, u_m) \in \mathbb{R}^m, \sum_{i=1}^m u_i X_i(x) = v \right\} & \text{if } v \in \text{span}(X_i(x), i \in \{1, \dots, m\}), \\ +\infty & \text{if not.} \end{cases}$$

This defines for any $x \in \mathcal{M}$ a positive definite quadratic form $g(x, \cdot)$ on the horizontal space

$$\text{span}(X_1(x), \dots, X_m(x)).$$

Remark that, if finite, the infimum is in fact a minimum, and is realized by a vector $(u_1, \dots, u_m) \in \mathbb{R}^m$. Given $\gamma : [0, 1] \rightarrow \mathcal{M}$ an absolutely continuous path, we define its length accordingly by

$$\text{length}(\gamma) := \int_0^1 \sqrt{g(\gamma(t), \dot{\gamma}(t))} dt.$$

The fact that this quantity is finite implies that $\dot{\gamma}(t) \in \text{span}(X_1(\gamma(t)), \dots, X_m(\gamma(t)))$ for almost all $t \in [0, 1]$. Note also that when the vectors are linearly independent, the infimum is among one unique u realizing the decomposition. Also, it is always finite if γ is a horizontal path (in the sense of Definition 4.3).

Then, this allows to define a sub-Riemannian (also called Carnot-Carathéodory) distance on \mathcal{M} by

$$d_{\mathcal{L}}(x_0, x_1) = \inf \{ \text{length}(\gamma) \mid \gamma \text{ horizontal path, } \gamma(0) = x_0, \gamma(1) = x_1 \}, \quad x_0, x_1 \in \mathcal{M}.$$

The Chow-Rashevski Theorem 4.2 implies that, under Assumption 4.1, the distance $d_{\mathcal{L}}$ is always finite on $\mathcal{M} \times \mathcal{M}$. We also define accordingly $d_{\mathcal{L}}(x_0, E) = \inf_{x_1 \in E} d_{\mathcal{L}}(x_0, x_1)$ for a point $x_0 \in \mathcal{M}$ and a subset $E \subset \mathcal{M}$.

With these definitions in hand, we may now state our main result, which concerns the quantitative unique continuation (or quantitative approximate observability) for the Hypoelliptic wave equation

$$\begin{cases} \partial_t^2 u + \mathcal{L}u = 0 \\ (u, \partial_t u)|_{t=0} = (u_0, u_1). \end{cases} \quad (4.5)$$

Theorem 4.8. *Let \mathcal{L} as above satisfying Assumptions 4.1 and 4.2. Assume that ω is a non empty open set of \mathcal{M} and let $T > \sup_{x \in \mathcal{M}} d_{\mathcal{L}}(x, \omega)$. Then, there exist $\kappa, C, \mu_0 > 0$ such that we have*

$$\|(u_0, u_1)\|_{L^2 \times \mathcal{H}_{\mathcal{L}}^{-1}} \leq C e^{\kappa \mu^k} \|u\|_{L^2(\cdot) - T, T(\times \omega)} + \frac{1}{\mu} \|(u_0, u_1)\|_{\mathcal{H}_{\mathcal{L}}^1 \times L^2} \quad (4.6)$$

for all $\mu \geq \mu_0$, for any $(u_0, u_1) \in \mathcal{H}_{\mathcal{L}}^1 \times L^2$, and associated u solution of (4.5) on $]-T, T[$.

As before, this estimate could be stated equivalently rewritten under one of the following two formulations: for all $(u_0, u_1) \in \mathcal{H}_{\mathcal{L}}^1 \times L^2 \setminus \{(0, 0)\}$, one has

$$\|(u_0, u_1)\|_{\mathcal{H}_{\mathcal{L}}^1 \times L^2} \leq C e^{c\Lambda^k} \|u\|_{L^2(\cdot, T, \times \omega)}, \quad \text{with } \Lambda = \frac{\|(u_0, u_1)\|_{\mathcal{H}_{\mathcal{L}}^1 \times L^2}}{\|(u_0, u_1)\|_{L^2 \times \mathcal{H}_{\mathcal{L}}^{-1}}}, \quad (4.7)$$

or

$$\|(u_0, u_1)\|_{L^2 \times \mathcal{H}_{\mathcal{L}}^{-1}} \leq C \frac{\|(u_0, u_1)\|_{\mathcal{H}_{\mathcal{L}}^1 \times L^2}}{\log\left(1 + \|(u_0, u_1)\|_{\mathcal{H}_{\mathcal{L}}^1 \times L^2} / \|u\|_{L^2(\cdot, T, \times \omega)}\right)^{1/k}}, \quad (4.8)$$

where, in the last expression, the function $x \mapsto (\log(1 + 1/x))^{-1/k}$ has to be extended by zero at $x = 0^+$.

Again, in the particular situation of Example 4.6, the sequence of eigenfunctions of Proposition 4.7 shows that the exponent $e^{\kappa\mu^k}$ in (4.6) (resp. $e^{c\Lambda^k}$ in (4.7) and $\log^{-1/k}$ in (4.8)) cannot be improved in general.

Moreover, the assumption on the time $T > \sup_{x \in \mathcal{M}} d_{\mathcal{L}}(x, \omega)$ is optimal. Indeed, the hypoelliptic wave equation (4.5) also satisfies finite speed of propagation. The formulation of this result is similar to the one associated to the classical wave equation, but with the Riemannian distance replaced by the sub-Riemannian distance $d_{\mathcal{L}}$.

As a corollary of this result (see [Rob95] or [LL20a, Appendix]), we obtain the approximate controllability of the Hypoelliptic wave equation, as well as an estimate of the cost of approximate controls. Here, we only state approximate controllability to zero, which is equivalent to approximate controllability to the whole state space $\mathcal{H}_{\mathcal{L}}^1 \times L^2$ on account to the reversibility of the equation.

Corollary 4.9 (Cost of approximate control). *For any $T > 2 \sup_{x \in \mathcal{M}} d_{\mathcal{L}}(x, \omega)$, there exist $C, c > 0$ such that for any $\varepsilon > 0$ and any $(u_0, u_1) \in \mathcal{H}_{\mathcal{L}}^1 \times L^2$, there exists $g \in L^2((0, T) \times \omega)$ with*

$$\|g\|_{L^2((0, T) \times \omega)} \leq C e^{c/\varepsilon^k} \|(u_0, u_1)\|_{\mathcal{H}_{\mathcal{L}}^1 \times L^2},$$

such that the solution of

$$\begin{cases} (\partial_t^2 + \mathcal{L})u = \mathbf{1}_{\omega} g & \text{in } (0, T) \times \mathcal{M}, \\ (u, \partial_t u)|_{t=0} = (u_0, u_1), & \text{in } \mathcal{M}, \end{cases}$$

satisfies $\|(u, \partial_t u)|_{t=T}\|_{L^2 \times \mathcal{H}_{\mathcal{L}}^{-1}} \leq \varepsilon \|(u_0, u_1)\|_{\mathcal{H}_{\mathcal{L}}^1 \times L^2}$.

To the authors' knowledge, these results are the first ones concerning the approximate observability/controllability of hypoelliptic waves. They furnish not only the approximate observability/controllability but also an (optimal in general) estimate of the cost. Moreover, in this context, even qualitative unique continuation did not seem to be known.

In the elliptic case $k = 1$, these can be obtained by the theory developed by Lebeau in [Leb92] (even on a manifold with boundary). However, in this (elliptic) case, the analyticity assumption can be removed, as explained in Section 3.

Finally, we shall see that we prove actually a more general statement in which the term $\|(u_0, u_1)\|_{\mathcal{H}_{\mathcal{L}}^1 \times L^2}$ in the right-handside of Estimate (4.6) can be changed into $\|(u_0, u_1)\|_{\mathcal{H}_{\mathcal{L}}^s \times \mathcal{H}_{\mathcal{L}}^{s-1}}$ for any $s > 0$, if changing the power of μ accordingly.

4.2.3 Quantitative approximate observability of the hypoelliptic heat equation

We now turn to the study of observability properties for solutions of the hypoelliptic heat equation

$$\begin{cases} \partial_t y + \mathcal{L}y = 0, & \text{in } (0, T) \times \mathcal{M}, \\ y(0) = y_0 & \text{in } \mathcal{M}, \end{cases} \quad (4.9)$$

from a subdomain $\omega \subset \mathcal{M}$. By duality, we are equivalently concerned here with different controllability properties of the following system

$$\begin{cases} (\partial_t + \mathcal{L})u = \mathbb{1}_\omega g, & \text{in } (0, T) \times \mathcal{M}, \\ u(0) = u_0, & \text{in } \mathcal{M}. \end{cases} \quad (4.10)$$

We provide with two main results, still under Assumptions 4.1 and 4.2:

1. For any $k \in \mathbb{N}^*$, we prove an approximate observability result in any time $T > 0$ with a frequency-dependent constant of order $Ce^{c\Lambda^k}$, where $\Lambda = \|y_0\|_{\mathcal{H}_L^1} / \|y_0\|_{L^2}$, or, equivalently, approximate controllability with cost e^{c/ε^k} . This is Theorem 4.10 below which is the analogues of Theorem 4.8 for parabolic equations.
2. Finally, in the very particular case $k = 2$ (including Grushin and Heisenberg operators), we prove an approximate observability/controllability property *to trajectories* in large time with a polynomial cost. This is Theorem 4.12 below and may be interpreted as a counterpart of the exact controllability to trajectories for the heat equation [LR95, FI96] (case $k = 1$). There is no similar result if $k > 2$, except if we restrict to more regular (Gevrey-type) data.

The first result we obtain provides the cost of approximate observability of the whole state space $L^2(\mathcal{M})$. There is no restriction for the hypoellipticity index k , but the (exponential) cost depends on this parameter.

Theorem 4.10. *For all $T > 0$, there exist $C, c > 0$ such that for any $y_0 \in \mathcal{H}_L^1$ and associated solution y of (4.9), we have*

$$\|y_0\|_{L^2}^2 \leq C e^{c\Lambda^k} \int_0^T \int_\omega |y(t, x)|^2 dx dt, \quad \Lambda = \frac{\|y_0\|_{\mathcal{H}_L^1}}{\|y_0\|_{L^2}}, \quad (4.11)$$

and, for any $\mu > 0$,

$$\|y_0\|_{L^2}^2 \leq C e^{c\mu^k} \int_0^T \int_\omega |y(t, x)|^2 dx dt + \frac{1}{\mu^2} \|y_0\|_{\mathcal{H}_L^1}^2. \quad (4.12)$$

Again, in the particular situation of Example 4.6, the sequence of eigenfunctions of Proposition 4.7 shows that the exponent $e^{\kappa\mu^k}$ in (4.12) (resp. $e^{c\Lambda^k}$ in (4.11)) cannot be improved in general.

This theorem generalizes the results of Fernandez-Cara-Zuazua and Phung [FCZ00, Phu04] in the elliptic case $k = 1$. Yet, in this framework, the analyticity was not necessary (as in all above stated results in the case $k = 1$) and the setting can be relaxed (uniform dependence of the constants with respect to lower order terms and to the time T , boundary value problems...).

As a corollary (see [LL20a, Appendix]), we obtain, given an initial state and a target state both belonging to the space $L^2(\mathcal{M})$, and given a precision ε , the existence of a control function bringing the initial state in an ε -neighborhood of the target (in appropriate topology). We obtain as well an estimate of the cost of the control.

Corollary 4.11 (Cost of approximate control to the state space). *For any $T > 0$, there exist $C, c > 0$ such that for any $\varepsilon > 0$ and any $u_0 \in L^2(\mathcal{M}), u_1 \in L^2(\mathcal{M})$, there exists $g \in L^2((0, T) \times \omega)$ with*

$$\|g\|_{L^2((0, T) \times \omega)} \leq C e^{c/\varepsilon^k} \|e^{-T\mathcal{L}}u_0 - u_1\|_{L^2(\mathcal{M})},$$

such that the solution of (4.10) issued from u_0 satisfies

$$\|u(T) - u_1\|_{\mathcal{H}_{\mathcal{L}}^{-1}} \leq \varepsilon \|e^{-T\mathcal{L}}u_0 - u_1\|_{L^2(\mathcal{M})}.$$

In this statement, $e^{-T\mathcal{L}}u_0$ stands for the solution at time T to Equation (4.10) with $g = 0$.

Our second result concerning the hypoelliptic heat equation is, as opposed to the previous one, concerned with final state approximate observability (or equivalently an approximate controllability to trajectories) with a *polynomial* cost, and is restricted to the case $k = 2$.

Theorem 4.12. *Assume that $k = 2$. There exist T_0 so that for $T > T_0$, there exists $C > 0$ and $\beta > 0$ such that for all $\eta > 0$ and $y_0 \in L^2(\mathcal{M})$ and associated solution y to (4.9),*

$$\|y(T)\|_{L^2}^2 \leq \frac{C}{\varepsilon^\beta} \int_0^T \int_\omega |y(t, x)|^2 dt dx + \varepsilon \|y_0\|_{L^2}^2.$$

This result gives directly the following corollary concerning approximate controllability to trajectories (or, equivalently, to zero) at a polynomial cost (see again [LL20a, Appendix]).

Corollary 4.13 (Cost of approximate control to trajectories if $k = 2$). *Assume that $k = 2$, and let $T_0 > 0$ as in Theorem 4.12. For any $T > T_0$, there exists $C > 0$, and $\beta > 0$ so that for all $\varepsilon > 0$, we have the following statement: for any $u_0, \tilde{u}_0 \in L^2$, there exists $g \in L^2((0, T) \times \omega)$ with*

$$\|g\|_{L^2((0, T) \times \omega)} \leq \frac{C}{\varepsilon^\beta} \|u_0 - \tilde{u}_0\|_{L^2},$$

such that the associated solution u of (4.10) satisfies

$$\|u(T) - e^{-T\mathcal{L}}\tilde{u}_0\|_{L^2(\mathcal{M})} \leq \varepsilon \|u_0 - \tilde{u}_0\|_{L^2}.$$

4.2.4 Decay of damped hypoelliptic equations

We finally present a result concerning the damped hypoelliptic wave equation

$$\begin{cases} (\partial_t^2 + \mathcal{L} + 1_\omega \partial_t)u = 0, & \text{on } (0, +\infty) \times \mathcal{M}, \\ (u, \partial_t u)|_{t=0} = (u_0, u_1), & \text{on } \mathcal{M}, \end{cases} \quad (4.13)$$

where $\omega \subset \mathcal{M}$ is a non-empty open set. Solutions to (4.13) enjoy formally the following dissipation identity (obtained by taking the inner product of (4.13) with $\partial_t u$ and integrating on $(0, T)$):

$$E(u(T)) - E(u(0)) = - \int_0^T \int_\omega |\partial_t u(t, x)|^2 ds(x) dt, \quad E(u) = \frac{1}{2} \left(\sum_{i=1}^m \|X_i u\|_{L^2(\mathcal{M})}^2 + \|\partial_t u\|_{L^2(\mathcal{M})}^2 \right).$$

An important question is then to understand at which rate the energy decays. We denote the damped wave operator $\mathcal{A} = \begin{pmatrix} 0 & \text{Id} \\ -\mathcal{L} & -1_\omega(x) \end{pmatrix}$. In this context, we obtain a logarithmic decay rate with power k , which can be proved to be optimal in general.

Theorem 4.14 (Decay rates for damped hypoelliptic waves). *Assume Assumptions 4.1 and 4.2. Then, for all $(u_0, u_1) \in \mathcal{H}_{\mathcal{L}}^1 \times L^2$, the associated solution to (4.13) satisfies $E(u(t)) \rightarrow 0$. Moreover, for all $j \in \mathbb{N}^*$, there exists $C_j > 0$ such that for all $(u_0, u_1) \in D(\mathcal{A}^j)$, the associated solution to (4.13) satisfies*

$$E(u(t))^{\frac{1}{2}} \leq \frac{C_j}{\log(t+2)^{j/k}} \|\mathcal{A}^j(u_0, u_1)\|_{\mathcal{H}_{\mathcal{L}}^1 \times L^2}, \quad \text{for all } t \geq 0.$$

This result is the analogue in the hypoelliptic setting of the Lebeau theorem [Leb96] (case $k = 1$).

We obtain similar results for damped Schrödinger equation $(i\partial_t + \mathcal{L} + i1_\omega \partial_t)v = 0$ and the damped plate equation $(\partial_t^2 + \mathcal{L}^2 + 1_\omega \partial_t)v = 0$. In order to keep the article reasonably short, we refer to [LL20b] for these results.

4.3 Idea of the proofs

4.3.1 Quantitative unique continuation for the hypoelliptic wave equation

The proof of Theorem 4.8 is based on the general strategy described in Section 2 for quantifying and propagating unique continuation properties. We only use here the “Holmgren-John” case, i.e. when the operator has analytic coefficients as described in Remark 2.3.

Here, when compared to the case of the classical wave equation described in Section 3, two additional difficulties arise: one being of geometric nature, and the other one related to the compatibility between the energy spaces associated to \mathcal{L} and those dealt with in [LL19].

Let us first describe the geometric difficulty. The proof is inspired by the case of the classical wave equation explained in Section 3: the idea is, given a point $x_0 \in \mathcal{M}$, to take any path $\gamma : [0, 1] \rightarrow \mathcal{M}$ with $\gamma(0) = x_0$ and $\gamma(1) \in \omega$ (observation set), of length sufficiently small, and then to construct a family of appropriate non characteristic hypersurfaces in these coordinates near $[-T, T] \times \gamma$. There, we apply the general Theorem 2.5, which allows to bound the solution u to $(\partial_t^2 - \Delta)u = 0$ in a neighborhood of $(t, x) = (0, x_0)$ by u in $[-T, T] \times \omega$.

Here, due to the non definiteness/ellipticity of the operator \mathcal{L} , we are not able to construct *global* coordinates near *any* path γ together with appropriate noncharacteristic hypersurfaces, in which to apply the results of [LL19]. To overcome this difficulty, we do not consider *any* path between x_0 and ω , but rather only so called *normal geodesics*, that is, projections on \mathcal{M} of hamiltonian curves of the principal symbol of the operator \mathcal{L} . The existence of such paths γ (minimizing the sub-Riemannian distance) from any point x_0 to ω is a well-known result in sub-Riemannian geometry, proved by Rifford and Trélat [RT05]. Then, locally near a point of γ , the introduction of normal geodesic coordinates allows us to define local coordinates in which to apply a *local* version of a slight variant of Theorem 2.5.

Note that this single geometric construction, combined with the usual Holmgren-John theorem would be enough to prove the qualitative uniqueness statement.

These arguments eventually allows to prove an estimate of the form

$$\|u\|_{L^2(\cdot)_{-\varepsilon, \varepsilon}(\times \mathcal{M})} \leq C e^{\kappa \mu} \|u\|_{L^2(\cdot)_{-T, T}(\times \omega)} + \frac{C}{\mu} \|u\|_{H^1(\cdot)_{-T, T}(\times \mathcal{M})}, \quad (4.14)$$

for μ large and u solution to $(\partial_t^2 + \mathcal{L})u = 0$. This estimate is the same as (3.2) (after energy estimates) for the wave equation.

This leads us to the second main difficulty we have to face in the proof of Theorem 4.8. Whereas the left hand-side of (4.14) is bounded from below by the natural $L^2 \times \mathcal{H}_{\mathcal{L}}^{-1}$ norm of the data, the right hand-side is not directly linked to their $\mathcal{H}_{\mathcal{L}}^1 \times L^2$ norm. More precisely, the hypoelliptic estimates of Rothschild and Stein [RS76] of Theorem 4.4 imply that $\|u\|_{H^1(\cdot)_{-T, T}(\times \mathcal{M})} \leq C \|(u_0, u_1)\|_{\mathcal{H}_{\mathcal{L}}^k \times \mathcal{H}_{\mathcal{L}}^{k-1}}$.

This provides a weaker version of Theorem 4.8 which has exactly the same form as in the case of the wave equation (cost $e^{\kappa\mu}$), but with the norm $\|(u_0, u_1)\|_{\mathcal{H}_{\mathcal{L}}^k \times \mathcal{H}_{\mathcal{L}}^{k-1}}$ in the right hand-side. This weaker version is however interesting for itself since the proof is much less involved.

To obtain the estimate of Theorem 4.8 (and in fact a family of such estimates with any $\mathcal{H}_{\mathcal{L}}^s \times \mathcal{H}_{\mathcal{L}}^{s-1}$, $s > 0$, in the right hand-side), we thus need to work with a version of (4.14) still containing frequency cutoff localization and an $e^{-c\mu}$ small remainder (instead of the $1/\mu$ one). These low-frequency-with-exponentially-small-remainder estimates are then combined with the spectral representation of solutions to $(\partial_t^2 + \mathcal{L})u = 0$ in order to gain back derivatives in the remainder term.

4.3.2 From waves to eigenfunction tunneling

Theorem 4.5 is simply deduced from Theorem 4.8 (under the equivalent form of estimate (4.7)) by using a particular solution to the wave equation (4.5), namely $u(t, x) = \cos(\sqrt{\lambda}t)\varphi(x)$. It only remains to notice that the frequency functions $\Lambda = \|(u_0, u_1)\|_{\mathcal{H}_{\mathcal{L}}^1 \times L^2} / \|(u_0, u_1)\|_{L^2 \times \mathcal{H}_{\mathcal{L}}^{-1}}$ is of order $\sqrt{\lambda}$ for $(u_0, u_1) = (\varphi, 0)$, where $\mathcal{L}\varphi = \lambda\varphi$.

4.3.3 From wave-like to heat-like equations

The proofs of Theorems 4.10 and 4.12 follow the general idea that the controllability/observability properties for hyperbolic equations implies controllability/observability properties for their parabolic counterpart, see [Rus73, Mil06, EZ11a, EZ11b] (see also [LR95]).

This has been named as “transmutation methods” by Luc Miller [Mil06]. Here, we use the method developed in [EZ11a] (itself relying on a Lebeau-Robbiano strategy [LR95]). In that paper, Ervedoza and Zuazua deduced the (exact final time) observability of the heat equation (known from [LR95, FI96]) from the approximate observability estimate for waves (namely the analogue of Theorem 4.8) as proved in [Phu10] (with loss) or above Theorem 3.1 (without loss). They prove the following result.

Proposition 4.15 ([EZ11a, EZ11b]). *Let $T, S > 0$ and $\alpha > 2S^2$. Then, there exists some kernel function $k_T(t, s)$ such that if y is solution of the heat equation (4.9), then $u(s) = \int_0^T k_T(t, s)y(t)dt$ is solution of*

$$\begin{cases} \partial_s^2 u + \mathcal{L}u = 0, & \text{for } s \in]-S, S[, \\ (u, \partial_s u)|_{s=0} = \left(0, \int_0^T \partial_s k_T(t, 0)y(t)dt\right) = \left(0, \int_0^T e^{-\alpha((1/t)+(1/(T-t)))} y(t)dt\right); \end{cases}$$

The authors also provide with useful estimates on this kernel.

Both proofs of Theorems 4.10 and 4.12, follow a similar strategy and apply Theorem 4.8 to a solution u of the wave equation constructed with Proposition 4.15. Yet, one important difference between Theorems 4.10 and 4.12 is that the first one is an estimate of the solution at initial time while the second one is at final time. The proofs therefore differ slightly depending on whether or not they use the natural decay of the heat equation. In the first case, Proposition 4.15 is applied directly to the solution of the heat equation that we want to observe. In the second case, Proposition 4.15 is applied to the low frequency part of the solution, more in the spirit of the “Lebeau-Robbiano” strategy [LR95]. Let us now give a few more details in each situation.

The proof of Theorem 4.10 relies on the transmutation technique of Proposition 4.15 applying directly the transmutation kernel $k_T(t, s)$ to the full solution y to the heat equation: $u(t) = \int_0^T k_T(t, s)y(s)ds$ is a solution to the wave equation. We then prove a fine asymptotic analysis of

$\int_0^T k_T(0, s)e^{-\lambda s} ds$ for high frequencies together with convexity estimates to bound the “frequency function” of $u(0)$, namely $\|u(0)\|_{\mathcal{H}_x^1} / \|u(0)\|_{L^2}$ by the frequency function of $y(0)$. The proof of this result via a direct transmutation method seems to be new, even for the classical heat equation. The usual proofs [FCZ00, Phu04] rather rely on the exact final time observability estimate, which does not hold here in general.

The proof of Theorems 4.12 is very close to that of [EZ11a]. Using again the transmutation result together with our estimate (4.7) for the wave equation, we obtain a “Lebeau-Robbiano-like” estimate, that is a low frequency observability with a good estimates of the cost with respect to the frequency. More precisely, defining the low frequency spaces $E_\lambda = \text{span}\{\varphi_j, \lambda_j \leq \lambda\}$, we obtain the following low-frequency observability estimate, with a precise estimation of the observability constant with respect to the cutoff frequency.

Lemma 4.16 (“Lebeau-Robbiano-like” estimates). *There exist $C, \gamma > 0$ such that for any $T > 0, \lambda \geq 0$, for every $y_0 \in E_\lambda$ and associated solution y to (4.9), we have*

$$\|y(T)\|_{L^2}^2 \leq \frac{C}{T} e^{(2\gamma\lambda^{k/2} + C/T)} \int_0^T \int_\omega |y(t, x)|^2 dt dx. \quad (4.15)$$

Moreover, there exists $c_0 > 0$ such that for any $T > 0$ there exists $C = C_T > 0$ such that for any $\lambda \geq 0$, any $y_0 \in E_\lambda$ and associated solution y to (4.9), we have

$$\|y_0\|_{L^2}^2 \leq C e^{2c_0\lambda^{k/2}} \int_0^T \int_\omega |y(t, x)|^2 dt dx. \quad (4.16)$$

Note the difference between (4.15) and (4.16): the former estimates the *final* data at time T (with explicit dependence with respect to time) whereas the latter estimates the *initial* data. Estimate (4.16) is used in the proof of Theorem 4.10 whereas Estimate (4.15) is used in the proof of Theorem 4.12.

The cost $e^{2c_0\lambda^{k/2}}$ in these low-frequency observability estimates has to be compared to the dissipation for high frequencies $\sqrt{\lambda_j} \geq \sqrt{\lambda}$, namely $e^{-t\lambda}$. Hence, we see that the cases $k = 1$ (classical heat equation, already discussed), $k = 2$ (Grushin, Heisenberg...), and $k > 2$ display very different features:

1. In case $k = 2$, the cost of observation of low frequencies $e^{c\lambda}$ and the parabolic dissipation for high frequencies $e^{-t\lambda}$ have the same strength: in this case, we need to wait a time long enough so that the dissipation “beats” the cost of the observability (essentially $t > c$). Moreover, the iterative procedure devised in [LR95] in order to control/observe *all* frequencies in finite time cannot converge here: each step would need a time $t > c$. Yet, it allows to obtain the approximate controllability result of Theorem 4.12.
2. In case $k > 2$, the dissipation for high frequencies $e^{-t\lambda}$ has no chance to compete with the cost of observation of low frequencies $e^{c\lambda^{k/2}}$. The only chance to obtain some positive result is to assume that the initial data are in some Gevrey-type space that allows to compensate for the cost of low frequencies $e^{c\lambda^{k/2}}$. This leads to approximate control results in Gevrey-type spaces (with polynomial cost) that we have chosen not to state here for simplicity. We refer to [LL17, Theorem 1.18] for more details.

4.3.4 From waves to resolvent estimates and decay for damped equations

The proof of Theorem 4.14 consists in several reductions to resolvent type estimates. First, we relate in an abstract setting decay of damped equations to resolvent estimates according

to [BD08]. More precisely, in the proof of Theorem 4.14, it suffices show an estimate of the form $\|(is - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H}_{\mathcal{L}}^1 \times L^2)} \leq Ce^{\kappa|s|^k}$ for all $|s| \geq 1$. We then prove that the latter is a consequence of the following estimate:

$$\|v\|_{L^2(\mathcal{M})} \leq Ce^{\kappa\lambda^k} (\|v\|_{L^2(\omega)} + \|(\mathcal{L} - \lambda^2)v\|_{L^2(\mathcal{M})}), \quad \text{for all } v \in \mathcal{H}_{\mathcal{L}}^2, \lambda \geq \lambda_0. \quad (4.17)$$

To obtain (4.17), for $v \in \mathcal{H}_{\mathcal{L}}^2$ with $(\mathcal{L} - \lambda^2)v = f$, we construct, as in Section 4.3.2 a particular solution to the wave equation with source term, namely $u(t, x) = \cos(\sqrt{\lambda}t)\varphi(x)$ solution of $(\partial_t^2 + \mathcal{L})u = \cos(\sqrt{\lambda}t)f$. A slight variant of Theorem 4.8 with source term applied to u allows us to obtain (4.17).

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