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GENERALIZED GRADIENT FLOW AND SINGULARITIES OF THE RIEMANNIAN DISTANCE FUNCTION

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ABSTRACT. Significant information about the topology of a bounded domain Ω of a Riemannian manifold M is encoded into the properties of the distance, $d_{\partial\Omega}$, from the boundary of Ω. We discuss recent results showing the invariance of the singular set of the distance function with respect to the generalized gradient flow of $d_{\partial\Omega}$, as well as applications to homotopy equivalence.

KEYWORDS: distance function, generalized characteristics, propagation of singularities, semiconcavity, Riemannian manifold, homotopy

CONTENTS

1. INTRODUCTION

It is well known that the distance function from the boundary of a bounded open set $\Omega \subset \mathbb{R}^n$, that is,

$$d_{\partial\Omega}(x) = \min_{y \in \partial\Omega} |y - x| \qquad \forall x \in \mathbb{R}^n \,,$$

is Lipschitz continuous in $\mathbb{R}^n$ with Lipschitz seminorm equal to one, and locally semiconcave in Ω[1]. Consequently, $d_{\partial\Omega}$ is differentiable in $\Omega \setminus \Sigma$, where $\Sigma \subset \Omega$ is a set of Lebesgue measure zero called the *singular set* of $d_{\partial\Omega}$. The structure of Σ has been investigated in several papers providing upper bounds for its Hausdorff dimension and also lower bounds in the form of conditions ensuring the propagation of singularities. It is the latter viewpoint we will be here concerned with in this paper.

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[1]This means that, for every convex compact set $G \subset \Omega$, there is a constant $K_G \in \mathbb{R}$ such that $x \mapsto d_{\partial\Omega}(x) - K_G|x|^2/2$ is concave on G.

For semiconcave functions, geometric conditions ensuring the propagation of singularities were obtained in [1]. In [2] and [7], propagation results were derived for viscosity solutions of the Hamilton-Jacobi equation

$$F(x, u, Du) = 0 \quad \text{in} \quad \Omega\,, \tag{1.1}$$

with $F(x, u, p)$ convex in p. More precisely, given a noncritical singular point x_0 of u, one can show that u fails to be differentiable on a nonconstant lipschitz arc $\gamma : [0, \sigma[\to \mathbb{R}^n$, starting at x_0, which satisfies the *generalized characteristic* inclusion

$$\gamma'(t) \in \text{ co } D_p F(\gamma(t), u(\gamma(t)), D^+ u(\gamma(t)))\,, \tag{1.2}$$

where $D^+ u$ denotes the superdifferential of u while 'co' stands for 'convex hull'. Observe that the above settings include the distance function, which solves the eikonal equation $|Dd_{\partial\Omega}|^2 = 1$. In this case, inclusion (1.2) reduces to the generalized gradient flow $\gamma' \in D^+ d_{\partial\Omega}(\gamma)$ up to rescaling.

The propagation of singularities along characteristics is a well-studied property of solutions to linear hyperbolic equations. In [9], for scalar hyperbolic conservation laws in one space dimension, Dafermos observed that singular arcs could be regarded as generalized solutions of the same differential equation governing classical characteristics.

The above considerations can be naturally extended to an open subset Ω of a Riemannian manifold M. In a local coordinate chart, the eikonal equation takes the form

$$\langle A^{-1}(x) Du(x), Du(x) \rangle = 1\,, \tag{1.3}$$

where $A(x)$ is related to the Riemannian scalar product g_x on the tangent space $T_x M$ by the formula

$$g_x(\xi, \zeta) = \langle A(x)\xi, \zeta \rangle \qquad \forall \xi, \zeta \in T_x M\,.$$

In this case, the equation of generalized characteristics is

$$\gamma'(t) \in A^{-1}(\gamma(t)) D^+ u(\gamma(t))\,. \tag{1.4}$$

In the aforementioned paper [9], singularities are shown to propagate along a generalized characteristic, forward in time up to infinity. Intuitively speaking, such a behaviour is related to the well-known interpretation of the entropy condition for solutions of conservation laws (with a convex flux), which ensures that characteristics can only go inside a singularity: a characteristic which enters the singular set remains "trapped" there.

Although the two-dimensional structure of the problem is essential for the proof of [9], in [3] it was proved that the same property holds for singular arcs of the euclidean distance function in arbitrary dimension, showing that Σ is invariant for the generalized gradient flow. Moreover, the approach of [3] applies to the Riemannian distance as well, even though $d_{\partial\Omega}^2$ fails to be semiconcave with $K = 2$, in general.

It is interesting to remark that the properties of the singular set of $d_{\partial\Omega}$ are also relevant to applied domains such as computer science (see [5, 11] and the references therein). Several authors have studied, at increasing levels of generality, the homotopy equivalence between Σ and Ω. If the boundary of Ω is smooth (or piecewise smooth in dimension 2), then a homotopy can be constructed by moving every point outside Σ by the smooth gradient

flow of the distance function until it reaches Σ. However, this method only works if the distance from a regular point p to Σ along the gradient of the distance is a continuous function of p. If Ω is a nonsmooth set of dimension at least 3, then this fails to be true and the construction of the homotopy becomes more involved [11]. An essential step of the procedure is to extend the gradient flow of the distance function past a singular point, ensuring that the corresponding generalized characteristic stays singular for all time. This is why the methods of [3] can be used to generalize the homotopy equivalence between Σ and Ω to complete Riemannian manifolds.

The above generalization is rich of important consequences even for problems which are naturally set in euclidean space. To clarify this point, we will describe the analysis of optimal exit time problems in $\mathbb{R}^n$ made in [3]. In this example, the minimum time function $T(x)$, which measures the minimum time needed to steer a point $x \in \Omega$ to $\partial\Omega$, can be interpreted as a Riemannian distance function. This connection allows to show the homotopy equivalence between Ω and the singular set of $T(\cdot)$ in a simple way, while such a result would be hard to derive in a euclidean setting.

2. SEMICONCAVE FUNCTIONS

In this section, we introduce semiconcave functions and recall some of their properties. Given $x, y \in \mathbb{R}^n$, we denote by $\langle x, y \rangle$ and by $|x|$ the euclidean scalar product and norm respectively. Let Ω be an open set in $\mathbb{R}^n$.

Definition 2.1. A function $u : \Omega \to \mathbb{R}$ is said to be *semiconcave* in Ω if there exists $K \geq 0$ such that

$$tu(x) + (1-t)u(y) - u(tx + (1-t)y) \leq t(1-t)K\frac{|x-y|^2}{2}$$

for any $x, y \in \Omega$ such that the segment from x to y is contained in Ω, and for any $t \in [0, 1]$. We call K a *semiconcavity constant* for u in Ω. We say that u is locally semiconcave in Ω if it is semiconcave on any subset $A \subset\subset \Omega$.

It is easy to see that u is semiconcave with constant K if and only if the function $u(x) - \frac{K}{2}|x|^2$ is concave or if $D^2u \leq K\,Id$ in the sense of distributions, where Id denotes the identity matrix.

The *(Fréchet) superdifferential* of a function $u : \Omega \to \mathbb{R}$ at a point $x \in \Omega$ is defined as the set

$$D^+u(x) = \left\{p \in \mathbb{R}^n \ : \ \limsup_{y \to x} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|y - x|} \leq 0\right\}.$$

In the case of a semiconcave function, the superdifferential enjoys the following properties; the proofs can be found in any textbook on convex analysis or in Chapter 3 of [6].

Proposition 2.2. *Let $u : \Omega \to \mathbb{R}$ be semiconcave.*
  (i) *The function u is locally Lipschitz continuous in Ω and differentiable almost everywhere.*
  (ii) *The superdifferential $D^+u(x)$ is nonempty for all $x \in \Omega$. It is a singleton if and only if u is differentiable at x, and in this case we have $D^+u(x) = \{Du(x)\}$, where $Du(x)$ is the standard gradient.*

(iii) *For any $x, y \in \Omega$ such that the segment from x to y is contained in Ω, for any $p \in D^+u(x)$ and $q \in D^+u(y)$, we have*

$$\langle q - p, y - x \rangle \leq K|y - x|^2, \tag{2.1}$$

where K is a semiconcavity constant of u.

(iv) *Given $\{x_i\} \subset \Omega$ such that $x_i \to \bar{x} \in \Omega$ and $p_i \in D^+u(x_i)$ such that $p_i \to \bar{p}$, we have that $\bar{p} \in D^+u(\bar{x})$.*

The singular set of u is defined as

$$\begin{aligned} \Sigma(u) &= \{x \in \Omega \mid \ u \text{ is not differentiable at } x\} \\ &= \{x \in \Omega \mid D^+u(x) \text{ contains more than one point}\}. \end{aligned}$$

General properties of semiconcave functions ensure that $\Sigma(u)$ is countably $(n-1)$-rectifiable that is, it can be covered by a countable family of Lipschitz hypersurfaces (see [4]).

3. EUCLIDEAN DISTANCE FUNCTION

We denote by d_C the euclidean distance function from a closed set $C \subset \mathbb{R}^n$ defined as

$$d_C(x) = \min_{y \in C} |y - x| \qquad \forall x \in \mathbb{R}^n \,,$$

and by $\text{proj}_C(x)$ the set of closest points in C to x:

$$\text{proj}_C(x) = \{y \in C \ : \ d_C(x) = |x - y|\} \qquad x \in \mathbb{R}^n \,.$$

The following properties of d_C are well known.

(i) d_C is differentiable at $x \notin C$ if and only if $\text{proj}_C(x)$ is a singleton and in this case

$$Dd_C(x) = \frac{x - y}{|x - y|} \tag{3.1}$$

where y is the unique element of $\text{proj}_C(x)$.

(ii) If $\text{proj}_C(x)$ is not a singleton then we have

$$D^+d_C(x) = \text{co} \left\{ \frac{x - y}{|x - y|} \ : \ y \in \text{proj}_C(x) \right\} = \frac{x - \text{co}\,(\text{proj}_C(x))}{d_C(x)}.$$

(iii) For any $x \notin C$ and any $y \in \text{proj}_C(x)$, d_C is differentiable along the segment $]x, y[$.

d_C is Lipschitz continuous in $\mathbb{R}^n$ with Lipschitz seminorm equal to one, and locally semiconcave in $\mathbb{R}^n \setminus C$ as we recall below (see [6, Proposition 2.2.2]).

Proposition 3.1. *Given any nonempty closed set $C \subset \mathbb{R}^n$, the distance function d_C is locally semiconcave on $\mathbb{R}^n \setminus C$. In addition, the squared distance function $d_C^2(\cdot)$ is semiconcave on all $\mathbb{R}^n$ with constant $K = 2$. Moreover, we have*

$$\langle d_C(x)p - d_C(y)q, x - y \rangle \leq |x - y|^2 \tag{3.2}$$

for all $x, y \in \mathbb{R}^n, p \in D^+d(x)$ and $q \in D^+d(y)$.

Proof. We begin by showing the semiconcavity of d_C^2. For any $x \in \mathbb{R}^n$ we have

$$d_C^2(x) - |x|^2 = \inf_{y \in C} |x - y|^2 - |x|^2 = \inf_{y \in C} \left(|y|^2 - 2\langle x, y \rangle \right).$$

Since the infimum of affine functions is concave we deduce that $u(x) := d_C^2(x) - |x|^2$ is concave on $\mathbb{R}^n$. This yields that d_C^2 is semiconcave on $\mathbb{R}^n$ with constant $K = 2$.

We now proceed to derive the local semiconcavity of d_C. Let us first observe that, given $z, h \in \mathbb{R}^n$, $z \neq 0$, we have

$$(|z + h| + |z - h|)^2$$
$$\leq \quad 2(|z + h|^2 + |z - h|^2) = 4(|z|^2 + |h|^2) \leq \left(2|z| + \frac{|h|^2}{|z|} \right)^2.$$

Thus

$$|z + h| + |z - h| - 2|z| \leq \frac{|h|^2}{|z|}. \tag{3.3}$$

Let S be a set with positive distance from C. For any x, h such that the line segment $[x - h, x + h]$ is contained in S, let $\bar{x} \in C$ be such that $d_C(x) = |x - \bar{x}|$. Then

$$d_C(x + h) + d_C(x - h) - 2d_C(x)$$
$$\leq \quad |x + h - \bar{x}| + |x - h - \bar{x}| - 2|x - \bar{x}|$$
$$\leq \quad \frac{|h|^2}{|x - \bar{x}|}.$$

Moreover $|x - \bar{x}| = d_C(x) \geq \mathrm{dist}\,(S, C)$. We conclude that d_C satisfies the desired property.

Finally, Estimate (3.2) follows from Proposition 2.2(iii), observing that $D^+ d_C^2(x) = 2d_C(x) D^+ d_C(x)$. □

Consequently, d_C satisfies the *eikonal equation*

$$|Du(x)| = 1 \quad \text{for a.e. } x \in \mathbb{R}^n \setminus C. \tag{3.4}$$

4. GENERALIZED CHARACTERISTICS OF EIKONAL TYPE EQUATIONS

In the following, we will consider semiconcave functions $u : \Omega \to \mathbb{R}$ which solve an equation of the form

$$\langle A^{-1}(x) Du(x), Du(x) \rangle = 1 \qquad (x \in \Omega \text{ a.e.}) \tag{4.1}$$

where $A(x)$ a symmetric positive definite $n \times n$ matrix with C^1 dependence on $x \in \Omega$ (the formulation with A^{-1}, rather than A, is natural in the context of Riemannian manifolds that we shall treat in the sequel).

It is well known (see e.g. [6, Prop. 5.3.1]) that, if u is locally semiconcave in Ω, then the following properties are equivalent:

- u satisfies (4.1);
- for every $x \in \Omega$ and $p \in D^+ u(x)$ we have $\langle A^{-1}(x)p, p \rangle \leq 1$;
- u is a viscosity solution of (4.1) (in the sense of [8]).

Throughout the paper, we call a *solution* of (4.1) a locally semiconcave function with any of the above properties.

Given a solution of (4.1), we consider the differential inclusion

$$\gamma'(t) \in A^{-1}(\gamma(t))D^+u(\gamma(t)). \tag{4.2}$$

A Lipschitz arc $\gamma : [0, t_0] \to \Omega$ is called a solution to the above problem if, for a.e. $t \in [0, t_0]$, it satisfies $\gamma'(t) = A^{-1}(\gamma(t))p(t)$ for some element $p(t) \in D^+u(\gamma(t))$. Such an arc will also be called a *generalized characteristic* of equation (4.1) associated with u.

We now recall some properties of generalized characteristics. The main part of the statement (in particular claim (iv) about the propagation of singularities) follows from the results first proved in [2] and then obtained with a simpler approach in [7, 14]. Here, we include some additional properties, not explicitly observed in the above references, which were derived in [3].

Theorem 4.1. *Let $u : \Omega \to \mathbb{R}$ be a solution of (4.1). Then, for every $x_0 \in \Omega$ there exists $t_0 > 0$ and a unique Lipschitz continuous arc $\gamma : [0, t_0[\to \Omega$ which satisfies (4.2) and the initial condition $\gamma(0) = x_0$. In addition, the right derivative $\gamma'_+(t)$ exists for every $t \in [0, t_0[$, and $p(t) := \gamma'_+(t)$ has the following properties:*

(i) *$p(t) \in A^{-1}(\gamma(t))D^+u(\gamma(t))$ for every $t \in [0, t_0[$ and*

$$\langle p(t), A(\gamma(t))p(t) \rangle \leq \langle q, A(\gamma(t))q \rangle, \ \forall q \in A^{-1}(\gamma(t))D^+u(\gamma(t)). \tag{4.3}$$

(ii) *$p(t)$ is continuous from the right for every $t \in [0, t_0[$ and, for all points t^* where it is discontinuous, we have*

$$\liminf_{t \to t^*-}\langle p(t), A(\gamma(t))p(t) \rangle \geq \langle p(t^*), A(\gamma(t^*))p(t^*) \rangle. \tag{4.4}$$

(iii) *For any $t \in [0, t_0[$, $\gamma(t) \in \Sigma(u)$ if and only if $\langle A(\gamma(t))p(t), p(t) \rangle < 1$.*

(iv) *If $x_0 \in \Sigma(u)$, then there exists $\sigma \in]0, t_0]$ such that $\gamma(t) \in \Sigma(u)$ for all $t \in [0, \sigma]$.*

(v) *For all $t \in [0, t_0[$ we have*

$$\frac{d}{dt^+}u(\gamma(t)) = \langle A(\gamma(t))p(t), p(t) \rangle, \tag{4.5}$$

where the symbol $\frac{d}{dt^+}$ denotes the derivative from the right.

We give below a continuous dependence result for generalized characteristics that is due to [3].

Lemma 4.2. *Under the above assumptions, for any $U \subset\subset \Omega$ there exist $C > 0$ and $t_0 > 0$ such that, if $x, y \in U$ and γ_x, γ_y are solutions of (4.2) with initial conditions $\gamma_x(0) = x$ and $\gamma_y(0) = y$ respectively, then*

$$|\gamma_x(t) - \gamma_y(t)| \leq C|x - y|, \qquad t \in [0, t_0]. \tag{4.6}$$

In particular, the above results apply to the distance from the boundary of Ω, $d_{\partial\Omega}$. In this case, the flow associated with generalized characteristics is called the *generalized gradient flow*.

Corollary 4.3. *Let $\Omega \subset \mathbb{R}^n$ be an open set. Then, for every $x \in \Omega$ there exists a unique Lipschitz continuous arc $\gamma : [0, \infty[\to \Omega$ such that*

$$\gamma'(t) \in D^+d_{\partial\Omega}(\gamma(t)) \quad t \in [0, \infty[\ a.e. \qquad \gamma(0) = x. \tag{4.7}$$

In addition, for any $t_0 \geq 0$ such that $\gamma(t_0) \in \Sigma(d_{\partial\Omega})$ there exists $\sigma > 0$ such that $\gamma(t) \in \Sigma(d_{\partial\Omega})$ for all $t \in [t_0, t_0 + \sigma[$. Finally, the derivative from the right $\gamma'_+(t)$ exists for all $t \in [0, +\infty)$ and satisfies the properties described in Theorem 4.1, with $u(x) = d_{\partial\Omega}(x)$ and $A(x) \equiv Id$.

Proof. The statement follows directly from Theorem 4.1, provided we show that the maximal interval of existence of γ is $[0, +\infty[$. To see this we note that, if such an interval is $[0, T[$ with $T \neq +\infty$, then necessarily $\gamma(t)$ approaches $\partial\Omega$ as $t \to T$, that is, $d_{\partial\Omega}(\gamma(t)) \to 0$ as $t \to T$, in contrast with the property that $d_{\partial\Omega}(\gamma(t))$ is positive and nondecreasing in t by (4.5). □

5. Gradient flow and singularities of the euclidean distance

We have just seen that generalized characteristics starting from a singular point stay singular up to some time $T > 0$. This raises the question whether, at list in some case, one can have that $T = \infty$. In other terms, we would like to investigate the invariance of the singular set under the generalized characteristic flow. An interesting result in this direction is the following one, obtained in [3], the proof of which we describe below.

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set, let $x \in \Omega$, and let $\gamma(\cdot)$ be the solution of (4.7) given by Corollary 4.3. If $\gamma(t_0) \in \Sigma(d_{\partial\Omega})$ for some $t_0 \geq 0$, then $\gamma(t) \in \Sigma(d_{\partial\Omega})$ for all $t \in [t_0, +\infty[$.*
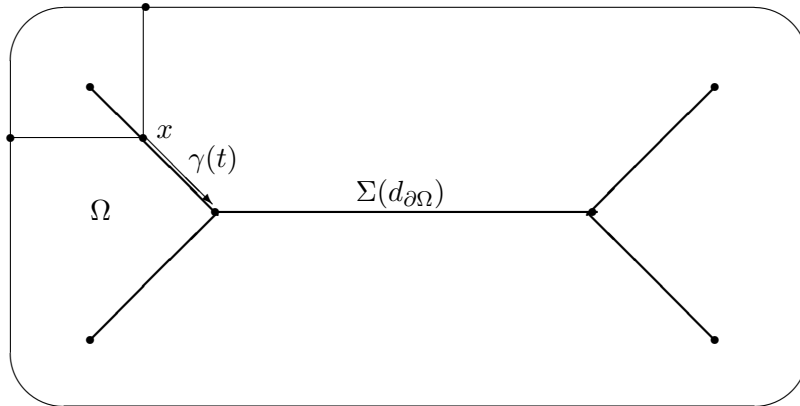

FIGURE 1. Invariance of $\Sigma(d_{\partial\Omega})$ under gradient flow

Proof. For simplicity of notation, we suppose $t_0 = 0$. Set

$$p(t) := \gamma'_+(t), \qquad \delta(t) := d_{\partial\Omega}(\gamma(t)) \qquad t \in [0, \infty[.$$

From Theorem 4.1 and Corollary 4.3 we know that $p(t) \in D^+ d_{\partial\Omega}(\gamma(t))$ for all t; in addition $\gamma(t) \in \Sigma(d_{\partial\Omega})$ and $|p(t)| < 1$ for all $t > 0$ in a right neighbourhood of 0. Our aim is to show that $|p(t)| < 1$ holds for every $t \geq 0$; by part (iii) of Theorem 4.1, this will prove our assertion.

Let $0 \leq s < t$. By Proposition 3.1, we have

$$\delta(t)\langle p(t) - p(s), \gamma(t) - \gamma(s)\rangle$$
$$\leq |\gamma(t) - \gamma(s)|^2 - (\delta(t) - \delta(s))\langle p(s), \gamma(t) - \gamma(s)\rangle. \quad (5.1)$$

Now, set $t = s + h$ $(h > 0)$,

$$\gamma_h(s) = \frac{\gamma(s+h) - \gamma(s)}{h} \qquad \text{and} \qquad \delta_h(s) = \frac{\delta(s+h) - \delta(s)}{h}.$$

We observe that $|\gamma_h| \leq 1, |\delta_h| \leq 1$, since γ and δ are both Lipschitz functions of constant 1. From (5.1) we obtain

$$\left\langle \gamma_h'(s), \gamma_h(s) \right\rangle \leq \frac{1}{\delta(s+h)}[|\gamma_h(s)|^2 - \delta_h(s)\langle p(s), \gamma_h(s)\rangle]. \quad (5.2)$$

Here and in the rest of the proof, we use for simplicity the notation of the ordinary derivative to mean the derivative from the right of expressions involving γ_h. Observe that, by Theorem 4.1, we have for all $s \geq 0$

$$\lim_{h \to 0^+} \gamma_h(s) = p(s), \qquad \lim_{h \to 0^+} \delta_h(s) = |p(s)|^2. \quad (5.3)$$

From a heuristic point of view, it is useful to take the limit as $h \downarrow 0$ in (5.2). We obtain

$$\frac{d}{ds}|p(s)|^2 \leq \frac{2}{\delta(s)}|p(s)|^2(1 - |p(s)|^2). \quad (5.4)$$

Such an inequality implies that, if $|p(t_0)| < 1$, then $|p(s)| < 1$ for all $s > t_0$. However, such a reasoning is only formal, because we cannot say anything about the differentiability of $p(\cdot)$. It is interesting to observe that the crucial constant 1, in the expression $1 - |p(s)|^2$ above, arises from the previous computations as $K/2$, where $K = 2$ is the semiconcavity constant of $d_{\partial\Omega}^2$. Although the above argument is not rigorous, it suggests that $\gamma_h(\cdot)$ can be estimated by a suitable adaptation of the separation of variables procedure which could be used to integrate (5.4). We refer the reader to [3] for the detailed resoning. $\qquad\qquad\square$

6. The Riemannian case

Let us now consider a complete Riemannian manifold M, possibly noncompact. For any $x \in M$, we denote by T_xM the tangent space to M at x and by T_x^*M the cotangent space. For simplicity, we use the same symbol $\langle \cdot, \cdot \rangle$ to denote the scalar product of two vectors of T_xM, or of two elements of T_x^*M, or the pairing of a form in T_x^*M and a vector in T_xM. The Riemannian distance between two points $x, y \in M$ will be denoted by $d(x,y)$. If u is a function on a Riemannian manifold M, we denote by $Du(x) \in T_xM$ its gradient and by $du(x) \in T_x^*M$ its differential. In addition, we denote by D^2u the hessian of u, interpreted as a linear operator from T_xM to itself, as for example in [13, Ch. 14].

If C is a nonempty closed subset of M, we denote by $d_C(x)$ the *distance function* from C, defined as

$$d_C(x) = \min_{y \in C} d(y, x). \quad (6.1)$$

In order to extend the techniques of the previous section, we first need to recall some basic properties of parallel transport and geodesic curves. On M, there is a canonical notion of derivative of a vector field, called covariant derivative. Using this definition, a vector field is called *parallel* along a curve γ if its derivative in direction $\gamma'(t)$ is zero for all t. If we have a curve $\gamma : [a, b] \to M$ and a vector $v \in T_{\gamma(a)}M$, there is a unique vector field $v(t) \in T_{\gamma(t)}M$, with $t \in [a, b]$, which is parallel along γ; such a field $v(t)$ is called the *parallel transport* of v along $\gamma(t)$. Parallel transport preserves the scalar product and therefore gives an isometry between the tangent spaces at different points. The *geodesics* on M can be defined equivalently as the curves γ such that the speed $\gamma'(t)$ is parallel along the curve γ itself or as the curves which are stationary for the energy functional. Geodesics have constant speed and are curves of minimal length between two endpoints if these points are close enough to each other.

Given a point $x \in M$, we denote by $\exp_x(\cdot)$ the exponential map at x. We recall that, given a tangent vector $v \in T_xM$, $\exp_x(v)$ is the point reached at $t = 1$ by the geodesic $\gamma(t)$ starting with $\gamma(0) = x$ and $\gamma'(0) = v$. If f is a smooth function and $df(x) \in T_x^*M$ is its differential at x, we have

$$f(\exp_x(v)) - f(x) = \langle df(x), v \rangle + o(|v|), \qquad v \in T_xM, \quad v \to 0.$$

Let us now consider a function $u : M \to \mathbb{R}$ not necessarily smooth. We say that $p \in T_x^*M$ belongs to $d^+u(x)$, the *superdifferential* of u at x, if

$$u(\exp_x(v)) - u(x) \leq \langle p, v \rangle + o(|v|), \qquad v \in T_xM, \quad v \to 0.$$

This is equivalent to saying that there exists a smooth function f touching u from above at x such that $df(x) = p$. It is easy to see that, if $p \in d^+u(x)$ and if $\gamma : [-a, a] \to M$ is any smooth curve such that $\gamma(0) = x$ (not necessarily a geodesic), then

$$\limsup_{h \to 0} \frac{u(\gamma(h)) - u(\gamma(0))}{h} \leq \langle p, \gamma'(0) \rangle. \tag{6.2}$$

We recall that a subset $U \subset M$ is called convex if any distance minimizing geodesic between two points in U is contained in U. The notion of semiconcavity can be extended to Riemannian manifolds as follows.

Definition 6.1. A function $u : U \to \mathbb{R}$, with $U \subset M$ convex, is called *semiconcave* in U with constant K if, for every geodesic $\gamma : [0, 1] \to U$ and $t \in [0, 1]$, we have

$$(1 - t)u(\gamma(0)) + tu(\gamma(1)) - u(\gamma(t)) \leq t(1 - t) \, K \, \frac{d(\gamma(0), \gamma(1))^2}{2} \, . \tag{6.3}$$

A detailed exposition of the basic properties of semiconcave functions on a manifold is given in [13]. Notice that, in such a reference, functions satisfying (6.3) are called "semiconcave with modulus $\omega(t) = Kt^2/2$".

It can be checked (see Proposition 10.12 and inequality (10.14) in [13]) that, if u is semiconcave with constant K, then its superdifferential is nonempty at each point. In addition, any $p \in d^+u(x)$ satisfies

$$u(\exp_x(v)) - u(x) \leq \langle p, v \rangle + K \frac{|v|^2}{2}$$

for all $x \in U, v \in T_xM$ such that $\exp_x(v) \in U$.

Denote by $\gamma(\cdot)$ the geodesic $\gamma(t) = \exp_x(tv)$ starting at x with speed v. If we set $y = \gamma(1)$ and $w = \gamma'(1) \in T_y M$, we have $x = \exp_y(-w)$. Thus, if $q \in d^+u(y)$, we find $u(x) - u(y) \leq -\langle q, w \rangle + K|w|^2/2$. Since $|w| = d(x, y) = |v|$, we can sum up with the previous inequality to obtain

$$\langle q, w \rangle - \langle p, v \rangle \leq K|v|^2. \tag{6.4}$$

Denote by $\Pi : T_x M \to T_y M$ the parallel transport along the geodesic γ. Since parallel transport preserves the scalar product and γ is a geodesic, we have $w = \Pi v$ and $\langle q, w \rangle = \langle \Pi^{-1}(q), v \rangle$. We conclude that the above inequality can be rewritten as

$$\langle \Pi^{-1}(q) - p, v \rangle \leq K|v|^2 \tag{6.5}$$

for all $v \in T_x M$ with $\exp_x(v) \in U$, and any $p \in d^+u(x)$ and $q \in d^+u(\exp_x(v))$.

It is well known that the properties of the hessian of the distance function in a Riemannian manifold are closely related with the curvature of the manifold. Roughly speaking, positive curvature decreases the hessian of the distance function (i.e., gives a "stronger" semiconcavity), while negative curvature increases it. In particular, it can be proved that the square of the distance function is semiconcave with constant 2 only if the manifold has nonnegative sectional curvature. Even in the case when the curvature has arbitrary sign, however, it turns out that we can replace the square by another function of the distance which enjoys the properties we need for our application. The crucial result for our purposes is the following (see [3] for a proof).

Theorem 6.2. *Let M be a Riemannian manifold and let $\Omega \subset M$ be any open set (not necessarily smooth). Suppose that all sectional curvatures κ at any point of Ω satisfy $\kappa \geq -\alpha^2$ for some $\alpha > 0$ and define $v(x) = \cosh(\alpha d_{\partial\Omega}(x))$. Then, given any convex compact set $C \subset \Omega$, the function v is semiconcave on C with constant $K = \alpha^2 \max_C v(x)$.*

Remark 6.3. We mention that, if the infimum of the sectional curvature is zero or positive, then it is possible to use functions different from the hyperbolic cosine which give a sharper semiconcavity estimate (e.g. the square of the distance in the euclidean case); the result of Theorem 6.2, however, suffices for the purposes of this paper. This theorem also imples that the distance function itself is locally semiconcave in Ω. However, the semiconcavity constant in general becomes unbounded as $\partial\Omega$ is approached.

We now proceed to show how Theorem 4.1 and Corollary 4.3 have been extended to manifolds in [3].

Theorem 6.4. *For a given open bounded subset $\Omega \subset M$, let us set $u(x) = d_{\partial\Omega}(x)$ for $x \in \Omega$. For every $x_0 \in \Omega$ there exists a unique lipschitz continuous arc $\gamma : [0, +\infty[\to \Omega$ such that*

$$\gamma'(t) \in d^+u(\gamma(t)) \quad t \in [0, +\infty[\ a.e. \qquad \gamma(0) = x_0. \tag{6.6}$$

The arc γ satisfies properties analogous to the ones of Theorem 4.1 and Corollary 4.3 in the euclidean case. In particular, the right derivative $\gamma'_+(t)$

exists for every $t \geq 0$, is continuous from the right and satisfies (6.6) everywhere. The derivative of u along γ satisfies

$$\frac{d}{dt^+}u(\gamma(t)) = |\gamma'_+(t)|^2, \qquad t \in [0, +\infty[. \qquad (6.7)$$

Moreover, for any $t_0 \geq 0$ such that $\gamma(t_0) \in \Sigma(u)$ there exists $\sigma = \sigma(t_0) > 0$ such that $\gamma(t) \in \Sigma(u)$ for all $t \in [t_0, t_0 + \sigma[$.

Notice that, since $\gamma'(t) \in T_{\gamma(t)}M$ and $d^+u(\gamma(t)) \subset T^*_{\gamma(t)}M$, in (6.6) the two spaces are identified via the canonical isomorphism.

Proof. The result can be easily deduced from the euclidean case by using a local coordinate chart. In fact, if $\phi : U \to M$ is a local chart around x_0, where $U \subset \mathbb{R}^n$, and $G(x)$ is the matrix associated to the scalar product on T_xM in the chart ϕ, then it is easy to see that the function $\bar{u} := u \circ \phi : U \to \mathbb{R}$ satisfies

$$\sum_{i,j=1}^n g^{ij}(x)\frac{\partial \bar{u}}{\partial x_i}\frac{\partial \bar{u}}{\partial x_j} = 1,$$

where $g^{ij}(x)$ are the entries of the inverse matrix $G^{-1}(x)$. Thus, the assertions of the theorem follow from the corresponding ones of Theorem 4.1. We observe, in particular, that the equation satisfied by the generalized characteristics can be written in local coordinates as

$$\gamma'(t) \in G^{-1}(\gamma(t))D^+\bar{u}(\gamma(t)), \qquad (6.8)$$

where $D^+\bar{u}$ is the euclidean superdifferential of $\bar{u}$. Finally, the property that $\gamma(t)$ can be defined for $t \in [0, +\infty)$ is obtained by the same argument of Corollary 4.3. □

Following [3], we now explain how Theorem 5.1 extends to manifolds.

Theorem 6.5. *Let M be any smooth complete Riemannian manifold, let $\Omega \subset M$ any bounded open set, let $u(\cdot) = d_{\partial\Omega}(\cdot)$ and let $\gamma(\cdot)$ be the arc of Theorem 6.4. If $\gamma(t_0) \in \Sigma(u)$ for some $t_0 \geq 0$ then $\gamma(t) \in \Sigma(u)$ for all $t \in [t_0, +\infty[$.*

Proof. Since Ω is bounded, we can find a finite value $\alpha > 0$ such that the sectional curvature is everywhere greater than $-\alpha^2$ on Ω and Theorem 6.2 can be applied.

Let $x_0 \in \Omega$ and let $\gamma(t)$ be the solution of the differential inclusion (6.6). Let us set

$$p(t) = \gamma'_+(t), \qquad \delta(t) = u(\gamma(t)).$$

We now suppose that $\gamma(t_0) \in \Sigma(u)$. We then fix any $s \geq t_0$, and consider $t > s$ close enough to s so that $\gamma(s)$ and $\gamma(t)$ both belong to a neighborhood where any two points are connected by a unique minimal geodesic.

Let us call $v(s,t)$ the vector in $T_{\gamma(s)}M$ such that $\gamma(t) = \exp_{\gamma(s)}(v(s,t))$. Also, we denote by $\Pi_{s,t} : T_{\gamma(s)}M \to T_{\gamma(t)}M$ the isometry induced by the parallel transport along the geodesic connecting $\gamma(s)$ to $\gamma(t)$, and we set $\Pi_{t,s} = \Pi_{s,t}^{-1}$ to denote the inverse map, which is associated to the same geodesic with the opposite direction.

Using Gauss Lemma (see e.g. Lemma 3.3.5 in [10]), we obtain that

$$\frac{\partial}{\partial t^+}\frac{d(\gamma(s),\gamma(t))^2}{2} = \langle \Pi_{s,t}v(s,t),\gamma_+'(t)\rangle = \langle v(s,t),\Pi_{t,s}p(t)\rangle,$$

and, similarly,

$$\frac{\partial}{\partial s^+}\frac{d(\gamma(s),\gamma(t))^2}{2} = \langle -v(s,t),p(s)\rangle.$$

It follows, for $h > 0$ small enough,

$$\frac{d}{ds^+}\frac{d(\gamma(s),\gamma(s+h))^2}{2} = \langle \Pi_{s+h,s}p(s+h) - p(s), v(s,s+h)\rangle. \quad (6.9)$$

Let us now set $\phi(\tau) = \cosh(\alpha\tau)$, for $\tau \in \mathbb{R}$. Then, Theorem 6.2 gives a semiconcavity estimate for the function $\phi \circ u$. Let us also observe that, by the definition of superdifferential,

$$q \in d^+u(x) \iff \phi'(u(x))\, q \in d^+(\phi \circ u)(x). \quad (6.10)$$

Let us denote by $C \subset M$ the spherical neighbourhood of radius h centered at $\gamma(s)$. If h is sufficiently small, C is a convex set which includes the point $\gamma(s+h)$, and in addition $\sup_C u(x) \le u(\gamma(s))+h$. By Theorem 6.2, we have that $(\phi\circ u)(x)$ is semiconcave on C with constant given by $K = \alpha^2\phi(\delta(s)+h)$. Therefore, using (6.5), (6.6), (6.10) and the property that $\alpha^2\phi = \phi''$, we find

$$\langle \phi'(\delta(s+h))\,\Pi_{s+h,s}p(s+h) - \phi'(\delta(s))\,p(s), v(s,s+h)\rangle$$
$$\le \phi''(\delta(s)+h)\,d(\gamma(s),\gamma(s+h))^2.$$

We rewrite the above inequality as

$$\phi'(\delta(s+h))\,\langle \Pi_{s+h,s}p(s+h) - p(s), v(s,s+h)\rangle$$
$$\le \phi''(\delta(s)+h)\,d(\gamma(s),\gamma(s+h))^2 \quad (6.11)$$
$$-[\phi'(\delta(s+h)) - \phi'(\delta(s))]\,\langle p(s), v(s,s+h)\rangle.$$

Let us set

$$v_h(s) = \frac{v(s,s+h)}{h}, \qquad \delta_h(s) = \frac{\delta(s+h) - \delta(s)}{h}.$$

Dividing inequality (6.11) by h^2, we obtain, using formula (6.9) and the fact that $d(\gamma(s),\gamma(s+h)) = |v(s,s+h)| = h|v_h(s)|$,

$$\frac{\phi'(\delta(s+h))}{2}\frac{d}{ds^+}|v_h'(s)|^2 \le \phi''(\delta(s)+h)|v_h(s)|^2$$

$$(6.12)$$

$$-\frac{\phi'(\delta(s+h)) - \phi'(\delta(s))}{h}\langle p(s), v_h(s)\rangle.$$

We rewrite for simplicity this inequality as

$$\frac{d}{ds^+}|v_h(s)|^2 \le \psi_h(s), \quad (6.13)$$

where we have set

$$\psi_h(s) \;=\; \frac{2}{\phi'(\delta(s+h))}\left[\phi''(\delta(s)+h)|v_h(s)|^2 \right.$$

$$\left. -\frac{\phi'(\delta(s+h))-\phi'(\delta(s))}{h}\langle p(s),v_h(s)\rangle\right]. \tag{6.14}$$

Now, we observe that

$$\lim_{h\downarrow 0} v_h(s) = \gamma'_+(s) = p(s), \tag{6.15}$$

for a.e. s, which can be easily checked for instance by using local coordinates around $\gamma(s)$. More formally, the above relation follows from the fact that $\lim_{h\to 0} v_h(s) = \frac{d}{dh}\big|_{h=0} v(s,s+h)$, that $\exp_{\gamma(s)} v(s,s+h) = \gamma(s+h)$ and that the differential of exp at zero is the identity (see the proof of Proposition 18 in Ch. 5 of [12]); therefore, the derivatives $\frac{d}{dh}\big|_{h=0} v(s,s+h)$ and $\frac{d}{dh}\big|_{h=0} \gamma(s+h)$ coincide.

In addition, using Theorem 6.4, we find that for a.e. s

$$\lim_{h\to 0} \frac{\phi'(\delta(s+h))-\phi'(\delta(s))}{h}$$

$$= \;\phi''(\delta(s))\lim_{h\to 0}\frac{\delta(s+h)-\delta(s)}{h} = \phi''(\delta(s))|p(s)|^2.$$

Therefore, we see that the function $\psi_h(s)$ defined in (6.14) is uniformly bounded for $h>0$ small and s varying in a bounded interval. Moreover, for all $s\geq 0$,

$$\lim_{h\downarrow 0}\psi_h(s) \;=\; 2\frac{\phi''(\delta(s))}{\phi'(\delta(s))}\left(|p(s)|^2-|p(s)|^4\right)$$

$$= \;\frac{2\alpha}{\tanh(\alpha\,\delta(s))}\left(|p(s)|^2-|p(s)|^4\right). \tag{6.16}$$

From this point on, the proof proceeds as in the euclidean case. □

7. Homotopy equivalence

In this section, we discuss and application of the invariance result described above. We begin by recalling the well known notion of homotopy equivalence.

Definition 7.1. Let X and Y be two topological spaces and let

$$f : X \to Y \quad\text{and}\quad g : X \to Y$$

be two continuous maps. We say that f and g are *homotopic* if there exists a continuous map $H : X \times [0,1] \to Y$, called homotopy, such that

$$H(0,\cdot) = f(\cdot) \quad\text{and}\quad H(1,\cdot) = g(\cdot).$$

Furthermore, we say that X and Y have the same *homotopy type* if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are homotopic to the identity on X and Y, respectively.

The following result is a direct consequence of Definition 7.1.

Lemma 7.2. *Let $Y \subset X$. If there exists a continuous map*

$$H : X \times [0,1] \to X$$

such that

    (a) $H(x,0) = x$, *for every* $x \in X$,
    (b) $H(x,1) \in Y$, *for every* $x \in X$, *and*
    (c) $H(x,t) \in Y$, *for every* $(x,t) \in Y \times [0,1]$,

then X and Y have the same homotopy type.

We are now ready to explain how Theorem 6.5 can be used to obtain the following homotopy equivalence result which holds under no regularity assumption on $\partial\Omega$.

Theorem 7.3. *Let Ω be a bounded open subset of a smooth Riemannian manifold M. Then Ω has the same homotopy type as $\Sigma(d_{\partial\Omega})$.*

Proof. In view of Lemma 7.2 it suffices to construct a continuous map

$$H : \Omega \times [0,1] \to \Omega$$

satisfying conditions $(a), (b)$, and (c) above with $Y = \Sigma(d_{\partial\Omega})$.

For any $x \in \Omega$, let $\gamma_x(\cdot)$ be the generalized characteristic starting at x. We claim that

$$\exists\, T > 0 \ : \ \forall x \in \Omega \quad \gamma_x(T) \in \Sigma(d_{\partial\Omega}). \tag{7.1}$$

Indeed, set $T = 2\operatorname{diam}(\Omega)$ and let $x \in \Omega$. Arguing by contradiction, suppose $\gamma_x(T) \notin \Sigma(d_{\partial\Omega})$. Then, in light of Theorem 6.5, $\gamma_x(t) \notin \Sigma(d_{\partial\Omega})$ for all $t \in [0,T]$. So, $|\gamma_x'(t)| = 1$ for every $t \in [0,T]$. Hence, owing to (6.7), we have

$$d_{\partial\Omega}(\gamma_x(T)) = d_{\partial\Omega}(x) + \int_0^T |\gamma_x'(t)|^2 \, dt = d_{\partial\Omega}(x) + T.$$

Then,

$$2\operatorname{diam}(\Omega) = T = d_{\partial\Omega}(\gamma_x(T)) - d_{\partial\Omega}(x) \leq \operatorname{diam}(\Omega).$$

The above contradiction shows that (7.1) holds true. Next, define

$$H(x,t) = \gamma_x(tT) \qquad (x,t) \in \Omega \times [0,1].$$

We point out that H is a locally Lipschitz continuous map in view of Lemma 4.2. Moreover, on account of (7.1) and Theorem 6.5, H satisfies conditions (b) and (c) of Lemma 7.2. This completes the proof. $\qquad\square$

8. Singularities of the minimum time function

The time optimal control problem we discuss below is hard to treat arguing just inside the euclidean framework. The analysis we have developed for Riemannian manifolds, however, turns out to be useful for this purpose.

Let $F : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ be a smooth function such that $\det F(x) \neq 0$ for all x and let $\Omega \subset \mathbb{R}^n$ be a bounded open set. For any given $x \in \Omega$ we consider the control system

$$\begin{cases} y'(t) = F(y(t))\alpha(t), \\ y(0) = x, \end{cases} \tag{8.1}$$

where $\alpha : [0, +\infty) \to B_1(0)$ is a measurable function called the control. We denote by $y(\cdot; x, \alpha)$ the trajectory of (8.1), and we define the exit time from Ω of the trajectory as

$$\tau(x, \alpha) = \inf\{t > 0 \ : \ y(t; x, \alpha) \in \partial\Omega\} \in (0, +\infty].$$

The *minimum time function* is defined as

$$T(x) = \inf_{\alpha} \tau(x, \alpha), \qquad x \in \Omega.$$

Under our hypotheses, it is well known that the infimum is attained and that $T(\cdot)$ is a semiconcave solution of the Hamilton–Jacobi–Bellman equation

$$H(x, DT(x)) = 1, \qquad x \in \Omega, \tag{8.2}$$

where H is defined as

$$H(x, p) = \langle F(x)F^*(x)p, p\rangle$$

with F^* the transpose matrix.

Let us consider the Riemannian metric g on $\mathbb{R}^n$ induced by the scalar product with matrix

$$G(x) := (F^*)^{-1}(x)F^{-1}(x).$$

Then, using the subscripts $_e$ and $_g$ to distinguish between the euclidean and Riemannian metrics, we have

$$|v|_g \leq 1 \iff \langle G(x)v, v\rangle_e \leq 1 \iff |F^{-1}(x)v|_e \leq 1,$$

which shows that an arc $y(\cdot)$ is an admissible trajectory for the control system (8.1) if and only $|y'(t)|_g \leq 1$. It follows that $T(x) \equiv d_{\partial\Omega}(x)$, where the distance function $d_{\partial\Omega}$ is taken with respect to the Riemannian metric g.

Thus, the previous analysis can be applied to the singular set $\Sigma(T)$ of the minimum time function. In particular, recalling also (6.8), we obtain that $\Sigma(T)$ is invariant under the flow induced by the differential inclusion

$$\gamma'(t) \in G^{-1}(\gamma(t))D^+T(\gamma(t)).$$

The above inclusion, up to a factor 2, can be written equivalently as

$$\gamma'(t) \in D_pH(\gamma(t), D^+T(\gamma(t))),$$

which is the equation of the characteristics associated with (8.2). Therefore, Theorem 7.3 yields the following.

Corollary 8.1. Ω *and* $\Sigma(T)$ *have the same homotopy type.*

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