Séminaire Laurent Schwartz

EDP et applications

Année 2021-2022

Charles Bertucci

On monotone solutions of mean field games master equations

Séminaire Laurent Schwartz — EDP et applications (2021-2022), Exposé n° XIV, 13 p.

https://doi.org/10.5802/slsedp.153

Institut des hautes études scientifiques
Le Bois-Marie • Route de Chartres
F-91440 BURES-SUR-YVETTE
http://www.ihes.fr/

Centre de mathématiques Laurent Schwartz
CMLS, École polytechnique, CNRS, Université Paris-Saclay
F-91128 PALAISEAU CEDEX
http://www.math.polytechnique.fr/

Publication membre du
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 2266-0607

Séminaire Laurent-Schwartz — EDP et applications
Centre de mathématiques Laurent Schwartz, 2021-2022
Exposé n^{o} XIV, 1-13

ON MONOTONE SOLUTIONS OF MEAN FIELD GAMES MASTER EQUATIONS

CHARLES BERTUCCI

Abstract. This note presents the concept of monotone solutions of mean field games master equations, in several cases. The first case that I treat is the one in which the underlying game has only a finite state space. The other are the case of a continuous state space and the so-called Hilbertian approach. Most of the results presented here come from the two papers [1, 2], except for results concerning the Hilbert space case and the case of general monotone operators which are new.

Contents

1. Introduction 1
2. Mean field games master equations and monotonicity 2
2.1. Master equations in finite state space 2
2.2. Master equations on the space of probability measure 3
2.3. Master equations on Hilbert spaces 4
3. Monotone solutions of master equations 4
3.1. Derivation of a notion of weak solutions 4
3.2. Monotone solutions for master equations on infinite dimensional spaces 6
3.3. Uniqueness of monotone solutions 7
3.4. Stability of monotone solutions 10
4. Extensions to general monotone operators 11
References 12

1. Introduction

The study of mean field games (MFG in short) master equations is getting more and more attention in the recent years. MFG are differential games involving an infinite number of non atomic agents. These games are now a central piece of the mathematical modeling toolbox. We do not elaborate too much on this. The master equation is the partial differential equation (PDE in short) satisfied by the value function of a MFG. This value is the function which associates to a state of the game the value that the players are going to get when the game starts from this state. Quite often, the state of the game is simply the time and the measure describing the repartition of players. In general, such a concept of value has no reason to be well defined since there is, in general, no guarantee that a particular outcome of the game can be selected. This problem of definition usually translates as a creation of singularities at the level of the master equation.

The problem of solving MFG master equations in a general setting is by now still an open problem. For quite some time, the most satisfying results on MFG master equations were the one established in [8, 6]. Those results stated that in the so-called monotone regime, under additional smoothness assumptions, a theory of classical solutions of master equations can be established. The monotone regime is a type of assumptions under which the underlying game has an adversarial structure which is strong enough so that a unique Nash equilibrium exists. Because of this uniqueness result, we can define a notion of value and verify that, when it is smooth, it is indeed a solution of the master equation. Other regimes have been studied in the literature and we refer to the references in [1, 2] for more details on these regimes.

In these notes, we are concerned with proposing a weak notion of solution for master equations, in a monotone regime. Hence, we want to weaken the regularity assumptions on the datas, that were needed in [6] for instance, while preserving the key structural assumptions on the master equations. This notion was introduced in [1, 2]. It has been used in [5] to study more singular master equations and in [3] to study the convergence of a discretized master equation.

2. Mean field games master equations and monotonicity

We present here several models of MFG master equations, without detailing the underlying MFG they model.

2.1. **Master equations in finite state space.** The typical example of a MFG master equation in finite state space is the PDE satisfied by $U : [0,\infty) \times \mathbb{O} \to \mathbb{R}^d$

$$\partial_t U + (F(x,U)\cdot\nabla_x)U = G(x,U), \text{ in } (0,\infty)\times\mathbb{O}, \tag{2.1}$$

$$U|_{t=0} = U_0 \text{ in } \mathbb{O}, \tag{2.2}$$

where the data of the problem are $F, G : \mathbb{O}\times\mathbb{R}^d \to \mathbb{R}^d$ and $U_0 : \mathbb{O} \to \mathbb{R}^d$. The previous PDE is set on $\mathbb{O}$ which is assumed to be a convex compact set of $\mathbb{R}^d$, and to be the closure of its interior. We shall assume that $\mathbb{O}$ satisfies almost everywhere the following stability assumption

$$\forall x \in \partial\mathbb{O}, \langle F(x,p), \eta(x)\rangle \geq 0, \tag{2.3}$$

where $\eta(x)$ is the unit outward vector to $\partial\mathbb{O}$ at x, and the previous inequality is asked only when it is well defined. The notation $\langle , \rangle$ stands for the usual scalar product of $\mathbb{R}^d$. In this context, we say that the monotone regime is in force when

Hypothesis 1. *The function $(G,F) : \mathbb{O}\times\mathbb{R}^d \to \mathbb{R}^{2d}$ is monotone. The function $U_0 : \mathbb{O} \to \mathbb{R}^d$ is monotone.*

Recall that a function $f : \Omega \subset \mathbb{R}^d \to \mathbb{R}^d$ is called monotone when for all $x, y \in \Omega$,

$$\langle f(x) - f(y), x - y\rangle \geq 0. \tag{2.4}$$

More general type of equations are also of interest, especially ones involving non-local terms, but we refer to [1] for more details on this. Recall that this type of master equations is studied for MFG in which the state space of the players is finite.

2.2. **Master equations on the space of probability measure.** When the state space of the player is infinite, the typical form of the master equation is

$$\begin{aligned}\partial_t U - \sigma\Delta_x U + H(x, \nabla_x U) - \langle \nabla_m U(x,m,\cdot), \operatorname{div}(D_p H(\cdot, \nabla U(\cdot, m))m)\rangle \\ - \sigma \langle \nabla_m U(x,m,\cdot), \Delta m\rangle = f(x,m), \text{ in } (0,\infty)\times\mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d)\end{aligned} \tag{2.5}$$

$$U(0,x,m) = U_0(x,m) \text{ in } \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d). \tag{2.6}$$

In the previous, $\mathbb{T}^d$ stands for the d dimensional torus, $\mathcal{P}(\mathbb{T}^d)$ for the set of probability measures on it, $\sigma > 0$, $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ and $f : \mathcal{P}(\mathbb{T}^d) \to (\mathbb{T}^d \to \mathbb{R})$ are the data of the problem. The solution U of this PDE is a function $[0,\infty)\times\mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ and $\langle,\rangle$ denotes here the extension of the $L^2(\mathbb{T}^d, \mathbb{R})$ scalar product.

The derivative with respect to the measure argument is defined as usual for a smooth function $\phi : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ by

$$\forall x \in \mathbb{T}^d, \nabla_m \phi(x) = \lim_{\theta\to 0} \frac{\phi((1-\theta)m + \theta\delta_x) - \phi(m)}{\theta}, \tag{2.7}$$

where δ_x is the Dirac mass at $x \in \mathbb{T}^d$. In particular, $\frac{\delta\phi}{\delta m} : \mathbb{T}^d \to \mathbb{R}$.

In the context of (2.5), we are in the monotone regime when

Hypothesis 2. *The Hamiltonian H is convex in its second argument and f is a strongly monotone operator for the duality product $\langle,\rangle$ in the sense that*

$$\begin{aligned}\forall \mu, \nu \in \mathcal{P}(\mathbb{T}^d), \langle f(\mu) - f(\nu), \mu - \nu\rangle \geq 0, \\ \langle f(\mu) - f(\nu), \mu - \nu\rangle = 0 \Rightarrow f(\mu) = f(\nu).\end{aligned} \tag{2.8}$$

Even if the link between this setting and the previous might not be obvious at first, (2.1) and (2.5) are somehow of the same nature. Indeed, if we define the operators $\mathcal{G}$ and $\mathcal{F}$ by

$$\begin{aligned}\mathcal{G}(m,\varphi) &= f(m) + \sigma\Delta\varphi - H(\cdot, \nabla_x\varphi), \\ \mathcal{F}(m,\varphi) &= -\sigma\Delta m - \operatorname{div}(D_p H(\cdot, \nabla_x \varphi)m),\end{aligned} \tag{2.9}$$

then (2.5) can be rewritten as

$$\partial_t U(t,m) + \langle \mathcal{F}(m,U), \nabla_m\rangle U = \mathcal{G}(m,U) \text{ in } (0,\infty)\times\mathcal{P}(\mathbb{T}^d). \tag{2.10}$$

In the previous equation, U is seen as $U : [0,\infty)\times\mathcal{P}(\mathbb{T}^d) \to (\mathbb{T}^d \to \mathbb{R})$. Finally, let me remark that stability conditions such as (2.3) are automatically verified here.

2.3. **Master equations on Hilbert spaces.** The last example of master equations we present here are the ones which are posed on a Hilbert space. This approach was introduced by P.-L. Lions in [8] to develop another framework to study (2.5), namely but not only, in the case $\sigma = 0$. His idea was, forgetting the time dependence, to lift an element from $\mathcal{P}(\mathbb{T}^d) \to (\mathbb{T}^d \to \mathbb{R}^d)$ to a one from $L^2(\Omega, \mathbb{R}^d) \to L^2(\Omega, \mathbb{R}^d)$, where $(\Omega, \mathcal{A}, \mathbb{P})$ is a standard probability space. Considering $\phi : \mathcal{P}(\mathbb{T}^d) \to (\mathbb{T}^d \to \mathbb{R}^d)$, its lifting V is defined by

$$\forall X \in L^2, V(X) = \phi(\pi(X), \mathcal{L}(\pi(X))), \tag{2.11}$$

where $\mathcal{L}(X) \in \mathcal{P}(\mathbb{T}^d)$ is the law of X and $\pi(X)$ is a $\mathbb{T}^d$ random variable such that $\pi(X) = X$ modulo the quotient $\mathbb{R}^d/\mathbb{Z}^d$. For simplicity I now denote $\mathcal{H} = L^2(\Omega, \mathbb{R}^d)$.

In the case $\sigma = 0$ and $\nabla_x H = 0$, the natural lifting of (2.5) is established by considering the lifting V of $\nabla_x U$, if U is a solution of (2.5), and writing the PDE satisfied by V

$$\begin{aligned} &\partial_t V(t,X) + \langle D_p H(V(t,X)), \nabla\rangle V(t,X) \\ &\qquad\qquad = \nabla_x f(X, \mathcal{L}(X)) =: \mathcal{A}(X), \text{ in } (0,\infty) \times \mathcal{H}, \end{aligned} \tag{2.12}$$

$$V|_{t=0}(X) = \nabla_x U_0(X, \mathcal{L}(X)) =: \mathcal{U}_0(X) \text{ in } \mathcal{H}. \tag{2.13}$$

In the previous equation, $\langle , \rangle$ stands for the usual scalar product of $\mathcal{H}$. In this last situation, we are in the monotone regime if

Hypothesis 3. *The Hamiltonian H is convex and $\mathcal{A}$ and $\mathcal{U}_0$ are monotone.*

Note that the monotone regime holds for different set of assumptions for (2.5) and (2.12), even if they somehow model the same problem.

3. MONOTONE SOLUTIONS OF MASTER EQUATIONS

In this section, I introduce briefly the notion of monotone solution and present results of uniqueness and stability of such solutions. I shall not be concerned with result of existence here, and refer the reader to [1, 2] for such questions.

3.1. **Derivation of a notion of weak solutions.** One of the main contributions of J.-M. Lasry and P.-L. Lions in [7] was to identify structural assumptions (the monotone regimes) under which a concept of value can be defined for MFG. In terms of master equations, this translates into the fact that, in all the regimes mentioned above, the master equations propagates the monotonicity of the initial condition. We now sketch the main idea of the proof of this statement, on the case of (2.1).

Proposition 3.1. *Under Hypothesis 1, there exists at most one smooth solution U of (2.1). If it exists, it satisfies $U(t) : \mathbb{O} \to \mathbb{R}^d$ is monotone for all $t \geq 0$.*

Proof. Consider two such solutions U and V. Let us define $W : [0,\infty) \times \mathbb{O}^2 \to \mathbb{R}$ by

$$W(t,x,y) = \langle U(t,x) - V(t,y), x - y\rangle. \tag{3.1}$$

Let me remark that

$$\nabla_x W(t,x,y) = U(t,x) - V(t,y) + \langle D_x U(t,x), x-y\rangle, \tag{3.2}$$

and that the analogous relation holds for $\nabla_y W$. From the previous relation and the fact that U and V are smooth solutions of (2.1), I obtain that, on $(0,\infty)\times\mathbb{O}^2$,

$$\begin{aligned}\partial_t W + \langle F(x,U), \nabla_x W\rangle + \langle F(y,V), \nabla_y W\rangle \\ = \langle G(x,U) - G(y,V), x-y\rangle + \langle F(x,U) - F(y,V), U-V\rangle.\end{aligned} \tag{3.3}$$

From Hypothesis 1, I can deduce that

$$\begin{aligned}\partial_t W + \langle F(x,U), \nabla_x W\rangle + \langle F(y,V), \nabla_y W\rangle \geq 0,\\ W|_{t=0} \geq 0.\end{aligned} \tag{3.4}$$

Hence using a comparison principle, which I do not detail here[1], I deduce that $W \geq 0$ for all time. Take now x in the interior of $\mathbb{O}$ and $\epsilon > 0$ such that $x + \epsilon B_1(0) \subset \mathbb{O}$. It then follows that

$$-\epsilon\langle U(t,x) - V(t,x+\epsilon y), y\rangle = W(t,x,x+\epsilon y) \geq 0. \tag{3.5}$$

Dividing by ϵ and letting ϵ go to 0, we deduce that $U(t,x) = V(t,x)$, from the continuity of U and V, I finally deduce that $U = V$, and thus that $U(t)$ is monotone (from the non-negativity of W) for all $t \geq 0$. □

Remark 3.1. For the previous result, the proof is obviously more important than the statement, as for regular solutions, Hypothesis 1 is not needed. See for instance [4] for more details on this question.

This elementary proof illustrates the fact that Hypothesis 1 leads to both the propagation of monotonicity and the uniqueness of solutions of (2.1).

The fundamental idea behind the notion of monotone solution, is that very few regularity was needed on the solutions U and V in the previous proof. Indeed, if regularity is needed for W to be a classical solution of (3.3), fewer regularity is sufficient to establish a comparison principle, which is the main argument of the proof. Let me recall that this observation is the main idea behind the notion of viscosity solution.

Hence, in the monotone regime, the regularity needed to maintain the previous proof of uniqueness is lower semi continuity of W with respect to both x and y. A priori, at the level of U (and V), continuity has to be required in order to guarantee the lower semi continuity of W.

Now that the question of regularity has been treated, it remains to understand in which sense a continuous function can be a solution of (2.1). Observe that in the previous proof, the equation was only used at points of minimum of W. Moreover, at these points of minima, taking $\nabla_x W = 0$ in (3.2) allows to replace $\langle D_x U(t,x), x-y\rangle$ with $U-V$. Thus, we only need to know an information on (2.1), once the scalar product with $x-y$ has been taken, at points of minima of W, and

[1] A more general result is proven below.

at these points we can replace the terms involving D_xU with terms which do not involve derivative. This leads to the

Definition 1. A continuous function $U : [0,\infty) \times \mathbb{O} \to \mathbb{R}^d$ is a monotone solution of (2.1) if: for any $T > 0, V \in \mathbb{R}^d$, $y \in \mathbb{O}$, $\theta : [0,T] \to \mathbb{R}$ smooth function and (t_0, x_0) point of strict minimum of

$$\begin{aligned} &[0,T] \times \mathbb{O} \to \mathbb{R}, \\ &\qquad (t,x) \to \langle U(t,x) - V, x - y\rangle - \theta(t), \end{aligned} \tag{3.6}$$

the following holds

$$\frac{d\theta}{dt}(t_0) \geq \langle G(x_0, U(t_0,x_0)), x_0 - y\rangle + \langle F(x_0, U(t_0,x_0)), U(t_0,x_0) - V\rangle. \tag{3.7}$$

Remark 3.2. The way I treat the time derivative is the usual way to deal with this kind of term in the viscosity solution theory.

Remark 3.3. I only ask for information at points of strict minimum for reasons of stability that will be more apparent later on.

Remark 3.4. We only have an inequality and not an equality because the minimum can be reached on the boundary of $\mathbb{O}$.

3.2. **Monotone solutions for master equations on infinite dimensional spaces.** A remarkable feature of the previous notion of solution, is that it can be easily adapted to infinite dimensional cases, as show the following definitions.

Definition 2. A continuous function $U : [0,\infty) \times \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$, $\mathcal{C}^2$ in its second argument, is a monotone solution of (2.5) if: for any $T > 0, \phi \in \mathcal{C}^2, \mu \in \mathcal{M}(\mathbb{T}^d), \theta :$ $[0,T] \to \mathbb{R}$ smooth function and (t_0, m_0) point of strict minimum of

$$\begin{aligned} &[0,T] \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}, \\ &\qquad (t,m) \to \langle U(t,\cdot,m) - \phi, m - \mu\rangle - \theta(t), \end{aligned} \tag{3.8}$$

the following holds

$$\frac{d\theta}{dt}(t_0) \geq \langle \mathcal{G}(m_0, U(t_0,m_0)), m_0 - \mu\rangle + \langle \mathcal{F}(m_0, U(t_0,m_0)), U(t_0,m_0) - \phi\rangle. \tag{3.9}$$

Let me insist on the fact that in this setting, the regularity of U with respect to x is necessary. Because we are here in the case $\sigma > 0$, it is in general true and thus does not raise too much issue. In some sense we are seeing the value function as $U : [0,\infty) \times \mathcal{P}(\mathbb{T}^d) \to \mathcal{C}^2(\mathbb{T}^d)$. Moreover, the "test" function ϕ also has to be assumed $\mathcal{C}^2$ here.

Remark 3.5. Note that in the Definition, I take $\nu \in \mathcal{M}(\mathbb{T}^d)$ and not only $\nu \in \mathcal{P}(\mathbb{T}^d)$. The reason why shall be make precise in the proof of the uniqueness result below.

For the case of (2.12), we provide a slightly different notion of solution, to allow to establish the forthcoming results more easily.

Definition 3. A continuous function $U : [0,\infty) \times \mathcal{H} \to \mathcal{H}$ is a monotone solution of (2.12) if: for any $T > 0, V \in \mathcal{H}, Y, Z \in \mathcal{H}, \theta : [0,T] \to \mathbb{R}$ smooth function and (t_0, X_0) point of strict minimum of

$$\begin{aligned} &[0,T] \times \mathcal{H} \to \mathbb{R}, \\ &\qquad (t, X) \to \langle U(t,X) - V, X - Y\rangle - \theta(t) - \langle Z, X\rangle, \end{aligned} \tag{3.10}$$

the following holds

$$\begin{aligned} &\frac{d\theta}{dt}(t_0) + \langle D_pH(U(t_0,X_0)), Z\rangle \\ &\qquad \geq \langle \mathcal{A}(X_0), X_0 - Y\rangle + \langle D_pH(U(t_0,X_0)), U(t_0,X_0) - V\rangle. \end{aligned}$$

Remark that the question of regularity seems simpler in this Hilbertian approach.

Remark 3.6. In this case, I introduce a more restrictive definition than the previous ones by the means of the term Z. The previous definitions correspond to the case $Z = 0$. This addition shall be helpful later on.

A similar notion of monotone solution was introduced in [5] in a different context. In this paper, the authors use what we could call a semi-Hilbertian approach in which $U : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ is lifted to $V : \mathbb{T}^d \times \mathcal{H} \to \mathbb{R}$.

3.3. **Uniqueness of monotone solutions.** Since the definition of monotone solutions is based on a proof of uniqueness, their uniqueness in the monotone regime should be no surprise to the reader. We can provide the following results of uniqueness for the case of (2.5) and (2.12). The case of (2.1) can be treated in a similar fashion.

Theorem 1. *Under Hypothesis 2, there exists at most one monotone solution of* (2.5) *in the sense of Definition 2.*

I reproduce here the proof of [2].

Proof. Let us consider U and V two such solutions. We define W by

$$\begin{aligned} W(t,s,\mu,\nu) &= \langle U(t,\cdot,\mu) - V(s,\cdot,\nu), \mu - \nu\rangle \\ &:= \int_{\mathbb{T}^d} U(t,x,\mu) - V(s,x,\nu)(\mu - \nu)(dx). \end{aligned} \tag{3.11}$$

We want to prove that $W(t,t,\mu,\nu) \geq 0$ for all $t \geq 0$, $\mu,\nu \in \mathcal{P}(\mathbb{T}^d)$. Assume it is not the case, hence there exists $t_*, \delta, \bar{\epsilon} > 0$, such that for all $\epsilon \in (0,\bar{\epsilon}), \alpha > 0, \phi, \psi \in C^2$ such that $\|\phi\|_2 + \|\psi\|_2 \leq \epsilon$ and $\gamma_1, \gamma_2 \in (\frac{\bar{\epsilon}}{2}, \bar{\epsilon})$,

$$\inf_{\substack{t,s\in[0,t_*]\\ \mu,\nu\in\mathcal{P}(\mathbb{T}^d)}} \left\{ W(t,s,\mu,\nu) + \langle \phi,\mu\rangle + \langle\psi,\nu\rangle + \frac{1}{2\alpha}(t-s)^2 + \gamma_1 t + \gamma_2 s \right\} \leq -\delta. \tag{3.12}$$

From Stegall's Lemma [9, 10], we know that there exists (for any value of α) ϕ, ψ, γ_1 and γ_2 such that $(t,s,\mu,\nu) \to W(t,s,\mu,\nu) + \langle\phi,\mu\rangle + \langle\psi,\nu\rangle + \frac{1}{2\alpha}(t-s)^2 + \gamma_1 t + \gamma_2 s$ has a strict minimum on $[0,t_*]^2 \times \mathcal{P}(\mathbb{T}^d)^2$ at (t_0, s_0, μ_0, s_0).

We assume first that $t_0 > 0$ and $s_0 > 0$. Using the fact that U is a monotone solution of (2.5) we obtain that

$$
\begin{aligned}
-\gamma_1 - \frac{t_0 - s_0}{\alpha} + \langle -\sigma\Delta U(\mu_0) + H(\cdot, \nabla_x U), \mu_0 - \nu_0\rangle &\geq \langle f(\cdot,\mu_0), \mu_0 - \nu_0\rangle \\
&- \langle U(t_0,\mu_0) - V(s_0,\nu_0) + \phi, \operatorname{div}(D_pH(\nabla_x U)\mu_0)\rangle \\
&- \sigma\langle \Delta(U(t_0,\mu_0) - V(s_0,\nu_0) + \phi), \mu_0\rangle,
\end{aligned}
\tag{3.13}
$$

and similarly for V :

$$
\begin{aligned}
-\gamma_2 - \frac{s_0 - t_0}{\alpha} &+ \langle -\sigma\Delta V(s_0,\nu_0) + H(\cdot, \nabla_x V), \nu_0 - \mu_0\rangle \\
&\geq \langle f(\cdot,\nu_0), \nu_0 - \mu_0\rangle - \langle V(s_0,\nu_0) - U(t_0,\mu_0) + \phi, \operatorname{div}(D_pH(\nabla_x V)\nu_0)\rangle \\
&\qquad - \sigma\langle \Delta(V(s_0,\nu_0) - U(t_0,\mu_0) + \phi), \nu_0\rangle.
\end{aligned}
\tag{3.14}
$$

Summing the two previous relations, using the monotonicity of f and the convexity of H, we deduce that

$$
\begin{aligned}
&-\gamma_1 - \gamma_2 \\
&\geq -\langle \phi, \operatorname{div}(D_pH(\nabla_x U)\mu_0)\rangle - \sigma\langle \Delta\psi, \nu_0\rangle - \langle \psi, \operatorname{div}(D_pH(\nabla_x V)\nu_0)\rangle - \sigma\langle \Delta\phi, \mu_0\rangle.
\end{aligned}
\tag{3.15}
$$

The previous relation is a contradiction (provided that ϵ had been chosen sufficiently small compared to $\bar{\epsilon}$).

Let us now turn to the case $t_0 = 0$ (the case $s_0 = 0$ being treated in exactly the same fashion). By construction s_0 satisfies $|s_0 - t_0| \leq C\sqrt{\alpha}$ for some $C > 0$ independent of ϵ. Thus choosing $\alpha > 0$ sufficiently small, we easily manage to contradict (3.12).

Hence we have proven that $W(t,t,\mu,\nu) \geq 0$ for $t \geq 0, \mu, \nu \in \mathcal{P}(\mathbb{T}^d)$. Making the same sort of computation as the ones at the end of the proof of Proposition 3.1, we obtain that there exists $c : [0,\infty) \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ such that $V = U - c$. Hence, in this case, the non-negativity of W do not exactly yields the equality between U and V. To obtain such an equality, we have to use the strong monotonicity of f.

Assume that there exists t_*, ν_* such that $c(t_*,\nu_*) = -\delta_0 < 0$ and consider a non-negative non-zero measure $\rho \in \mathcal{M}(\mathbb{T}^d)$. Because the initial condition is satisfied for both U and V, we know that $t_* > 0$. Furthermore, from Stegall's Lemma, we know that for any $\epsilon > 0$ there exists $t_0, \delta, \bar{\epsilon} > 0$, such that for all $\epsilon' \in (0,\bar{\epsilon}), \alpha > 0, \phi, \psi \in C^2$ such that $\|\phi\|_2 + \|\psi\|_2 \leq \epsilon'$ and $\gamma_1, \gamma_2 \in (\bar{\epsilon}/2, \bar{\epsilon})$,

$$
\begin{aligned}
\inf_{\substack{t,s\in[0,t_0] \\ \mu,\nu\in\mathcal{P}(\mathbb{T}^d)}} \Big\{ &\langle U(t,\mu) - V(s,\nu), \mu - \nu + \epsilon\rho\rangle + \langle \phi, \mu\rangle + \langle \psi, \nu\rangle + \\
&+ \frac{1}{2\alpha}(t-s)^2 + \gamma_1 t + \gamma_2 s \Big\} \leq -\epsilon\delta_0/2.
\end{aligned}
\tag{3.16}
$$

and the infimum is attained at a unique point $(t_\epsilon, s_\epsilon, \mu_\epsilon, \nu_\epsilon)$. Moreover, we can choose $\bar{\epsilon}$ such that $\bar{\epsilon}/\epsilon \to \kappa > 0$ as $\epsilon \to 0$. Proceeding as we did in the first part of the proof in the case $s_\epsilon, t_\epsilon > 0$ for almost all $\epsilon > 0$, we arrive at the relation

$$-\gamma_1 - \gamma_2 \geq \langle f(\mu_\epsilon) - f(\nu_\epsilon), \mu_\epsilon - \nu_\epsilon + \epsilon\rho\rangle + o(\epsilon). \tag{3.17}$$

Letting $\epsilon \to 0$, we deduce that $\limsup\langle f(\mu_\epsilon)-f(\nu_\epsilon), \mu_\epsilon-\nu_\epsilon\rangle \leq 0$. Hence, using the strong monotonicity of f, we deduce that, extracting a subsequence if necessary, $(\mu_\epsilon)_\epsilon$ and $(\nu_\epsilon)_{\epsilon>0}$ converges toward the same limit. Hence, using this information in (3.17), we arrive at a contradiction. The case where either s_ϵ or t_ϵ is equal to 0 for sufficiently many $\epsilon > 0$ can be treated in a similar way. Thus we have proved that $c \geq 0$ and thus by symmetry that $U = V$. □

Remark 3.7. The use of Stegall's Lemma is fundamental in the previous proof. This is mainly because we are only asking information at points of strict minimum in the definitions of monotone solutions. Creating those points of strict minimum is in general not trivial and need the use of Stegall's Lemma. To be complete, let me state that in the previous proof, we did not exactly use the results from Stegall but rather an adaptation which is proved in [2].

Theorem 2. *Under Hypothesis 3, there exists at most one bounded monotone solution of* (2.12)*.*

Proof. Consider two solutions U and V and assume that they are not equal. Hence there exists $\delta > 0, T > 0$ such that for any $\beta \geq 0$

$$\inf_{\substack{s,t\leq T\\ X,Y\in\mathcal{H}}} \langle U(t,X) - V(s,Y), X-Y\rangle + \beta(t-s)^2 < -\delta. \tag{3.18}$$

From this we deduce that for $\alpha, \kappa > 0$ (independently) sufficiently small, for any $\gamma_1, \gamma_2 \in (\kappa/2, \kappa)$,

$$\inf_{\substack{s,t\leq T\\ X,Y\in\mathcal{H}}} \langle U(t,X)-V(s,Y), X-Y\rangle+\beta(t-s)^2+\alpha(|X|^2+|Y|^2)+\gamma_1 t+\gamma_2 s < -\delta. \tag{3.19}$$

Moreover from Stegall's Lemma, for any $\epsilon > 0$, there exists $\xi, \xi' \in \mathcal{H}, \eta, \eta' > 0$, $|\xi| + |\xi'| + \eta + \eta' \leq \epsilon$ such that

$$\begin{aligned}(t,s,X,Y) \to \langle U(t,X) - V(s,Y), X-Y\rangle + \beta(t-s)^2 + \alpha(|X|^2+|Y|^2)\\ + (\gamma_1+\eta)t + (\gamma_2+\eta')s + \langle \xi, X\rangle + \langle \xi', Y\rangle\end{aligned} \tag{3.20}$$

has a strict minimum on $[0,T]^2 \times \mathcal{H}^2$ at some point (t_0, s_0, X_0, Y_0). This is mainly due to the fact that, since U and V are bounded, and there are quadratic terms, the infimum of the previous function can be approximated by only considering points in a bounded set.

Assume first that $t_0, s_0 > 0$. Using the fact that U is a monotone solution of (2.12), we obtain

$$\begin{aligned}&-\gamma_1 + \eta + \langle D_pH(U(t_0,X_0)), 2\alpha X_0 + \xi\rangle\\ &\qquad \geq \langle \mathcal{A}(X_0), X_0 - Y_0\rangle + \langle D_pH(U(t_0,X_0)), U(t_0,X_0) - V(s_0,Y_0)\rangle.\end{aligned} \tag{3.21}$$

The same relation for V reads

$$(3.22)\quad -\gamma_2 + \eta' + \langle D_pH(V(s_0,Y_0)), 2\alpha Y_0 + \xi'\rangle \geq \langle \mathcal{A}(Y_0), Y_0 - X_0\rangle + \langle D_pH(U(s_0,Y_0)), V(s_0,Y_0) - U(t_0,X_0)\rangle.$$

Summing the two previous relation and using Hypothesis 3 yields

$$(3.23)\quad -\gamma_1 - \gamma_2 + \eta + \eta' \geq -\langle D_pH(U(t_0,X_0)), 2\alpha X_0 + \xi\rangle - \langle D_pH(V(s_0,Y_0)), 2\alpha Y_0 + \xi'\rangle.$$

As usual in viscosity solutions estimate, $\alpha(|X_0| + |Y_0|) \to 0$ as $\alpha \to 0$[2]. Hence, using the boundness of U and V and setting $\alpha, \epsilon \to 0$, we obtain that

$$(3.24)\quad -\gamma_1 - \gamma_2 \geq -\kappa \geq 0,$$

which is a contradiction. The cases $t_0 = 0$ and $s_0 = 0$ can be treated as in the previous proof. Hence $(t, X, Y) \to \langle U(t,X) - V(t,Y), X - Y\rangle \geq 0$ from which we deduce that $U = V$. □

The main difference with the previous proof is that, because $\mathcal{H}$ is not bounded, one had to add the terms in α in order to guarantee that the points of (strict) minimum indeed exist. The addition of those terms is the reason why I had to reinforce the definition of monotone solution by adding the term Z in Definition 3.

3.4. **Stability of monotone solutions.** I now present a result of stability of monotone solutions. I only focus on the case of (2.1) here, the other being similar. One of the main interest of the following result is that it allows to obtain results of existence of monotone solutions, as it is done in [1].

Theorem 3. *Assume that there is a sequence $(G_n, F_n)_{n\geq 0}$ converging locally uniformly toward $(G, F) : \mathbb{O} \times \mathbb{R}^d \to \mathbb{R}^{2d}$. Assume that there is a sequence $(U_n)_{n\geq 0}$ of monotone solutions of* (2.1) *such that for any $n \geq 0$, U_n is the solution associated to (G_n, F_n), and that this sequence $(U_n)_{n\geq 0}$ converges uniformly toward some function U. Then U is a monotone solution of* (2.1)*.*

Proof. Consider $T > 0, V \in \mathbb{R}^d, y \in \mathbb{O}$, a smooth function $\theta : [0,T] \to \mathbb{R}$ and (t_*, x_*) point of strict minimum of $(t, x) \to \langle U(t,x) - V, x - y\rangle - \theta(t)$ on $[0,T] \times \mathbb{O}$. For any $n \geq 0$, thanks to Stegall's Lemma, there exist $\delta_n > 0, \xi_n \in \mathbb{R}^d, \delta_n + |\xi_n| \leq n^{-1}$ such that $(t,x) \to \langle U_n(t,x) - V, x - y\rangle - \theta(t) + \langle \xi_n, x\rangle + \delta_n t$ has a strict minimum on $[0,T] \times \mathbb{O}$ at some point (t_n, x_n). Because U_n is a monotone solution of a certain master equation, it follows that

$$(3.25)\quad \frac{d\theta}{dt}(t_n) - \delta_n \geq \langle G_n(x_n, U_n(t_n,x_n)), x_n - y\rangle + \langle F_n(x_n, U_n(t_n,x_n)), U(t_n,x_n) - V + \xi_n\rangle.$$

Moreover, since (t_*, x_*) is a strict minimum of $(t,x) \to \langle U(t,x) - V, x - y\rangle - \theta(t)$ on $[0,T] \times \mathbb{O}$, we deduce that $(t_n, x_n) \to (t_*, x_*)$ as $n \to \infty$. Hence passing to the limit $n \to \infty$ in (3.25) yields the required result. □

[2]Observe that X_0 and Y_0 actually depends on α.

Remark 3.8. Let me insist on the facts that: i) the fact that we only ask for information at points of strict minimum was crucial, ii) no monotonicity is required to obtain this stability result.

4. Extensions to general monotone operators

I end these notes with an extension of the previous notion of solution, in the case of (2.1), to more singular operators. As the notion of monotone solution is mainly based on the monotonicity assumed in Hypothesis 1, it is very tempting to assume that F and G are only monotone operators, and not necessary functions. Let me insist on the fact that there exist numerous applications in which this type of master equations is at interest. The simplest examples are maybe the ones in which the players of the MFG have some sort of singular actions, like in the case of optimal stopping for example [1].

In the following, I assume that $G, F : \mathbb{O} \times \mathbb{R}^d \rightrightarrows \mathbb{R}^{2d}$, i.e. they are set valued maps. In this context, we would still like to be able to talk about a solution of

$$\partial_t U + \langle F(x, U), \nabla_x \rangle U = G(x, U) \tag{4.1}$$

in the sense that for any $t, x \in (0, \infty) \times \mathbb{O}$, there exist

$$(A, B) \in (G(x, U(t, x)), F(x, U(t, x)))$$

such that

$$\partial_t U(t, x) + \langle B, \nabla_x \rangle U(t, x) = A. \tag{4.2}$$

In particular this implies that $U(t, x)$ has to be such that $U(t, x)$ is in the domain of (G, F). The initial condition U_0 is still assumed to be a function here. The natural notion of monotone solution here is

Definition 4. A continuous function $U : (0, \infty) \times \mathbb{O} \to \mathbb{R}^d$ is a monotone solution of (4.1), if

- For any $(t, x) \in (0, \infty) \times \mathbb{O}, (x, U(t, x)) \in Dom(G, F)$.
- For any $T > 0, V \in \mathbb{R}^d, y \in \mathbb{O}, \theta : [0, T] \to \mathbb{R}$ smooth function and (t_0, x_0) point of strict minimum of

$$\begin{aligned} &[0, T] \times \mathbb{O} \to \mathbb{R}, \\ &\qquad (t, x) \to \langle U(t, x) - V, x - y \rangle - \theta(t), \end{aligned} \tag{4.3}$$

  there exist $(A, B) \in (G(x, U(t, x)), F(x, U(t, x)))$ such that

$$\frac{d\theta}{dt}(t_0) \geq \langle A, x_0 - y \rangle + \langle B, U(t_0, x_0) - V \rangle. \tag{4.4}$$

- The initial condition holds.

The following result holds.

Theorem 4. *Assume that (G, F) is a monotone operator and that U_0 is monotone, then there exists at most one monotone solution of* (4.1).

Proof. This proof is very similar to the previous ones hence I do not reproduce it here. ☐

A result of stability can also be established under some additional assumption.

Theorem 5. *Assume that* $(G_n, F_n)_{n\geq 0}$ *is a sequence of locally uniformly bounded operators. That is, for any bounded set* B *of* $\mathbb{O} \times \mathbb{R}$*, there exists a bounded set* C *such that for any* $n \geq 0, (G_n, F_n)(B) \subset C$*. Assume that the sequence* $(G_n, F_n)_{n\geq 0}$ *converges in the sense of graphs toward* $(G, F) : \mathbb{O} \times \mathbb{R}^d \rightrightarrows \mathbb{R}^{2d}$*. Assume that there is a sequence* $(U_n)_{n\geq 0}$ *of monotone solutions of* (4.1) *such that for any* $n \geq 0$*,* U_n *is the solution associated to* (G_n, F_n)*, and that this sequence* $(U_n)_{n\geq 0}$ *converges uniformly toward some function* U*. Then* U *is a monotone solution of* (4.1)*.*

Proof. The fact that $(x, U(t, x)) \in Dom(G, F)$ is immediate. Consider $T > 0, V \in \mathbb{R}^d, y \in \mathbb{O}$, a smooth function $\theta : [0, T] \to \mathbb{R}$ and (t_*, x_*) point of strict minimum of $(t, x) \to \langle U(t, x) - V, x - y\rangle - \theta(t)$ on $[0, T] \times \mathbb{O}$. For any $n \geq 0$, thanks to Stegall's Lemma, there exist $\delta_n > 0, \xi_n \in \mathbb{R}^d, \delta_n + |\xi_n| \leq n^{-1}$ such that $(t, x) \to \langle U_n(t, x) - V, x - y\rangle - \theta(t) + \langle \xi_n, x\rangle + \delta_n t$ has a strict minimum on $[0, T] \times \mathbb{O}$ at some point (t_n, x_n). Because U_n is a monotone solution of a certain master equation, it follows that there exists $(A_n, B_n) \in (G_n, F_n)(x_n, U_n(t_n, x_n))$ such that

$$\frac{d\theta}{dt}(t_n) - \delta_n \geq \langle A_n, x_n - y\rangle + \langle B_n, U_n, (t_n, x_n) - V + \xi_n\rangle. \tag{4.5}$$

Since (t_*, x_*) is a strict minimum of $(t, x) \to \langle U(t, x) - V, x - y\rangle - \theta(t)$ on $[0, T] \times \mathbb{O}$, we deduce that $(t_n, x_n) \to (t_*, x_*)$ as $n \to \infty$. Moreover, because of the assumption we made on the boundedness of $(G_n, F_n)_{n\geq 0}$, $(A_n, B_n)_{n\geq 0}$ is a bounded sequence. Hence, extracting a subsequence if necessary, it converges toward some limit (A_*, B_*). Because of the convergence of $(G_n, F_n)_{n\geq 0}$ toward (G, F) in the sense of graphs, we deduce that $(A_*, B_*) \in (G, F)(x_*, U(t_*, x_*))$. Passing to the limit in (4.5) yields

$$\frac{d\theta}{dt}(t_*) - \delta_n \geq \langle A_*, x_* - y\rangle + \langle B_*, U(t_*, x_*) - V\rangle, \tag{4.6}$$

which proves the claim. ∎

References

[1] Charles Bertucci. Monotone solutions for mean field games master equations: finite state space and optimal stopping. *Journal de l'École polytechnique — Mathématiques*, 8:1099–1132, 2021.

[2] Charles Bertucci. Monotone solutions for mean field games master equations: continuous state space and common noise, 2021. `arXiv:2107.09531`.

[3] Charles Bertucci and Alekos Cecchin. Mean field games master equations: from discrete to continuous state space, 2022. Available at `charles-bertucci@github.io`.

[4] Charles Bertucci, Jean-Michel Lasry, and Pierre-Louis Lions. On lipschitz solutions of mean field games master equations, 2022. Forthcoming.

[5] Pierre Cardaliaguet and Panagiotis Souganidis. Monotone solutions of the master equation for mean field games with idiosyncratic noise. *SIAM Journal on Mathematical Analysis*, 54(4):4198–4237, 2022.

[6] Pierre Cardaliaguet, François Delarue, Jean-Michel Lasry, and Pierre-Louis Lions. *The Master Equation and the Convergence Problem in Mean Field Games:(AMS-201)*, volume 201. Princeton University Press, 2019.

[7] Jean-Michel Lasry and Pierre-Louis Lions. Mean field games. *Japanese Journal of Mathematics*, 2(1):229–260, 2007.

[8] Pierre-Louis Lions. Cours au College de France, 2011. `www.college-de-france.fr`.

[9] Charles Stegall. Optimization of functions on certain subsets of banach spaces. *Mathematische Annalen*, 236(2):171–176, 1978.

[10] Charles Stegall. Optimization and differentiation in banach spaces. *Linear Algebra and Its Applications*, 84:191–211, 1986.

CMAP, Ecole Polytechnique, UMR 7641, 91120 Palaiseau, France