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Andrew Lawrie

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Institut des hautes études scientifiques
Le Bois-Marie • Route de Chartres
F-91440 BURES-SUR-YVETTE
http://www.ihes.fr/

Centre de mathématiques Laurent Schwartz
UMR 7640 CNRS/École polytechnique
F-91128 PALAISEAU CEDEX
http://www.math.polytechnique.fr/

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STABLE SOLITON RESOLUTION FOR EQUIVARIANT WAVE MAPS EXTERIOR TO A BALL

ANDREW LAWRIE

Abstract. In this report we review the proof of the stable soliton resolution conjecture for equivariant wave maps exterior to a ball in $\mathbb{R}^3$ and taking values in the 3-sphere. This is joint work with Carlos Kenig, Baoping Liu, and Wilhelm Schlag.

1. Introduction

This report describes recent work of the author with Wilhelm Schlag [15], Carlos Kenig and W. Schlag [12], and C. Kenig, Baoping Liu, and W. Schlag [10, 11]. We establish stable soliton resolution for equivariant wave maps

$$U : \mathbb{R}^{1+3}_{t,x} \backslash (\mathbb{R} \times B(0,1)) \to \mathbb{S}^3,$$

with a Dirichlet condition on the boundary of the unit ball $B(0,1) \subset \mathbb{R}^3$ and initial data of finite energy.

To be precise, consider the Lagrangian action

$$\mathcal{L}(U, \partial_t U) = \int_{\mathbb{R}^{1+3}_{t,x} \backslash (\mathbb{R} \times B(0,1))} \frac{1}{2} \Big(- |\partial_t U|_g^2 + \sum_{j=1}^3 |\partial_x U|_g^2 \Big) dt dx,$$

where g is the round metric on $\mathbb{S}^3$, and where we only consider functions for which the boundary of the unit cylinder $\mathbb{R} \times B(0,1)$ gets mapped to a fixed point on the 3-sphere, i.e, $U(t, \partial B(0,1)) = N$, where $N \in \mathbb{S}^3$ is say, the north pole. Under the usual ℓ-equivariant assumption, for $\ell \in \mathbb{N}$, the Euler-Lagrange equation associated with this action reduces to an equation for the azimuth angle ψ measured from the north pole on $\mathbb{S}^3$, namely

$$\psi_{tt} - \psi_{rr} - \frac{2}{r}\psi_r + \frac{\ell(\ell+1)}{2r^2} \sin(2\psi) = 0.$$

The Dirichlet boundary condition then becomes $\psi(t,1) = 0$ for all $t \in \mathbb{R}$ and thus the Cauchy problem under consideration is,

$$\begin{aligned} &\psi_{tt} - \psi_{rr} - \frac{2}{r}\psi_r + \frac{\ell(\ell+1)}{2r^2} \sin(2\psi) = 0, \quad r \geq 1, \\ &\psi(0,r) = \psi_0(r), \ \psi_t(0,r) = \psi_1(r), \quad \text{and} \quad \psi(t,1) = 0, \ \forall t, \end{aligned} \tag{1.1}$$

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and solutions $\vec{\psi}(t) := (\psi(t), \psi_t(t))$ to (1.1) will be referred to as ℓ-equivariant exterior wave maps. The conserved energy for (1.1) is given by

$$\mathcal{E}_\ell(\psi, \psi_t) = \int_1^\infty \frac{1}{2}\left(\psi_t^2 + \psi_r^2 + \frac{\ell(\ell+1)\sin^2\psi}{r^2}\right) r^2\, dr.$$

A simple analysis of the last term in the integrand above yields topological information on the wave map if we require the energy to be finite. Indeed, smooth $\vec{\psi}(t,r)$ on a time interval $I = (t_0, t_1)$ with finite energy must satisfy $\psi(t,\infty) = n\pi, \forall t \in I$, where $n \in \mathbb{Z}$. Given the fact that ψ measures the azimuth angle from the north pole, and $\psi(t,1) = 0$ for all $t \in I$, this means that the integer $|n|$ measures the *topological degree* of the map. Note that the case $n \geq 0$ covers the entire range $n \in \mathbb{Z}$ by the symmetry $\psi \mapsto -\psi$.

In what follows we will refer to $n \geq 0$ as the *degree* of the map, and we will denote by $\mathcal{E}_{\ell,n}$ the connected component of the metric space of all initial data (ψ_0, ψ_1) with finite energy, obeying the boundary condition $\psi_0(1) = 0$ and of degree n, i.e.,

$$\mathcal{E}_{\ell,n} = \Big\{(\psi_0, \psi_1) \mid \mathcal{E}_\ell(\psi_0, \psi_1) < \infty,\ \psi_0(1) = 0,\ \lim_{r\to+\infty} \psi_0(r) = n\pi\Big\}.$$

There are several appealing features of this model that make it an ideal setting in which to study soliton resolution. First, by removing the unit ball in $\mathbb{R}^3$ and imposing the Dirichlet boundary condition, we break the scaling symmetry. This removes the super-criticality at $r = 0$ of the $3d$ wave maps problem and effectively renders the problem subcritical relative to the energy. Global well-posedness in the energy space is then an immediate consequence. Second, the removal of the unit ball also gives rise to an infinite family of stationary solutions $(Q_{\ell,n}(r), 0)$, indexed by their topological degree $n \in \mathbb{N}$; see Section 2.1 for more. In particular, the solution $(Q_{\ell,n}(r), 0)$ satisfies

$$Q_{\ell,n}(1) = 0, \quad \lim_{r\to\infty} Q_{\ell,n}(r) = n\pi.$$

Moreover, $(Q_{\ell,n}(r), 0)$ minimizes the energy in $\mathcal{E}_{\ell,n}$ and is the unique stationary solution in this degree class. Both of these features are in stark contrast to the same equation on $\mathbb{R}^{1+3}$ which is super-critical relative to the energy, is known to develop singularities in finite time, and has no nontrivial finite energy stationary solutions, see for example Shatah [16], and Shatah and Struwe [17].

For a fixed equivariance class $\ell \in \mathbb{N}$, the natural topology in which to place a degree $n = 0$ solution is the *energy space* $\mathcal{H}_{\ell,0} = \dot{H}_0^1 \times L^2(\mathbb{R}^3_*)$ with norm

$$\|\vec{\psi}\|^2_{\mathcal{H}_0} := \int_1^\infty (\psi_t^2 + \psi_r^2)\, r^2\, dr, \qquad \vec{\psi} = (\psi, \psi_t). \tag{1.2}$$

Here $\mathbb{R}^3_* := \mathbb{R}^3 \backslash B(0,1)$, and $\dot{H}_0^1(\mathbb{R}^3_*)$ is the completion of smooth functions on $\mathbb{R}^3_*$ with compact support under the first norm on the right-hand side of (1.2). For $n \geq 1$, denote by $\mathcal{H}_n := \mathcal{E}_{\ell,n} - (Q_{\ell,n}, 0)$ with "norm"

$$\|\vec{\psi}\|_{\mathcal{H}_n} := \|\vec{\psi} - (Q_{\ell,n}, 0)\|_{\mathcal{H}_0}.$$

We remark that the boundary condition at $r = \infty$ is now $\vec{\psi} - (Q_{\ell,n}, 0) \to 0$ as $r \to \infty$ with this notation.

The exterior model was first introduced in the physics literature in [2], as an easier alternative to the Skyrmion equation. Recently, (1.1) was proposed by Bizon, Chmaj, and Maliborski in [3] as a model to study the problem of relaxation to the ground states given by various equivariant harmonic maps. Both [2, 3] stress the analogy of the stationary equation with that of the damped pendulum by demonstrating the existence and uniqueness of the ground state harmonic maps via a phase-plane analysis. The numerical simulations in [3] indicate that for each equivariance class $\ell \geq 1$, and each topological class $n \geq 0$, every solution scatters to the unique harmonic map $Q_{\ell,n}$ that lies in $\mathcal{E}_{\ell,n}$, giving evidence that the soliton resolution conjecture holds true in this exterior model. This conjecture was verified for 1-equivariant (or co-rotational) exterior wave maps with topological degree $n = 0$ by the author, and Schlag in [15], for $\ell = 1$ and all topological degrees $n \geq 0$ by Kenig, the author, and Schlag [12], and finally for all remaining equivaraince classes $\ell \geq 2$ by Kenig, the author, Liu, and Schlag in [11].

The main result is as follows.

Theorem 1.1 (Stable Soliton Resolution [15, 12, 11])**.** *Let $\ell \geq 1$ and $n \geq 0$ be arbitrary integers. For any smooth energy data in $\mathcal{E}_{\ell,n}$ the corresponding wave map $\vec{\psi}(t)$ is globally regular and scatters to the harmonic map $(Q_{\ell,n}, 0)$ as $t \to \pm\infty$.*

Here "scattering to the harmonic map $(Q_{\ell,n}, 0)$" means that for each solution $\vec{\psi}(t)$ to (1.1) we can find solutions $\vec{\varphi}_L^{\pm}$ to the linear equation

$$\varphi_{tt} - \varphi_{rr} - \frac{2}{r}\varphi_r + \frac{\ell(\ell+1)}{r^2}\varphi = 0, \quad r \geq 1, \quad \varphi(t,1) = 0.$$

so that

$$\vec{\psi}(t) = (Q_{\ell,n}, 0) + \vec{\varphi}_L^{\pm}(t) + o_{\mathcal{H}_0}(1), \quad \text{as} \quad t \to \pm\infty.$$

In other words, Theorem 1.1 is a verification of the *soliton resolution conjecture* for (1.1).

We emphasize that only the scattering statement in Theorem 1.1 is difficult to prove. We employ the concentration compactness/rigidity method developed by the Kenig and Merle in [13, 14]. After proving a suitable small data/pertubative theory, and carrying out the concentration compactness procedure, one reduces the proof of Theorem 1.1 to a rigidity argument, where the goal is to show that any solution to (1.1) with a pre-compact trajectory in the energy space must be a harmonic map. To prove this we use a version of the 'channels of energy' argument introduced by Duyckaerts, Kenig, and Merle in [7, 8]. The proof relies crucially on *exterior energy estimates* for the free radial wave equation in dimension $d = 2\ell + 3$ where ℓ is the equivariance class. These estimates were established in [4] for dimension $d = 3$, in [12] for dimension $d = 5$, and [10] for *all odd dimensions*; see Theorem 4.1.

2. Harmonic maps, high dimensional reduction

We briefly cover a few basic properties of the harmonic maps $Q_{\ell,n}$, and reduce the ℓ-equivariant wave map problem to an exterior semi-linear wave equation in $\mathbb{R}^d_* := \mathbb{R}^d \backslash B(0,1)$, with a Dirichlet boundary condition at $r = 1$, and with $d := 2\ell + 3$.

2.1. **Exterior harmonic maps.** In each energy class $\mathcal{E}_{\ell,n}$, there is a unique finite energy exterior harmonic map, which is a minimizer of the energy $\mathcal{E}_{\ell,n}$ and also a static solution to (1.1), i.e.

$$\begin{aligned} &\partial_{rr} Q_{\ell,n} + \frac{2}{r}\partial_r Q_{\ell,n} = \frac{\ell(\ell+1)}{2r^2}\sin(2Q_{\ell,n}) \\ &Q_{\ell,n}(1) = 0, \quad \lim_{r\to\infty} Q_{\ell,n}(r) = n\pi \end{aligned} \tag{2.1}$$

As in [12] we change variables, setting $s := \log r$, and $\phi(s) := Q_{\ell,n}(r)$. The equation (2.1) becomes

$$\phi_{ss} + \phi_s = \frac{\ell(\ell+1)}{2}\sin(2\phi), \qquad \phi(0) = 0, \qquad \phi(\infty) = n\pi, \tag{2.2}$$

Noting that (2.2) can be written as an autonomous system in the plane, we can perform a standard analysis of the phase portrait to deduce the following result.

Lemma 2.1. *For all $\alpha \in \mathbb{R}$, there exists a unique solution $Q_{\ell,\alpha} \in \dot{H}^1(\mathbb{R}^3_*)$ to* (2.1) *with*

$$Q_{\ell,\alpha}(r) = n\pi - \frac{\alpha}{r^{\ell+1}} + O(r^{-3(\ell+1)}) \qquad \textit{as } r \to \infty \tag{2.3}$$

The $O(\cdot)$ is uniquely determined by α and vanishes for $\alpha = 0$. Moreover, there exist a unique $\alpha_0 > 0$ such that $Q_{\alpha_0}(1) = 0$, we will denote it as $Q_{\ell,n}$.

2.2. **Reduction to an exterior wave equation in high dimensions.** At this point we fix an arbitrary equivariance class $\ell \geq 1$ and topological degree $n \geq 0$. We reduce (1.1) to a semi-linear equation in $\mathbb{R}^{2\ell+3}_*$. To perform this reduction, we first linearize (1.1) about the unique ℓ-harmonic map of degree n, $Q_{\ell,n}$. As we have fixed ℓ and n, we will simplify notation by writing $Q = Q_{\ell,n}$ and we note that when $n = 0$ we have $Q \equiv 0$.

For each solution $\vec{\psi}$ to (1.1) we define $\vec{\varphi}$ by

$$\vec{\psi} := (Q, 0) + \vec{\varphi}.$$

Using the equations for $\vec{\psi}$ and for Q we see that $\vec{\varphi}$ solves

$$\begin{aligned} &\varphi_{tt} - \varphi_{rr} - \frac{2}{r}\varphi_r + \frac{\ell(\ell+1)\cos(2Q)}{r^2}\varphi = Z(r,\varphi) \\ &\varphi(t,1) = 0, \quad \varphi(t,\infty) = 0 \qquad \forall t, \\ &\vec{\varphi}(0) = (\psi_0 - Q, \psi_1), \end{aligned} \tag{2.4}$$

where here

$$Z(r,\varphi) := \frac{\ell(\ell+1)}{2r^2}[2\varphi - \sin(2\varphi)]\cos(2Q) + (1 - \cos(2\varphi))\sin 2Q$$

The left-hand-side of (2.4) has more dispersion than a wave equation in $3d$ due to the strong repulsive potential

$$\frac{\ell(\ell+1)\cos(2Q)}{r^2} = \frac{\ell(\ell+1)}{r^2} + O(r^{-2\ell-4}) \quad \text{as} \quad r \to \infty$$

where we have used the asymptotic behavior of Q from (2.3) in the expansion above. Indeed, the coefficient $\ell(\ell+1)$ in front of the r^{-2} term indicates that we have the the same dispersion as a $d = 2\ell+3$-dimensional wave equation. This is made precise by the following standard reduction. We define $\vec{u}$ by setting $\varphi = r^\ell u$. Then $\vec{u}$ solves the following equation.

$$\begin{aligned} &u_{tt} - u_{rr} - \frac{2\ell+2}{r}u_r + V(r)u = \mathcal{N}(r,u), \qquad r \geq 1 \\ &u(t,1) = 0, \ \forall t \in \mathbb{R}, \quad \vec{u}(0) = (u_0, u_1) \end{aligned} \tag{2.5}$$

where

$$V(r) := \frac{\ell(\ell+1)(\cos 2Q - 1)}{r^2}, \quad \mathcal{N}(r,u) := F(r,u) + G(r,u), \tag{2.6}$$

$$F(r,u) := \frac{\ell(\ell+1)}{r^{\ell+2}} \sin^2(r^\ell u) \sin 2Q, \ \ G(r,u) := \frac{\ell(\ell+1)}{2r^{\ell+2}}(2r^\ell u - \sin(2r^\ell u)) \cos 2Q$$

The potential $V(r)$ is real-valued, radial, bounded, and smooth. Using the asymptotics of Q in (2.3), we see that V has the asymptotics

$$V(r) = O(r^{-2\ell-4}), \qquad \text{as } r \to \infty \tag{2.7}$$

For the nonlinearity $\mathcal{N} = F + G$ we have

$$|F(r,u)| \leq C_0 r^{-3}|u|^2, \quad |G(r,u)| \leq C_0 r^{2\ell-2}|u|^3 \tag{2.8}$$

The constant C_0 here depends only on $d = 2\ell + 3$ and Q.

We will consider radial initial data

$$(u_0, u_1) \in \mathcal{H} := \dot{H}^1_0 \times L^2(\mathbb{R}^d_*),$$

where

$$\|(u_0,u_1)\|^2_{\mathcal{H}} := \int_1^\infty [(\partial_r u_0(r))^2 + u_1(r)^2]\, r^{2\ell+2}\, dr$$

and $\dot{H}^1_0(\mathbb{R}^d_*)$ is the completion under the first norm on the right-hand side above of all smooth radial compactly supported functions on $\mathbb{R}^d_*$, with $d = 2\ell + 3$.

From now on, we will work exclusively in the "u-formulation", (2.5), rather than with the ℓ-equivariant wave map angle $\psi(t,r)$ as in (1.1). In fact, the Cauchy problem (1.1) with data $(\psi_0, \psi_1) \in \mathcal{E}_{\ell,n}$ is equivalent to the problem (2.5) with initial data

$$\mathcal{H} \ni (u_0, u_1) := \frac{1}{r^\ell}(\psi_0 - Q, \psi_1).$$

We thus prove the analogous version of Theorem 1.1 in the "u-formulation." It is clear from the definition of $\vec{u}(t)$ that Theorem 1.1 is true if and only if every solution $\vec{u}(t)$ to (2.5) scatters as $t \to \pm\infty$. Scattering here means that solutions to (2.5) approach free waves in $\mathbb{R} \times \mathbb{R}^d_*$ in the space $\mathcal{H}$. A free wave in this context is a solution to (2.5) with $V = \mathcal{N} = 0$. We prove the following equivalent reformulation of Theorem 1.1.

Theorem 2.2. *For any initial data $\vec{u}(0) \in \mathcal{H}$, there exist a unique, global-in-time solution $\vec{u}(t) \in \mathcal{H}$ to (2.5). Moreover $\vec{u}(t)$ scatters to free waves as $t \to \pm\infty$.*

3. Small data theory and Concentration compactness

The proof of Theorem 2.2, and hence of the equivalent statement Theorem 1.1, proceeds via the concentration compactness/ rigidity method introduced by the Kenig and Merle in [13, 14]. The argument can be divided into three separate steps, namely (1) a small data theory, i.e., a proof of Theorem 2.2 for initial data with small enough $\mathcal{H}$-norm; (2) a concentration compactness argument. If Theorem 2.2 fails, then there exists a critical element, which is a minimal non-scattering solution with a pre-compact trajectory in $\mathcal{H}$. The critical element is nonzero, by step (1). The main ingredient here is an analogue of the nonlinear profile decomposition of Bahouri and Gerard, [1], adapted to (2.5) along with a nonlinear perturbation theory; and finally (3) a rigidity argument. Here one shows that any solution with a pre-compact trajectory as in step (2) must be identically $\equiv 0$. This contradicts the existence of the critical element from step (2) and completes the proof.

In this section we outline steps (1) and (2) above.

3.1. **Small data theory.** The first order of business is to establish a small data theory for (2.5). The main ingredients here are Strichartz estimates for the linear inhomogeneous wave equation perturbed by the radial potential V. Indeed, consider

$$\begin{aligned} &u_{tt} - u_{rr} - \frac{d-1}{r}u_r + V(r)u = \mathcal{N}, \qquad r \geq 1 \\ &u(t,1) = 0, \ \forall t, \quad \vec{u}(0) = (u_0, u_1) \in \mathcal{H}. \end{aligned} \tag{3.1}$$

The conserved energy for (3.1) with $\mathcal{N} = 0$ is

$$\mathcal{E}_L(u, u_t) = \frac{1}{2}\int_1^\infty (u_t^2 + u_r^2 + V(r)u^2)\, r^{d-1}\, dr$$

As shown in [15, 12] this energy has an important positive definiteness property, namely,

$$\mathcal{E}_L(u, u_t) = \frac{1}{2}(\|u_t\|_2^2 + \langle Hu \mid u\rangle), \quad H = -\Delta + V$$

It is shown in [3, 15] that H is a nonnegative self-adjoint operator in $L^2(\mathbb{R}^d_*)$ (with a Dirichlet condition at $r = 1$), and moreover, that the threshold energy zero is regular; this means that if $Hf = 0$ where $f \in H^2 \cap \dot{H}^1_0$ then $f = 0$. It is standard to conclude from this spectral information that for some constants $0 < c_1 < c_2$,

$$c_1\|f\|^2_{\dot{H}^1_0} \leq \langle Hf \mid f\rangle \leq c_2\|f\|^2_{\dot{H}^1_0} \quad \forall\, f \in \dot{H}^1_0(\mathbb{R}^d_*)$$

In the sequel we will sometimes write $\|\vec{u}\|^2_{\mathcal{E}} := \mathcal{E}_L(\vec{u})$, which satisfies

$$\|\vec{u}\|_{\mathcal{E}} \simeq \|\vec{u}\|_{\mathcal{H}} \quad \forall \vec{u} \in \mathcal{H}(\mathbb{R}^d_*) \tag{3.2}$$

We call a triple (p, q, γ) admissible if

$$p > 2, q \geq 2, \quad \frac{1}{p} + \frac{d}{q} = \frac{d}{2} - \gamma, \quad \frac{1}{p} \leq \frac{d-1}{2}(\frac{1}{2} - \frac{1}{q})$$

Theorem 3.1 (Strichartz estimates [15])**.** *Let* $(p, q, \gamma), (r, s, \rho)$ *be admissible triples, then any solution* u *to* (3.1) *with radial initial data satisfies*

$$\||\nabla|^{-\gamma}\nabla u\|_{L^p_t L^q_x(\mathbb{R}^d_*)} \lesssim \|\vec{u}(0)\|_{\mathcal{H}} + \||\nabla|^{\rho}\mathbb{N}\|_{L^{r'}_t L^{s'}_x(\mathbb{R}^d_*)},$$

where here r', s' *are the conjugates of* r, s*.*

Remark 1. The case when potential $V = 0$ is proved in [9]. The case with V as in (2.6) can be proved by adapting the argument in [15, Proposition 5.1], which is performed for $\ell = 1$ to dimension $d = 2\ell + 3$. In [15] the proof is reduced to deducing localized energy estimates for (2.5) with V as in (2.6). The local energy estimates are proved using the distorted Fourier transform relative to the self-adjoint Schrödinger operator $H_V = -\Delta + V$ on $L^2(\mathbb{R}^d_*)$, and rely crucially on decay properties of the corresponding spectral measure. It is essential here H has no negative spectrum and that the edge of the continuous spectrum for H is neither an eigenvalue nor a resonance; see [15, Section 5] for more details.

A standard consequence of the Strichartz estimates is the following small data scattering theory. For a time interval I, we denote by $S(I)$ the space $S(I) := L_t^{\frac{d+2}{d-2}}(I; L_x^{2\frac{d+2}{d-2}}(\mathbb{R}^d_*))$ with norm

$$\|u\|_{S(I)} := \|u\|_{L_t^{\frac{d+2}{d-2}}(I; L_x^{2\frac{d+2}{d-2}}(\mathbb{R}^d_*))} \tag{3.3}$$

Theorem 3.2. *The exterior Cauchy problem for* (2.5) *is globally well-posed in* $\mathcal{H} := \dot{H}^1_0 \times L^2(\mathbb{R}^d_*)$. *Moreover, a solution* $\vec{u}$ *scatters as* $t \to \pm\infty$ *to free waves, i.e., solutions* $\vec{u}_L^{\pm} \in \mathcal{H}$ *of*

$$\Box u_L^{\pm} = 0, \ r \geq 1, \ u_L^{\pm}(t, 1) = 0, \ \forall t \geq 0 \tag{3.4}$$

if and only if

$$\|u\|_{S(\mathbb{R}_\pm)} < \infty,$$

where $\mathbb{R}_+ := [0, \infty)$ *and* $\mathbb{R}_- := (-\infty, 0]$. *In particular, there exists a constant* $\delta_0 > 0$ *small so that if* $\|\vec{u}(0)\|_{\mathcal{H}} < \delta_0$, *then* $\vec{u}$ *scatters to free waves as* $t \to \pm\infty$.

3.2. **Concentration compactness.** By the concentration compactness methodology in [13, 14], we can perform the following reduction: If Theorem 1.1, (and hence Theorem 2.2), fails, we can construct a *critical element*, which is a global non-scattering solution to (2.5) with minimal energy and has a pre-compact trajectory in $\mathcal{H}$. Indeed, following the argument given in [12, Proof of Proposition 3.6] we deduce the following result.

Proposition 3.3. *Suppose that Theorem 2.2 fails. Then there exists a nonzero, global solution* $\vec{u}_*(t) \in \mathcal{H}$ *to* (2.5), *such that the trajectory*

$$\mathcal{K} := \{\vec{u}_*(t) | t \in \mathbb{R}\}$$

is pre-compact in $\mathcal{H} = \dot{H}^1 \times L^2(\mathbb{R}^d_*)$. *We call* $\vec{u}_*(t)$ *a critical element.*

The key ingredients in the proof of Proposition 3.3 are a Bahouri-Gérard profile decomposition and a nonlinear perturbation theory, see [12, Lemma 3.4 and Lemma 3.5]. We omit the details and just formulate the concentration compactness principle relative to the linear wave equation with a potential, i.e., (2.5) with $\mathcal{N} = 0$. We note that any solution to (2.5) with $\mathcal{N} = 0$, which is in $S(\mathbb{R})$ must scatter to "free" waves.

Lemma 3.4 ([12, Lemma 3.4], [1])**.** *Let* $\{u_n\}$ *be a sequence of radial solutions to* (2.5) *with* $\mathcal{N} = 0$, *which are uniformly bounded in* $\mathcal{H} = \dot{H}^1_0 \times L^2(\mathbb{R}^d_*)$. *After passing to a subsequence, there exists a sequence of solutions* V_L^j *to* (2.5) *with* $\mathcal{N} = 0$, *which are bounded in* $\mathcal{H}$, *and sequences of times* $\{t_n^j\} \subset \mathbb{R}$ *such that for errors* $w_{n,L}^k$ *defined by*

$$u_n(t) = \sum_{1 \leq j < k} V_L^j(t - t_n^j) + w_{n,L}^k(t)$$

we have for any $j < k$,

$$\vec{w}_{n,L}^k(t_n^j) \rightharpoonup 0$$

weakly in $\mathcal{H}$ *as* $n \to \infty$, *as well as*

$$\lim_{n \to \infty} |t_n^j - t_n^k| = \infty$$

and the errors w_n^k *vanish asymptotically in the sense that*

$$\lim_{k \to \infty} \limsup_{n \to \infty} \|w_{n,L}^k\|_{(L_t^\infty L_x^p \cap S)(\mathbb{R} \times \mathbb{R}^d_*)} = 0 \quad \forall \ \frac{2d}{d-2} < p < \infty$$

Finally, one has orthogonality of the free energy with a potential,

$$\|\vec{u}_n\|_{\mathcal{E}}^2 = \sum_{1 \leq j < k} \|\vec{V}_L^j\|_{\mathcal{E}}^2 + \|\vec{\gamma}_n^k\|_{\mathcal{E}}^2 + o_n(1)$$

as $n \to \infty$.

4. CHANNELS OF ENERGY

It remains to show that a nontrivial compact trajectory as in Proposition 3.3 cannot exist. This is referred to as the “rigidity” step in the Kenig-Merle scheme. Previous incarnations of this part of the argument, e.g., [13, 14], relied on virial or Morawetz type identities, which are obtained by contracting the stress energy tensor with appropriate vector fields. However, such dynamical identities are extremely sensitive to the precise structure of the particular nonlinear wave equation under consideration, and do not easily generalize more complicated nonlinearities. An important breakthrough was made by Duyckaerts, Kenig, and Merle, [4, 6, 5, 7, 8], who developed an alternative approach called the ‘channels of energy’ method for rigidity arguments.

Here we outline a version of this argument adapted to the present setting. The key new technical ingredient are exterior energy estimates for the free radial wave equation in odd dimensions proved for $d = 3$ in [4], for $d = 5$ in [12] and for $d \geq 7$ in [10].

4.1. **Exterior Energy Estimates.**

We now turn to the main new ingredient from the linear theory, which is Theorem 4.1. In order to motivate this result, we first review the analogous statements in dimensions $d = 1$ and $d = 3$.

Suppose $w_{tt} - w_{xx} = 0$ with smooth energy data $(w(0), \dot{w}(0)) = (f, g)$. Then by local energy conservation

$$\int_{x>a} \frac{1}{2}(w_t^2+w_x^2)(0,x)\,dx - \int_{x>T+a} \frac{1}{2}(w_t^2+w_x^2)(T,x)\,dx = \frac{1}{2}\int_0^T (w_t+w_x)^2(t,t+a)\,dt$$

for any $T > 0$ and $a \in \mathbb{R}$. Since $(\partial_t - \partial_x)(w_t + w_x) = 0$, we have that

$$\begin{aligned} \frac{1}{2}\int_0^T (w_t + w_x)^2(t, t + a)\,dt &= \frac{1}{2}\int_0^T (w_t + w_x)^2(0, a + 2t)\,dt \\ = \frac{1}{4}\int_a^{a+2T} (w_t + w_x)^2(0, x)\,dx &= \frac{1}{4}\int_a^{a+2T} (f_x + g)^2(x)\,dx \end{aligned}$$

Consequently,

$$\begin{aligned} &\int_{x>a} \frac{1}{2}(w_t^2 + w_x^2)(0, x)\,dx - \lim_{T\to\infty} \int_{x>T+a} \frac{1}{2}(w_t^2 + w_x^2)(T, x)\,dx \\ &= \frac{1}{4}\int_a^{\infty} (f_x + g)^2(x)\,dx \end{aligned}$$

and thus

$$\begin{aligned} &\min_{\pm} \Big[\int_{x>a} \frac{1}{2}(f_x^2 + g^2)(0, x)\,dx - \lim_{T\to\pm\infty} \int_{x>|T|+a} \frac{1}{2}(w_t^2 + w_x^2)(T, x)\,dx \Big] \\ &\leq \frac{1}{4}\int_a^{\infty} (f_x^2 + g^2)(x)\,dx \end{aligned}$$

whence

$$\max_{\pm} \lim_{T\to\pm\infty} \int_{x>|T|+a} \frac{1}{2}(w_t^2 + w_x^2)(T, x)\,dx \geq \frac{1}{4}\int_a^{\infty} (f_x^2 + g^2)(x)\,dx \tag{4.1}$$

Here we used that $t \mapsto -t$ leaves f unchanged, but turns g into $-g$.

Given $\Box u = 0$ radial in three dimensions, $w(t,r) = ru(t,r)$ solves $w_{tt} - w_{rr} = 0$. Consequently, (1.1) gives the following estimate from [4, Lemma 4.2], see also [5, 7, 8]: for any $a \geq 0$ one has

$$\begin{aligned} &\max_{\pm} \lim_{T\to\pm\infty} \int_{r>|T|+a} \frac{1}{2}((ru)_r^2 + (ru_t)^2)(T,r)\, dr \\ &\geq \frac{1}{4}\int_{r>a} ((rf)_r^2 + (rg)^2)(r)\, dr \end{aligned} \tag{4.2}$$

where $u(0) = f$, $\dot{u}(0) = g$. The left-hand side of (4.2) equals

$$\max_{\pm} \lim_{T\to\pm\infty} \int_{r>|T|+a} \frac{1}{2}(u_r^2 + u_t^2)(T,r)\, r^2 dr \tag{4.3}$$

by the standard dispersive properties of the wave equation. The right-hand side, on the other hand, exhibits the following dichotomy: if $a = 0$, then it equals half of the full energy

$$\frac{1}{4}\int_0^\infty (f_r^2 + g^2)(r)\, r^2 dr$$

However, if $a > 0$, then integration by parts shows that it equals (ignoring the constant from the spherical measure in $\mathbb{R}^3$)

$$\frac{1}{4}\int_{r>a} (f_r^2 + g^2)(r) r^2\, dr - \frac{1}{4} a f^2(a) = \frac{1}{4}\|\pi_a^\perp(f,g)\|^2_{\dot{H}^1\times L^2(r>a)}$$

where $\pi_a^\perp = \mathrm{Id} - \pi_a$ and π_a is the orthogonal projection onto the line

$$\{(cr^{-1}, 0) \mid c \in \mathbb{R}\} \subset \dot{H}^1 \times L^2(r > a).$$

The appearance of this projection is natural, in view of the fact that the Newton potential r^{-1} in $\mathbb{R}^3$ yields an explicit solution to $\Box u = 0$, $u(0,r) = r^{-1}$, $\dot{u}(0,r) = 0$: indeed, one has $u(r,t) = r^{-1}$ in $r > |t| + a$ for which (4.3) vanishes. Since $r^{-1} \notin L^2(r > 1)$ no projection appears in the time component. In contrast, the Newton potential in $\mathbb{R}^5$, viz. r^{-3}, does lie in $H^1(r > a)$ for any $a > 0$. This explains why in $\mathbb{R}^5$ we need to project away from a plane rather than a line. In higher dimensions, the dimension of the space of "bad solutions" generated by such examples grows.

Below we use the notation $[x]$ for the largest integer $k \in \mathbb{Z}, k \leq x$.

Theorem 4.1 ([10, Theorem 2])**.** *In any odd dimension $d > 0$, every radial energy solution of $\Box u = 0, u(0) = f, u_t(0) = g$ in $\mathbb{R}^{1+d}_{t,x}$ satisfies the following estimate: For every $R > 0$*

$$\max_{\pm} \lim_{t\to\pm\infty} \int_{r\geq|t|+R} |\nabla_{t,x} u(t,r)|^2\, r^{d-1}\, dr \geq \frac{1}{2}\|\pi_R^\perp(f,g)\|^2_{\dot{H}^1\times L^2(r\geq R; r^{d-1}dr)} \tag{4.4}$$

Here

$$P(R) := \mathrm{span}\left\{(r^{2k_1-d}, 0), (0, r^{2k_2-d}) \mid k_1 = 1, 2, \cdots [\frac{d+2}{4}]; k_2 = 1, 2, \cdots [\frac{d}{4}]\right\}$$

and $\pi_R^\perp$ denotes the orthogonal projection onto the complement of the plane $P(R)$ in $(\dot{H}^1 \times L^2)(r \geq R; r^{d-1}dr)$.

The inequality becomes an equality for data of the form $(0,g)$ and $(f,0)$. Moreover, the left-hand side of (4.4) vanishes exactly for all data in $P(R)$.

Remark 2. The elementary argument given in the beginning of this subsection does not seem to generalize to dimensions $d \geq 7$. Indeed, the proof of Theorem 4.1 given in [10] is performed on the "Fourier side" and involves computing explicit asymptotics for the left-hand-side of (4.4).

4.2. **Rigidity Argument.** With the linear theory in hand, one can now show that the critical element $\vec{u}_*$ from Proposition 3.3 does not exist. In particular, one has the following rigidity theorem.

Theorem 4.2 (Rigidity Theorem [12, 11])**.** *Let $\vec{u}(t) \in \mathcal{H} = \dot{H}^1 \times L^2(\mathbb{R}^d_*)$ be a global solution to* (2.5) *such that the trajectory*

$$\mathcal{K} := \{\vec{u}(t) | t \in \mathbb{R}\}$$

is pre-compact in $\mathcal{H}$. Then $u \equiv 0$.

We will use the hypothesis above in the following manner: the pre-compactness of the trajectory $\mathcal{K}$ implies that the $\mathcal{H}$-norm of $\vec{u}(t)$ decays on the exterior cone $\{r \geq R + |t|\}$ as $t \to \pm\infty$.

Corollary 4.3. *Given $\vec{u}(t)$ as in Theorem 4.2 and any $R \geq 1$, we have*

$$\|\vec{u}(t)\|_{\mathcal{H}(r \geq R+|t|)} \to 0 \text{ as } t \to \pm\infty. \tag{4.5}$$

The main idea is now as as follows: We can choose R large enough above so that the compact solution u has small energy outside the ball of radius R. This means that we can approximate the nonlinear solution in this exterior region by a free wave with the same data since the nonlinear terms are perturbative for small energies. One can then use Theorem 4.1 to deduce the following estimates for $\vec{u}(t)$.

Proposition 4.4. *Given a radial global solution $\vec{u}(t)$ to* (2.5) *with a pre-compact trajectory, there exists a number $R_0 > 1$ such that for every $R > R_0$, we have the following estimate uniformly in time $t \in \mathbb{R}$.*

$$\begin{aligned} \|\pi_R^\perp \vec{u}(t)\|_{\mathcal{H}(r \geq R)} \lesssim & R^{1-d} \|\pi_R \vec{u}(t)\|_{\mathcal{H}(r \geq R)} \\ & + R^{-\frac{d}{2}} \|\pi_R \vec{u}(t)\|^2_{\mathcal{H}(r \geq R)} + R^{-1} \|\pi_R \vec{u}(t)\|^3_{\mathcal{H}(r \geq R)} \end{aligned} \tag{4.6}$$

Here $\mathcal{H}(r \geq R) := \dot{H}^1 \times L^2(\mathbb{R}^d \backslash B(0, R))$, π_R and $\pi_R^\perp$ are defined as in Theorem 4.1.

Roughly, the above says that our compact trajectory stays uniformly close in $\mathcal{H}$ to the subspace $P(R)$ defined in Theorem 4.1. With a delicate, but elementary argument we can now establish the precise spacial asymptotic behavior of $\vec{u}(t)$. In particular one shows that $\vec{u}(0)$ has same the spacial decay as $\frac{1}{r}(Q(r) - n\pi)$ where Q is a solution to the elliptic equation (2.1) as in Lemma 2.1 – note that this is better decay than what is expected for generic energy data. Indeed, we prove that

$$\begin{aligned} u_0(r) &= \vartheta r^{2-d} + O(r^{3-2d}) \text{ as } r \to \infty \\ \int_r^\infty u_1(s) s^{2i-1} ds &= O(r^{2i+2-2d}) \text{ as } r \to \infty, \quad \forall 1 \leq i \leq k \end{aligned} \tag{4.7}$$

where ϑ is some constant, and $k := [\frac{d}{4}]$.

We then argue by contradiction to show that $\vec{u}(t) = (0, 0)$ is the only solution with pre-compact trajectory $\mathcal{K}$ as in Theorem 4.2 and data that decay like (4.7).

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Andrew Lawrie

Department of Mathematics, The University of California, Berkeley
970 Evans Hall #3840, Berkeley, CA 94720, U.S.A.
alawrie@math.berkeley.edu, sjoh@math.berkeley.edu