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Results on qualitative features of periodic solutions of KdV

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Consider the Korteweg-de Vries equation (KdV)

$$\partial_t u = -\partial_x^3 u + 6u\partial_x u \tag{1}$$

on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. It is globally in time well-posed on the Sobolev spaces $H^N \equiv H^N(\mathbb{T}, \mathbb{R})$ with $N \geq -1$. The aim of this paper is to describe new qualitative features of periodic solutions of KdV. First note that in contrast to solutions on the real line, periodic solutions do not have a special profile decomposition as $t \to \pm\infty$. Our main point of interest, related to the numerical experiments of Fermi, Pasta, and Ulam of particle chains, is to know how the distribution of energy among the Fourier modes evolves. A partial result in this direction says that due to the integrals provided by the KdV hierarchy, the Sobolev norms of smooth solutions stay bounded uniformly in time. In this paper we make further contributions to the study of how the Fourier coefficients $\hat{u}_n(t) = \int_0^1 u(t,x)e^{-2\pi inx}dx$ of a solution $u(t,x)$ of (1) evolve in time. Our first result aims at describing dispersion phenomena for solutions of KdV by studying how $\hat{u}_n(t)$ evolve for $|n|$ large. More precisely, we want to investigate if $\hat{u}_n(t)$ admits a WKB type expansion of the form

$$\hat{u}_n(t) = e^{iw_n t}\left(a_n(t) + \frac{b_n(t)}{n} + \dots\right) \tag{2}$$

where $e^{iw_n t}$ is a strongly oscillating phase factor with frequency w_n and the coefficients $a_n(t), b_n(t), \dots$ vary more slowly and satisfy the estimates

$$\sum n^{2N}|a_n(t)|^2 < \infty \quad \text{and} \quad \sum n^{2N}|b_n(t)|^2 < \infty.$$

To state our result more precisely, denote by ω_n, $n \geq 1$, the KdV frequencies of $u(t)$. Let us recall how they are defined. The KdV equation can be written as a Hamiltonian PDE with phase space L^2 and Poisson bracket

$$\{F, G\}(q) := \int_0^1 \partial F \, \partial_x \, \partial G dx \tag{3}$$

where F, G are C^1-functionals on L^2 and ∂F denotes the L^2-gradient of F. Then KdV takes the form $\partial_t u = \partial_x \partial_u \mathcal{H}$ where $\mathcal{H}$ is the KdV Hamiltonian

$$\mathcal{H}(q) := \int_0^1 \left(\frac{1}{2} (\partial_x q)^2 + q^3 \right) dx.$$

In terms of this set-up, the ω_n's are given by

$$\omega_n = \partial_{I_n} \mathcal{H}.$$

Here we use that $\mathcal{H}$ can be expressed as a real analytic function of the action variables I_n, $n \geq 1$, so that the partial derivatives $\partial_{I_n} \mathcal{H}$ are well defined – see below for more details. Alternatively, ω_n can be viewed as a function of q, which by a slight abuse of terminology, we also denote by ω_n. Clearly, for any $n \geq 1$, $\omega_n(u(t))$ is independent of t and depends in a nonlinear fashion on $u(0)$. It is convenient to introduce

$$\omega_{-n} := -\omega_n \quad \forall n \in \mathbb{Z}_{\geq 1} \quad \text{and} \quad \omega_0 := 0$$

and to denote the KdV flow by S^t, i.e., $S^t(u(0)) = u(t)$. In addition, let

$$R^t(u(0)) := S^t(u(0)) - \sum_{n \in \mathbb{Z}} e^{i\omega_n t} \hat{u}_n(0) e^{2\pi i n x}$$

where for any $n \in \mathbb{Z}$, $\omega_n = \omega_n(u(0))$.

Theorem 0.1. *For $q = u(0) \in H^N$, $N \in \mathbb{Z}_{\geq 0}$, the error $R^t(q)$ of the approximation $\sum_{n \in \mathbb{Z}} e^{i\omega_n t} \hat{q}_n e^{2\pi i n x}$ of the flow $S^t(q)$ has the following properties:*

(i) $R^t : H^N \to H^{N+1}$ is continuous;

(ii) for any $q \in H^N$, the curve $\{R^t(q) \mid t \in \mathbb{R}\}$ is relatively compact in H^{N+1};

(iii) for any $M > 0$, the set of curves $\{R^t(q) \mid t \in \mathbb{R}, q \in H^N, \|q\|_{H^N} \leq M\}$ is bounded in H^{N+1};

(iv) if in addition $N \in \mathbb{Z}_{\geq 1}$, then $\partial_t R^t : H^N \to H^{N-1}$ is continuous and for any $q \in H^N$, the orbit $\{\partial_t R^t(q) \mid t \in \mathbb{R}\}$ is relatively compact in H^{N-1}. Moreover for any $M > 0$ the set of orbits $\{\partial_t R^t(q) \mid t \in \mathbb{R}, q \in H^N, \|q\|_{H^N} \leq M\}$ is bounded in H^{N-1}.

Remark 0.1. *Actually, one can prove that for any $c \in \mathbb{R}$, the restrictions of R^t and $\partial_t R^t$ to the affine subspace $H_c^N = \{q \in H^N \mid \int_0^1 q(x) dx = c\}$ are real analytic.*

Remark 0.2. *In case $u(0)$ is a finite gap potential, there are formulas, due to Its-Matveev [3], for the frequencies ω_n in terms of periods of an Abelian differential, defined on the spectral curve associated to $u(0)$. These formulas can be extended to potentials in H^N, $N \geq -1$, – cf [12]. Alternative formulas can be found in [7], Appendix F.*

Remark 0.3. *Note that the frequencies ω_n depend on the initial conditions in a nonlinear way. The statement of Theorem 0.1 no longer holds if the KdV frequencies ω_n are replaced by their linearization at 0, i.e., by $(2\pi n)^3$, confirming the belief of experts in the field that solutions of KdV on the circle are not approximated by linear evolution over a time interval of infinite length – see [1] for results on linear approximations of solutions over* finite *time intervals.*

In terms of the above WKB ansatz (2), Theorem 0.1 says that with $w_n := \omega_n$ and $a_n(t) := \hat{u}_n(0)$, the remainder term

$$\rho_n(t) := b_n(t) + \cdots := n \cdot \left(e^{-i\omega_n t}\hat{u}_n(t) - \hat{u}_n(0)\right) = n\widehat{R}_n^t(u(0))e^{-i\omega_n t}$$

satisfies $\sum n^{2N}|\rho_n(t)|^2 < \infty$ and in case $N \in \mathbb{Z}_{\geq 1}$,

$$\sum n^{2(N-2)}|\partial_t \rho_n(t)|^2 < \infty. \tag{4}$$

As the asymptotics of the KdV frequencies are given by $\omega_n = 8\pi^3 n^3 + O(n)$ estimate (4) quantifies the assertion that $(\rho_n(t))_{n\in\mathbb{Z}}$ varies more slowly than $(\hat{u}_n(t))_{n\in\mathbb{Z}}$.

The second result we would like to describe concerns the approximation of KdV solutions by trigonometric polynomials. For any $L \in \mathbb{Z}_{\geq 1}$, denote by $P_L : L^2 \to L^2$ the L^2-orthogonal projection of $L^2 = H^0(\mathbb{T}, \mathbb{R})$ onto the $2L + 1$ dimensional $\mathbb{R}$-vector space generated by $e^{2\pi inx}$, $|n| \leq L$.

Theorem 0.2. *Let $N \in \mathbb{Z}_{\geq 0}$ be arbitrary. Then for any $M > 0$ and $\epsilon > 0$ there exists $L_{\epsilon,M} \geq 1$ such that for any $u(0) \in H^N$, with $\|u(0)\|_{H^N} \leq M$, $L \geq L_{\epsilon,M}$, and any $t \in \mathbb{R}$*

$$\|(Id - P_L)u(0)\|_{H^N} - \epsilon \leq \|(Id - P_L)u(t)\|_{H^N} \leq \|(Id - P_L)u(0)\|_{H^N} + \epsilon.$$

In particular, if $u(0)$ with $\|u(0)\|_{H^N} \leq M$ is a trigonometric polynomial of order $L_\star$, then for any $L \geq \max(L_\star, L_{\epsilon,M})$, $P_L u(t)$ approximates $u(t)$ uniformly in $t \in \mathbb{R}$ up an error of size ϵ.

Remark 0.4. *The proof of Theorem 0.2 shows that for any $|n| > L_{\epsilon,M}$ and $\|u(0)\|_{H^N} \leq M$,*

$$|\hat{u}_n(0)| - \epsilon \leq |\hat{u}_n(t)| \leq |\hat{u}_n(0)| + \epsilon \quad \forall t \in \mathbb{R}.$$

It means that for $|n|$ sufficiently large, the amplitude of the n'th Fourier mode is approximately constant, uniformly on bounded sets of H^N.

Remark 0.5. *It follows from the proof of Theorem 0.1 that corresponding results hold for the flow of any Hamiltonian in the Poisson algebra of KdV. In particular, this is true for the flows of Hamiltonians in the KdV hierarchy.*

Detailed proofs of Theorem 0.1 and Theorem 0.2 are presented in [10]. The main ingredient of these proofs are new asymptotics of the Birkhoff map of KdV. This map provides normal coordinates, allowing to solve KdV by quadrature.

Let us recall its set-up. First note that the average of any solution $u(t) \equiv u(t, x)$ of KdV in H^N is a conserved quantity. In particular, for any $c \in \mathbb{R}$, KdV leaves the subspaces $H_c^N \equiv H_c^N(\mathbb{T}, \mathbb{R})$ of H^N invariant where

$$H_c^N = \left\{ p(x) = \sum \hat{p}_n e^{2\pi i n x} \mid \hat{p}_0 = c; \|p\|_N < \infty; \hat{p}_{-n} = \overline{\hat{p}_n} \ \forall n \in \mathbb{Z} \right\}$$

with

$$\|p\|_N := \left(\sum |n|^{2N} |\hat{p}_n|^2 \right)^{\frac{1}{2}}.$$

In the case $N = 0$, we often write L_c^2 for H_c^0 and $\|p\|$ instead of $\|p\|_0$. To describe the normal coordinates of KdV, let us introduce for any $\alpha \in \mathbb{R}$ the $\mathbb{R}$-subspace $\mathfrak{h}^\alpha$ of $\ell^{2,\alpha}$, given by

$$\mathfrak{h}^\alpha := \left\{ z = (z_n)_{n \neq 0} \in \ell^{2,\alpha} \mid z_{-n} = \overline{z}_n \forall n \geq 1 \right\}$$

where

$$\ell^{2,\alpha} \equiv \ell^{2,\alpha}(\mathbb{Z}_0, \mathbb{C}) := \left\{ z = (z_n)_{n \neq 0} \mid \|z\|_\alpha < \infty \right\},$$

$\mathbb{Z}_0 := \mathbb{Z} \setminus \{0\}$, and

$$\|z\|_\alpha := \left(\sum_{n \neq 0} |n|^{2\alpha} |z_n|^2 \right)^{\frac{1}{2}}.$$

The space $\mathfrak{h}^\alpha$ is endowed with the standard Poisson bracket for which $\{z_n, z_{-n}\} = -\{z_{-n}, z_n\} = 2i$ for any $n \geq 1$ whereas all other brackets between coordinate functions vanish. Furthermore we denote by $H_{0,\mathbb{C}}^N \equiv H_0^N(\mathbb{T}, \mathbb{C})$, $L_{0,\mathbb{C}}^2 \equiv L_0^2(\mathbb{T}, \mathbb{C})$ and $\mathfrak{h}_\mathbb{C}^\alpha$ the complexification of the spaces H_0^N, L_0^2, and $\mathfrak{h}^\alpha$. Note that $\mathfrak{h}_\mathbb{C}^\alpha = \ell^{2,\alpha}(\mathbb{Z}_0, \mathbb{C})$. A detailed proof of the following result can be found in [7] – cf also [6].

Theorem 0.3. *There exist an open neighbourhood W of L_0^2 in $L_{0,\mathbb{C}}^2$ and a real analytic map $\Phi : W \to \mathfrak{h}_\mathbb{C}^{1/2}$ with the following properties:*

(BC1) *For any $N \in \mathbb{Z}_{\geq 0}$, the restriction of Φ to H_0^N is a canonical, bianalytic diffeomorphism onto $\mathfrak{h}^{N+1/2}$.*

(BC2) *When expressed in the new coordinates, the KdV-Hamiltonian $\mathcal{H} \circ \Phi^{-1}$, defined on $\mathfrak{h}^{3/2}$, is a real analytic function of the action variables $I_n = (z_n z_{-n})/2$, $n \geq 1$, alone.*

(BC3) *The differential $\Phi_0 \equiv d_0\Phi$ of Φ at 0 is the weighted Fourier transform,*

$$\Phi_0(h) = \left(\frac{1}{\sqrt{|n|\pi}} \hat{h}_n \right)_{n \neq 0} \tag{5}$$

The coordinates z_n, $n \neq 0$, are referred to as (complex) Birkhoff coordinates whereas Φ is called Birkhoff map. Note that in [7] the Birkhoff map is defined slightly differently by setting $\Phi(q)$ to be $(x_n, y_n)_{n \geq 1}$ where $x_n = (z_n + z_{-n})/2$

and $y_n = i(z_n - z_{-n})/2$. The fact that KdV admits globally defined Birkhoff coordinates is a very special feature of KdV. In more physical terms it says that KdV, when considered with periodic boundary conditions, is a system of infinitely many coupled oscillators.

Remark 0.6. *A result similar to the one of Theorem 0.3 holds for the defocusing NLS equation. A detailed proof can be found in [2]. Cf also [17].*

The key ingredient of the proofs of Theorem 0.1 and Theorem 0.2 is the following result on the asymptotics of the Birkhoff map, positively answering a question, raised by Kuksin and Perelman in [14].

Theorem 0.4. *For $N \in \mathbb{Z}_{\geq 0}$, there exists an open neighbourhood W_N of H_0^N in $H_{0,\mathbb{C}}^N \cap W$ so that $\Phi - \Phi_0$ maps W_N into $\mathfrak{h}_\mathbb{C}^{N+3/2}$ and, as a map from W_N to $\mathfrak{h}_\mathbb{C}^{N+3/2}$, is analytic. Here W is the neighbourhood of L_0^2 in $L_{0,\mathbb{C}}^2$ of Theorem 0.3. Furthermore, the restriction $A := (\Phi - \Phi_0)_{|H_0^N} : H_0^N \to \mathfrak{h}^{N+3/2}$ is a bounded map, i.e. it is bounded on bounded subsets of H_0^N.*

Remark 0.7. *In [19] a result similar to the one stated in Theorem 0.4 is proved for the Birkhoff map of KdV constructed in [5], where the phase space is endowed with the Poisson bracket introduced by Magri. As an application, a corresponding result is then derived for the modified Korteweg-de Vries equation (mKdV) on H^N with $N \geq 1$. Indeed, it was shown in [8] that the Miura map $f \mapsto f' + f^2$ canonically embeds the symplectic leaves of the phase space of mKdV, endowed with the Poisson bracket (3), into the phase space of KdV, endowed with the Magri bracket. (For a detailed study of the Miura map see [11].) As a consequence, results similar to the ones of Theorem 0.1 and Theorem 0.2 can be proved for mKdV – see [19].*

Remark 0.8. *We expect that similar results as the ones of Theorem 0.4 can be proved for the defocusing NLS equation. As a consequence, results similar to the ones of Theorem 0.1 and Theorem 0.2 are expected to hold for this equation.*

Remark 0.9. *Normalizing transformations such as the Birkhoff map are often viewed as nonlinear versions of the Fourier transform. In the case of KdV, Theorem 0.4 provides a qualitative statement in this respect, saying that Φ is a weakly nonlinear perturbation of the (weighted) Fourier transform.*

The proof of Theorem 1.4 is based on sharp asymptotic estimates of various spectral quantities of Schrödinger operators, some of which have not been considered before, which can be found in [9]. Some of these estimates improve on earlier results due to Marchenko [16] and others – see [18] and references therein.

Related results   Recently, Kuksin and Piatnitski initiated a study of random perturbations with damping of the KdV equation [15], [13]. More precisely they are interested, how the KdV-action variables evolve under certain perturbed equations. For this purpose they express the perturbed KdV equation in normal coordinates. Up to highest order, it is a linear differential equation if the

nonlinear part $\Phi - \Phi_0$ of the Birkhoff map is 1-smoothing, i.e. if it maps H_0^N to $\mathfrak{h}^{N+3/2}$ for any $N \geq 0$. In their recent paper, Kuksin and Perelman [14] succeeded in showing a local version of Theorem 0.4 near the equilibrium point $q = 0$. More preciseely they prove that on a neighbourhood U of the equilibrium point $q = 0$, there exists a canonical, real analytic diffeomorphism $\Psi : U \to V$ with $V \subseteq \mathfrak{h}^{1/2}$ a neighbourhood of 0 in $\mathfrak{h}^{1/2}$ providing Birkhoff coordinates for KdV so that $\Psi - \Psi_0$ is 1-smoothing where Ψ_0 denotes the linearization of Ψ at $q = 0$ and coincides with Φ_0. They obtain the map Ψ by generalizing Eliasson's construction of a Birkhoff map near an equilibrium point of a finite dimensional integrable system to a class of integrable PDEs including the KdV equation. In order to apply Eliasson's construction, Kuksin and Perelman need coordinates for the KdV equation, provided in [4], as a starting point. Eliasson's construction is based on Moser's path-method and, in general, cannot be extended to get global coordinates. However, for the study of random perturbations of KdV in [13], global Birkhoff coordinates for KdV are needed. In [14], it was conjectured that there exists a globally defined Birkhoff map Ψ so that $\Psi - \Psi_0$ is 1-smoothing. Note that Birkhoff maps are not uniquely determined. Theorem 0.4 confirms that this conjecture holds true and that Ψ can be chosen to be the Birkhoff map of Theorem 0.3.

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