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Abstract. In this paper, we develop and analyze a finite element fictitious domain approach based on Nitsche's method for the approximation of frictionless contact problems of two deformable elastic bodies. In the proposed method, the geometry of the bodies and the boundary conditions, including the contact condition between the two bodies, are described independently of the mesh of the fictitious domain. We prove that the optimal convergence is preserved. Numerical experiments are provided which confirm the correct behavior of the proposed method.

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1. Introduction

In the vast majority of finite element software, the contact conditions between deformable solids are taken into account through the introduction of Lagrange multipliers and/or penalization terms. The multipliers, which generally approximate the contact stresses, represent some additional unknowns. The approximated problem is then solved in a coupled way or iteratively on the multiplier using Uzawa's algorithm (see e.g. [27]). Recently in [6, 7], it has been proposed an extension to the contact conditions of Nitsche's method [24, 11, 17] which was originally dedicated to Dirichlet's condition. This method combines the advantages of both the penalty and Lagrange multiplier methods since it remains consistent, optimal and avoid the use of multipliers.

In a fictitious domain framework, this paper aims to adapt Nitsche's method to the case of frictionless contact of two elastic solids with the small deformations hypothesis. Frictionless contact is considered to keep the presentation as simpler as possible. However, the analysis extends without additional difficulties to the case of Tresca friction, in a similar way as in [5]. One of the advantages of the fictitious domain approach comes from the possibility to work with structured meshes regardless of the complexity of the geometry of the bodies and of the potential contact zone. This approach is particularly advantageous in the case of free boundary problems such as shape optimization and fluid-structure interaction. In that case, it prevents the consecutive remeshing which can be very costly, in particular for three-dimensional problems, and which may also generates some instabilities. More generally, a fictitious domain method may be used in the presence of complex or moving geometries to avoid meshing them.

The fictitious domain approach we consider in this work is the one using "cut elements" which is currently a subject of growing interest and is closely related to XFem approach introduced in [21] and widely studied since then (see for instance [20, 16, 26, 4, 23]). The case of a body with a Dirichlet (or transmission) condition with the use of cut-elements is studied in [16] when Lagrange multipliers and a Barbosa-Hughes stabilization are used, and in [14, 4, 1] when Nitsche's method and an additional interior penalty stabilization are considered. This fictitious domain method is to be compared with more classical strategies (see [19, 12, 13, 25, 2] and the references therein) where the elements are not cut. These more classical strategies offer the possibility to leave unchanged the stiffness matrix of the problem. The boundary conditions are then prescribed via additional penalty and Lagrange multiplier terms. However, in classical strategies, it is often quite difficult to obtain an optimal method regarding the convergence order which easily takes into account both Dirichlet and Neumann conditions. The Fictitious domain method with cut elements allows to consider both Dirichlet and Neumann conditions in a rather standard way. The main price to pay is the adaptation of integration methods on cut elements.

In that context of cut elements, our study is focused on the case of two bodies with Nitsche's method for both the Dirichlet condition and the frictionless contact condition.

The outline of the paper is the following. In Section 2, we introduce the contact problem and the fictitious domain situation. Then, in Section 3, the finite element approximation with the use of Nitsche's method is built. In particular, a specific, parameter free stabilization technique is introduced which is necessary to guarantee the optimal rate of convergence. The properties of the approximated problem are described in Section 4 including the existence and uniqueness of a solution to the discrete problem, the consistency and the *a priori* error analysis. Finally, in Section 5, some two and three-dimensional Hertz-type numerical experiments are presented which illustrate the optimality regarding the convergence of the method.

2. The unilateral contact problem in a fictitious domain framework

An example of fictitious domain situation is illustrated in Figure 2.1. Let Ω_i, $1 \leqslant i \leqslant 2$, be two possibly overlapping domains with piecewise $\mathscr{C}^1$ boundaries included in $\mathbb{R}^d$, $d = 2$ or 3, representing the reference configurations of two elastics bodies. Let Ω be a simple shaped polygonal fictitious domain (typically allowing the use of a structured mesh) containing both Ω_1 and Ω_2. The boundary Γ_1 of Ω_1 (respectively Γ_2 of Ω_2) is divided into three non overlapping parts: $\Gamma_{1,C}$ the slave potential zone of contact with $meas(\Gamma_{1,C}) > 0$ (respectively $\Gamma_{2,C}$ with $meas(\Gamma_{2,C}) > 0$); $\Gamma_{1,N}$ the Neumann part (respectively $\Gamma_{2,N}$) and $\Gamma_{1,D}$ the Dirichlet part with $meas(\Gamma_{1,D}) > 0$ (respectively $\Gamma_{2,D}$ with $meas(\Gamma_{2,D}) > 0$).

The two elastic bodies are subjected to volume forces $f = (f_1, f_2)$ on $\Omega_1 \times \Omega_2$, to surface loads $\ell = (\ell_1, \ell_2)$ on $\Gamma_{1,N} \times \Gamma_{2,N}$ and satisfy non homogeneous boundary Dirichlet conditions on $\Gamma_{1,D} \times \Gamma_{2,D}$, the displacement being prescribed to the given value $u_D = (u_{1,D}, u_{2,D})$. We assume small elastic deformation for the two bodies. The linearized strain tensor field is given by $\varepsilon(v) = \dfrac{1}{2}(\nabla v + \nabla v^T)$ and the stress tensor field $\sigma = (\sigma_{ij})_{1 \leqslant i,j \leqslant 2}$ is given by $\sigma(v) = A\varepsilon(v)$ where A is the fourth order symmetric elasticity tensor satisfying the usual uniform ellipticity and boundedness properties. Consequently, the displacement (u_1, u_2) on $\Omega_1 \times \Omega_2$ has to satisfy the following set of equations, apart for the contact condition which will be described later:

$$\begin{cases} \text{Find } u = (u_1, u_2) \text{ satisfying} & \\ \quad -\mathrm{div}\sigma(u_i) = f_i & \text{in} \quad \Omega_i, \\ \quad \sigma(u_i) = A\varepsilon(u_i) & \text{in} \quad \Omega_i, \\ \quad u_i = u_{i,D} & \text{on} \quad \Gamma_{i,D}, \\ \quad \sigma(u_i)n_i = \ell_i & \text{on} \quad \Gamma_{i,N}. \end{cases} \tag{2.1}$$

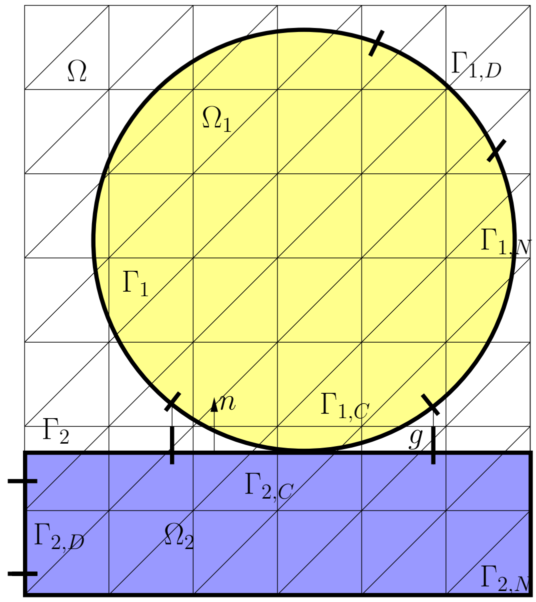


Figure 2.1. Example of fictitious domain situation for a contact problem between two elastics bodies with an example of structured mesh.

Now, concerning the contact conditions, let us define Π the orthogonal projection from the slave boundary $\Gamma_{1,C}$ on the master boundary $\Gamma_{2,C}$:

$$\Pi : \begin{array}{ccc} \Gamma_{1,C} & \rightarrow & \Gamma_{2,C} \\ x & \mapsto & \Pi(x). \end{array} \tag{2.2}$$

In order to simplify the mathematical analysis, the operator Π is assumed to be a $\mathscr{C}^1$ one to one correspondence on $\Pi(\Gamma_{1,C})$ (this hypothesis is satisfied, for instance, when $\Gamma_{i,C}$ are convex and $\mathscr{C}^1$ for $i \in \{1, 2\}$). The outward unit normal vector n for the contact condition is chosen to be the one of $\Gamma_{2,C}$:

$$n : \begin{array}{ccc} \Gamma_{1,C} & \rightarrow & \mathbb{R}^d \\ x & \mapsto & n_2(\Pi(x)). \end{array}$$

The initial gap g between $\Gamma_{1,C}$ and $\Gamma_{2,C}$ is defined to be the following distance function:

$$g : \begin{array}{ccc} \Gamma_{1,C} & \rightarrow & \mathbb{R} \\ x & \mapsto & (x - \Pi(x)) \cdot n. \end{array}$$

For (v_1, v_2) a displacement field defined on $\Omega_1 \times \Omega_2$, the normal jump is defined on the slave boundary Γ_1 for the normal displacement as follows:

$$[\![v \cdot n]\!] = (v_2 \circ \Pi - v_1) \cdot n.$$

Concerning the normal stress, we define

$$\sigma(v_1)n_1 = -\sigma_n(v_1)n + \sigma_t(v_1) \qquad \text{with } \sigma_n(v_1) = -\sigma(v_1)n_1 \cdot n$$

and

$$\sigma(v_2 \circ \Pi)n_2 \circ \Pi = \sigma_n(v_2 \circ \Pi)n + \sigma_t(v_2 \circ \Pi) \qquad \text{with } \sigma_n(v_2 \circ \Pi) = \sigma(v_2 \circ \Pi)n_2 \circ \Pi \cdot n.$$

This allows to define the normal stress jump as

$$[\![\sigma(u)n]\!] = \sigma(u_1)n_1 + \sigma(u_2 \circ \Pi)n_2 \circ \Pi \ \ |\det(J_\Pi)|\,,$$

with J_Π the Jacobian matrix of Π. This latter expression is derived accordingly with Newton's second law (action-reaction principle) which is expressed on arbitrary elementary surfaces (see Figure 2.2):

$$\forall \omega \subset \Gamma_{1,C}, \qquad \int_\omega \sigma(u_1)n_1 \, \mathrm{d}\Gamma = -\int_{\Pi(\omega)} \sigma(u_2)n_2 \mathrm{d}\Gamma = -\int_\omega \sigma(u_2\circ\Pi)n_2\circ\Pi \ \ |\det(J_\Pi)| \, \mathrm{d}\Gamma.$$

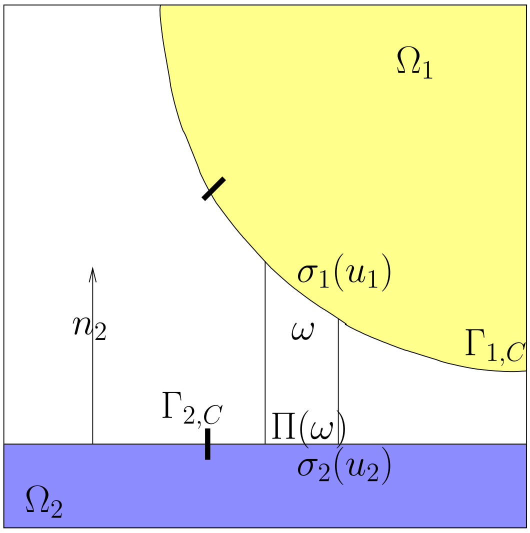


Figure 2.2. An example illustrating the action-reaction principle between the two bodies.

These jumps being defined, the unilateral frictionless contact conditions can be expressed on the slave boundary $\Gamma_{1,C}$ as follows:

$$\left\{\begin{array}{ll} [\![u\cdot n]\!] \leqslant g & (i),\\ \sigma_n(u_1)\leqslant 0 & (ii),\\ \sigma_n(u_1)([\![u\cdot n]\!]-g)=0 & (iii),\\ [\![\sigma(u)n]\!]=0 & (iv),\\ \sigma_t(u_1)=0 & (v). \end{array}\right. \tag{2.3}$$

Now, let us introduce the Hilbert space V and the convex cone K of admissible displacements:

$$V = H^1(\Omega_1)^d \times H^1(\Omega_2)^d,$$

$$K = \{v=(v_1,v_2)\in V \mid v_1 = u_{1,D} \text{ on } \Gamma_{1,D} \text{ and } v_2 = u_{2,D} \text{ on } \Gamma_{2,D} \mid [\![v\cdot n]\!] - g \leqslant 0 \text{ on } \Gamma_{1,C}\}.$$

We assume that f belongs to $L^2(\Omega_1)^d\times L^2(\Omega_2)^d$, ℓ belongs to $L^2(\Gamma_{1,N})^d\times L^2(\Gamma_{2,N})^d$ and u_D belongs to $H^{\frac32}(\Gamma_{1,D})^d \times H^{\frac32}(\Gamma_{2,D})^d$. We define the bilinear and the linear forms $a(.,.)$ and $L(.)$ by

$$a(u,v) = \sum_{i=1,2}\int_{\Omega_i} \sigma(u_i):\varepsilon(v_i)\, \mathrm{d}\Omega, \quad L(v) = \sum_{i=1,2}\int_{\Omega_i} f_i v_i \, \mathrm{d}\Omega + \sum_{i=1,2}\int_{\Gamma_{i,N}} \ell_i v_i \, \mathrm{d}\Gamma.$$

The weak formulation of Problem (2.1)-(2.3) as a variational inequality (see [10, 15, 18, 28]), reads:

$$\left\{\begin{array}{ll} \text{Find } u\in K \text{ such that} & \\ a(u,v-u)\geqslant L(v-u) & \forall v\in K. \end{array}\right. \tag{2.4}$$

Stampacchia's Theorem ensures that Problem (2.4) admits a unique solution.

3. A Nitsche-based finite element approximation

3.1. Nitsche's formulation

In this section, we assume that both the solution u and the test functions v are sufficiently regular (for instance, $(u, v) \in (H^{3/2+\nu}(\Omega_1)^d \times H^{3/2+\nu}(\Omega_2)^d)^2$ for $\nu > 0$). From the equilibrium equations and Green's formula, we obtain:

$$a(u, v) - \sum_{i=1,2} \int_{\Gamma_{i,D}} \sigma_n(u_i) n_i \cdot v_i \, \mathrm{d}\Gamma - \int_{\Gamma_{1,C}} \sigma_n(u_1) [\![v \cdot n]\!] \, \mathrm{d}\Gamma = L(v).$$

In order to build Nitsche's formulations for the contact and Dirichlet conditions, the contact conditions are expressed in an equivalent way by extending to our case the formulation given in [6, 7]. Denoting $z_+ = \max(z, 0)$ and for an arbitrary $\gamma > 0$, the contact conditions (2.3) on $\Gamma_{1,C}$ can be equivalently rewritten:

$$\sigma_n(u_1) = -\frac{1}{\gamma} [[\![u \cdot n]\!] - g - \gamma \sigma_n(u_1)]_+. \tag{3.1}$$

Let $\theta \in \mathbb{R}$ be a fixed parameter. This additional parameter for Nitsche's method determines the symmetry properties (see remarks (3.2) and [6, 7]). Then by using (3.1) and $[\![v \cdot n]\!] = ([\![v \cdot n]\!] - \theta\gamma\sigma_n(v)) + \theta\gamma\sigma_n(v))$, we obtain:

$$\begin{aligned} a(u, v) - \int_{\Gamma_{1,C}} \theta\gamma\sigma_n(u_1)\sigma_n(v_1) \, \mathrm{d}\Gamma - \sum_{i=1,2} \int_{\Gamma_{i,D}} \sigma_n(u_i) n_i \cdot v_i \, \mathrm{d}\Gamma \\ + \int_{\Gamma_{1,C}} \frac{1}{\gamma} [[\![u \cdot n]\!] - g - \gamma\sigma_n(u_1)]_+ ([\![v \cdot n]\!] - \theta\gamma\sigma_n(v_1)) \, \mathrm{d}\Gamma = L(v). \end{aligned}$$

Using contact conditions (2.3), it holds $\sigma_n(u_1) = \sigma_n(u_2 \circ \Pi) \; |\det(J_\Pi)|$. In order to ensure the stability, we introduce a stabilized formulation for elements having a small contribution [14, 4, 16]. We replace $\sigma_n(u_1)$ by a convex combination of $\sigma_n(u_1)$ and $\sigma_n(u_2 \circ \Pi) \; |\det(J_\Pi)|$. Namely, we define

$$\sigma_n(u) = t\sigma_n(u_2 \circ \Pi) \; |\det(J_\Pi)| + (1 - t)\sigma_n(u_1), \tag{3.2}$$

for a parameter $t \in [0, 1]$ which may be different for an element to an other for the finite element approximation. Note that a similar approach has been developed in [1] where an optimal choice of the fixed parameter $t \in [0, 1]$ is proposed. We obtain:

$$\begin{aligned} a(u, v) - \int_{\Gamma_{1,C}} \theta\gamma\sigma_n(u)\sigma_n(v) \, \mathrm{d}\Gamma - \sum_{i=1,2} \int_{\Gamma_{i,D}} \sigma_n(u) n_i \cdot v_i \, \mathrm{d}\Gamma \\ + \int_{\Gamma_{1,C}} \frac{1}{\gamma} [[\![u \cdot n]\!] - g - \gamma\sigma_n(u)]_+ ([\![v \cdot n]\!] - \theta\gamma\sigma_n(v)) \, \mathrm{d}\Gamma = L(v). \end{aligned}$$

We did not treat yet the Dirichlet conditions. In order to be coherent with the fictitious domain approach, we also describe the Dirichlet conditions thanks to Nitsche's method [14, 4, 17]. Then, writing $v_i = (v_i - \theta\gamma\sigma(v_i)n_i) + \theta\gamma\sigma(v_i)n_i$ as in the formulation for the contact conditions, we deduce:

$$\begin{aligned} & - \int_{\Gamma_{i,D}} \sigma(u_i) n_i \cdot v_i \, \mathrm{d}\Gamma \\ & \quad = \int_{\Gamma_{i,D}} \frac{1}{\gamma} (u_i - u_{i,D} - \gamma\sigma(u_i)n_i) \cdot (v_i - \gamma\theta\sigma(v_i)n_i) \, \mathrm{d}\Gamma - \int_{\Gamma_{i,D}} \theta\gamma\sigma(u_i)n_i \cdot \sigma(v_i)n_i \, \mathrm{d}\Gamma. \end{aligned} \tag{3.3}$$

We obtain the following weak formulation:

$$a(u,v)+\int_{\Gamma_{1,C}}\frac{1}{\gamma}[[\![u\cdot n]\!]-g-\gamma\sigma_n(u)]_+([\![v\cdot n]\!]-\theta\gamma\sigma_n(v))\ \mathrm{d}\Gamma$$

$$+\sum_{i=1,2}\int_{\Gamma_{i,D}}\frac{1}{\gamma}(u_i-u_{i,D}-\gamma\sigma(u_i)n_i)\cdot(v_i-\gamma\theta\sigma(v_i)n_i)\ \mathrm{d}\Gamma$$

$$-\int_{\Gamma_{1,C}}\theta\gamma\sigma_n(u)\sigma_n(v)\ \mathrm{d}\Gamma-\sum_{i=1,2}\int_{\Gamma_{i,D}}\theta\gamma\sigma(u_i)n_i\cdot\sigma(v_i)n_i\ \mathrm{d}\Gamma=L(v)\ \forall v\in V. \quad (3.4)$$

Finally, defining the bilinear form

$$A_{\theta\gamma}(u,v)=a(u,v)-\int_{\Gamma_{1,C}}\theta\gamma\sigma_n(u)\sigma_n(v)\ \mathrm{d}\Gamma-\sum_{i=1,2}\int_{\Gamma_{i,D}}\theta\gamma\sigma(u_i)n_i\cdot\sigma(v_i)n_i\ \mathrm{d}\Gamma,$$

our Nitsche-based method reads:

$$A_{\theta\gamma}(u,v)+\int_{\Gamma_{1,C}}\frac{1}{\gamma}[[\![u\cdot n]\!]-g-\gamma\sigma_n(u)]_+([\![v\cdot n]\!]-\theta\gamma\sigma_n(v))\ \mathrm{d}\Gamma$$

$$+\sum_{i=1,2}\int_{\Gamma_{i,D}}\frac{1}{\gamma}(u_i-u_{i,D}-\gamma\sigma(u_i)n_i)\cdot(v_i-\gamma\theta\sigma(v_i)n_i)\ \mathrm{d}\Gamma=L(v)\ \forall v\in V. \quad (3.5)$$

3.2. **Discrete Nitsche's formulation**

In what follows, Ciarlet's notations [8] are used. Let T_h be a family of triangulations of the fictitious domain Ω such that $\Omega=\bigcup_{K\in T_h}K$. Let h_K be the diameter of $K\in T_h$ and $h=\max_{K\in T_h}h_K$. The family of triangulations is assumed to be regular, i.e. it exists $C>0$ such that $\dfrac{h_K}{\rho_K}\leqslant C$ where ρ_K denotes the radius of the ball inscribed in K. We suppose that the mesh is quasi uniform in the sense that it exists $\zeta>0$ a constant such that $\forall K\in T_h,\ h_K\geqslant\zeta\, h$.

Let $\hat{K}$ be the fixed reference element (a triangle for $d=2$, a tetrahedron for $d=3$) and let T_K be the geometric transformation which satisfies $T_K(\hat{K})=K$. The family of triangulations is supposed affine, i.e. T_K reads as

$$\forall K\in T_h,\ T_K(\hat{x})=J_K\hat{x}+b_K,\ \hat{x}\in\hat{K},$$

where $J_K\in\mathbb{R}^{d,d}$ is the Jacobian matrix of T_K being invertible and $b_K\in\mathbb{R}^d$. Thus, we have:

$$|\det(J_K)|=\frac{\mathrm{mes}(K)}{\mathrm{mes}(\hat{K})},\qquad \|J_K\|\leqslant h_K/\rho_{\hat{K}},\qquad \left\|J_K^{-1}\right\|\leqslant h_{\hat{K}}/\rho_K.$$

Remark 3.1. The family of triangulations is regular and affine, so it holds:

$$|\det(J_K)|\leqslant Ch_K^d,\qquad \|J_K\|\leqslant Ch_K,\qquad \left\|J_K^{-1}\right\|\leqslant Ch_K^{-1}.$$

We introduce $U^h\subset H^1(\Omega)$ a family of finite element spaces indexed by h coming from some order $k\geqslant 1$ finite element method defined on T_h. Consequently, we suppose the existence of a global interpolation operator $\pi^h:\mathscr{C}^0(\overline{\Omega})\to U^h$ and a local one π_K^h on each element $K\in T_h$ such that:

$$\forall u\in\mathscr{C}^0(\overline{\Omega}),\qquad \pi^h(u)|_K=\pi_K^h(u_{|K})\qquad\text{and}\qquad\forall p\in P_k(K),\ \pi_K^h(p)=p.$$

We assume that the finite element method satisfies the following classical local interpolation error estimate for $k \geqslant l \geqslant 0$, $u \in H^{l+1}(\Omega)$:

$$\left\| u - \pi_K^h u \right\|_{m,K} \leqslant C h^{l+1-m} \left| u \right|_{l+1,K}, \qquad \text{with } \; 0 \leqslant m \leqslant l \leqslant k.$$

Note that, in particular, the classical P_k Lagrange finite element method [8] satisfies this estimate. The approximation spaces for our problem are defined by

$$V_1^h = (U^h)^d|_{\Omega_1} \quad , \quad V_2^h = (U^h)^d|_{\Omega_2} \quad \text{and} \quad V^h = (V_1^h \times V_2^h).$$

In the same way, we define the global operators

$$\Pi_i^h : H^{k+1}(\Omega)^d \to V_i^h, i = \{1,2\} \qquad \text{and} \qquad \Pi^h : H^{k+1}(\Omega)^d \times H^{k+1}(\Omega)^d \to V^h.$$

In order to write a discrete approximation of formulation (3.5), let us introduce the following discrete linear operators:

$$P_\tau^h : \begin{array}{ccl} V_1^h \times V_2^h & \to & L^2(\Gamma_{1,C}) \\ v & \mapsto & [\![v \cdot n]\!] - \tau \sigma_n(v), \end{array}$$

$$\overline{P}_{i,\tau}^h : \begin{array}{ccl} V_i^h & \to & L^2(\Gamma_{i,D})^d \\ v_i & \mapsto & v_i - \tau \sigma(v_i) n_i. \end{array}$$

Then, a finite element approximation of our Nitsche-based method reads as:

$$\left\{ \begin{array}{l} \text{Find } u^h \in V^h \text{ such that} \\ A_{\theta\gamma}(u^h, v^h) + \int_{\Gamma_{1,C}} \frac{1}{\gamma} [P_\gamma^h(u^h) - g]_+ P_{\theta\gamma}^h(v^h) \; \mathrm{d}\Gamma \\ + \sum_{i=1,2} \int_{\Gamma_{i,D}} \frac{1}{\gamma} (\overline{P}_{i,\gamma}^h(u_i^h) - u_{i,D}) \cdot \overline{P}_{i,\gamma\theta}^h(v_i^h) \; \mathrm{d}\Gamma = L(v^h) \qquad \forall v^h \in V^h. \end{array} \right. \tag{3.6}$$

In the following, we define $\gamma = \gamma_0 h_K$.

Remark 3.2. The additional parameter θ is aimed to be chosen in $[-1, 1]$. The following values of θ are of particular interest: for $\theta = 1$, we recover the symmetric method proposed and analyzed in [6]; for $\theta = 0$, we recover a non-symmetric version presented in [7] and for $\theta = -1$, we obtain a skew-symmetric version which has the remarkable property that convergence occurs for any value of γ_0 (see [7]).

Remark 3.3. Note that, concerning the Dirichlet conditions, we obtain Nitsche's classical reformulation since the terms on $\Gamma_{i,D}$ in (3.6) read

$$\int_{\Gamma_{i,D}} \frac{1}{\gamma} (u_i - u_{i,D}) \cdot v_i \; \mathrm{d}\Gamma - \theta \int_{\Gamma_{i,D}} (u_i - u_{i,D}) \cdot \sigma(v_i) n_i \; \mathrm{d}\Gamma - \int_{\Gamma_{i,D}} \sigma(u_i) n_i \cdot v_i \; \mathrm{d}\Gamma.$$

Indeed, the first term is a kind of penalty term for the Dirichlet condition, the second one ensure the symmetry when $\theta = 1$ and the third one ensure the consistency.

3.3. **Consistency**

The advantage of Nitsche's method, compared to penalization, is the consistency of the approximation in the following sense.

Theorem 3.4. *Let u be the solution to Problem* (2.1)-(2.3)*. Assume u is sufficiently regular (typically, $(u_1, u_2) \in H^{2+\nu}(\Omega_1)^d \times H^{2+\nu}(\Omega_2)^d$, for $\nu > 0$), then u is also a solution to the discrete problem* (3.6) *replacing u^h by u.*

Proof. Let u be the solution to (2.1)-(2.3) and take $v^h \in V^h$. We assume u sufficiently regular such that $\sigma_n(u) \in L^2(\Gamma_{1,C})$ and for $i=1,2$, $\sigma_n(u_i) \in L^2(\Gamma_{i,D})$. As a result, $P^h_{\theta\gamma}(u) \in L^2(\Gamma_{1,C})$, for $i=1,2$, $\overline{P}^h_{i,\theta\gamma}(u_i) \in L^2(\Gamma_{i,D})$ and $A_{\theta\gamma}(u,v^h)$ makes sense. On the one hand, we use the definition of $P^h_{\theta\gamma}$, $\overline{P}^h_{i,\theta\gamma}$, the reformulations (3.1) and (3.3) to obtain:

$$\begin{aligned}
A_{\theta\gamma}(u,v^h) &+ \int_{\Gamma_{1,C}} \frac{1}{\gamma}[P^h_\gamma(u)-g]_+ P^h_{\theta\gamma}(v^h)\ \mathrm{d}\Gamma + \sum_{i=1,2}\int_{\Gamma_{i,D}} \frac{1}{\gamma}(\overline{P}^h_{i,\gamma}(u_i)-u_{i,D})\cdot \overline{P}^h_{i,\gamma\theta}(v^h_i)\ \mathrm{d}\Gamma,\\
&= a(u,v^h) - \int_{\Gamma_{1,C}} \theta\gamma\sigma_n(u)\sigma_n(v^h)\ \mathrm{d}\Gamma - \sum_{i=1,2}\int_{\Gamma_{i,D}} \theta\gamma\sigma(u_i)n_i\cdot\sigma(v^h_i)n_i\ \mathrm{d}\Gamma\\
&\quad + \int_{\Gamma_{1,C}} \frac{1}{\gamma}[[\![u\cdot n]\!] - g - \gamma\sigma_n(u)]_+([\![v^h\cdot n]\!] - theta\gamma\sigma_n(v^h))\ \mathrm{d}\Gamma\\
&\quad + \sum_{i=1,2}\int_{\Gamma_{i,D}} \frac{1}{\gamma}(u_i - u_{i,D} - \gamma\sigma(u_i)n_i)\cdot(v^h_i - \gamma\theta\sigma(v^h_i)n_i)\ \mathrm{d}\Gamma,\\
&= a(u,v^h) - \int_{\Gamma_{1,C}} \theta\gamma\sigma_n(u)\sigma_n(v^h)\ \mathrm{d}\Gamma + \int_{\Gamma_{1,C}} \frac{1}{\gamma}(-\gamma\sigma_n(u))([\![v^h\cdot n]\!] - \theta\gamma\sigma_n(v^h))\ \mathrm{d}\Gamma\\
&\quad + \sum_{i=1,2}\int_{\Gamma_{i,D}} \frac{1}{\gamma}(-\gamma\sigma(u_i)n_i)\cdot(v^h_i - \gamma\theta\sigma(v^h_i)n_i)\ \mathrm{d}\Gamma - \sum_{i=1,2}\int_{\Gamma_{i,D}} \theta\gamma\sigma(u_i)n_i\cdot\sigma(v^h_i)n_i\ \mathrm{d}\Gamma,\\
&= a(u,v^h) - \int_{\Gamma_{1,C}} \sigma_n(u)[\![v^h\cdot n]\!]\ \mathrm{d}\Gamma - \sum_{i=1,2}\int_{\Gamma_{i,D}} \sigma(u_i)n_i\cdot v^h_i\ \mathrm{d}\Gamma,\\
&= a(u,v^h) - \int_{\Gamma_{1,C}} \sigma_n(u_1)[\![v^h\cdot n]\!]\ \mathrm{d}\Gamma - \sum_{i=1,2}\int_{\Gamma_{i,D}} \sigma(u_i)n_i\cdot v^h_i\ \mathrm{d}\Gamma.
\end{aligned}$$

On the other hand, multiplying by v^h_i and integrating (2.1), it holds:

$$-\sum_{i=1,2}\int_{\Omega_i} \mathrm{div}\sigma(u_i)v^h_i\ \mathrm{d}\Omega = \sum_{i=1,2}\int_{\Omega_i} f_i v^h_i\ \mathrm{d}\Omega.$$

Using Green's formula, we have:

$$-\int_{\Omega_i} \mathrm{div}\sigma(u_i)v^h_i\ \mathrm{d}\Omega = \int_{\Omega_i} \sigma(u_i):\varepsilon(v^h_i)\ \mathrm{d}\Omega - \int_{\Gamma_i} \sigma(u_i)n_i\cdot v^h_i\ \mathrm{d}\Gamma \qquad i=1,2,$$

with

$$\begin{aligned}
-\int_{\Gamma_i} \sigma(u_i)n_i\cdot v^h_i\ \mathrm{d}\Gamma &= -\int_{\Gamma_{i,D}} \sigma(u_i)n_i\cdot v^h_i\ \mathrm{d}\Gamma - \int_{\Gamma_{i,N}} \sigma(u_i)n_i\cdot v^h_i\ \mathrm{d}\Gamma - \int_{\Gamma_{i,C}} \sigma(u_i)n_i\cdot v^h_i\ \mathrm{d}\Gamma \quad i=1,2,\\
-\int_{\Gamma_i} \sigma(u_i)n_i\cdot v^h_i\ \mathrm{d}\Gamma &= -\int_{\Gamma_{i,D}} \sigma(u_i)n_i\cdot v^h_i\ \mathrm{d}\Gamma - \int_{\Gamma_{i,N}} \ell_i v^h_i\ \mathrm{d}\Gamma - \int_{\Gamma_{i,C}} \sigma(u_i)n_i\cdot v^h_i\ \mathrm{d}\Gamma \quad i=1,2.
\end{aligned}$$

Using the one to one correspondence of the projection, it holds:

$$\int_{\Gamma_{2,C}} \sigma(u_2)n_2\cdot v^h_2\ \mathrm{d}\Gamma = \int_{\Gamma_{1,C}} \sigma(u_2\circ\Pi)n_2\circ\Pi\cdot v^h_2\circ\Pi\ \ |\det(J_\Pi)|\ \mathrm{d}\Gamma.$$

Hence

$$\begin{aligned}
-\sum_{i=1,2}\int_{\Omega_i} \mathrm{div}\sigma(u_i)v^h_i\ \mathrm{d}\Omega &= \int_{\Omega_1} \sigma(u_1):\varepsilon(v^h_1)\ \mathrm{d}\Omega + \int_{\Omega_2} \sigma(u_2):\varepsilon(v^h_2)\ \mathrm{d}\Omega - \int_{\Gamma_{1,C}} \sigma(u_1)n_1\cdot v^h_1\ \mathrm{d}\Gamma\\
&\quad - \int_{\Gamma_{1,C}} \sigma(u_2\circ\Pi)n_2\circ\Pi\cdot v^h_2\circ\Pi\ \ |\det(J_\Pi)|\ \mathrm{d}\Gamma\\
&\quad - \sum_{i=1,2}\int_{\Gamma_{i,D}} \sigma(u_i)n_i\cdot v^h_i\ \mathrm{d}\Gamma - \int_{\Gamma_{1,N}} \ell_1 v^h_1\ \mathrm{d}\Gamma - \int_{\Gamma_{2,N}} \ell_2 v^h_2\ \mathrm{d}\Gamma.
\end{aligned}$$

Using (2.3), it holds:

$$-\sum_{i=1,2}\int_{\Omega_i}\mathrm{div}\sigma(u_i)v_i^h\ \mathrm{d}\Omega = a(u,v^h) - \sum_{i=1,2}\int_{\Gamma_{i,D}}\sigma(u_i)n_i\cdot v_i^h\ \mathrm{d}\Gamma - \int_{\Gamma_{1,N}}\ell_1 v_1^h\ \mathrm{d}\Gamma - \int_{\Gamma_{2,N}}\ell_2 v_2^h\ \mathrm{d}\Gamma$$
$$-\int_{\Gamma_{1,C}}\sigma_n(u_1)[\![v^h\cdot n]\!]\ \mathrm{d}\Gamma.$$

So

$$a(u,v^h) - \int_{\Gamma_{1,C}}\sigma_n(u_1)[\![v^h\cdot n]\!]\ \mathrm{d}\Gamma - \sum_{i=1,2}\int_{\Gamma_{i,D}}\sigma_n(u_i)n_i\cdot v_i^h\ \mathrm{d}\Gamma = L(v^h).$$

Which ends the proof. □

Moreover, formulation (3.5) is formally equivalent to (2.1) and (2.3) in the following sense.

Theorem 3.5. *Let $u\in H^2(\Omega_1)^d\times H^2(\Omega_2)^d$ be a solution to equation* (3.5) *then u is a solution to* (2.1) *and* (2.3).

Proof. For $u\in H^2(\Omega_1)^d\times H^2(\Omega_2)^d$ a solution to (3.5) and whatever $v\in H^2(\Omega_1)^d\times H^2(\Omega_2)^d$, it satisfies:

$$\int_{\Omega_i}(\mathrm{div}\sigma(u_i)+f_i)v_i\ \mathrm{d}\Omega = 0 \qquad \forall v_i\in H^2(\Omega_i)$$

i.e.

$$-\mathrm{div}\sigma(u_i) = f_i \qquad \text{a.e. in } \Omega_i,\ 1\leqslant i\leqslant 2.$$

We have, for all $v\in H^2(\Omega_1)^d\times H^2(\Omega_2)^d$:

$$\int_{\Gamma_{1,C}}\frac{1}{\gamma}[[\![u\cdot n]\!]-g-\gamma\sigma_n(u)]_+[\![v\cdot n]\!]\ \mathrm{d}\Gamma + \int_{\Gamma_{1,C}}\sigma(u_1)n_1\cdot v_1\ \mathrm{d}\Gamma + \int_{\Gamma_{2,C}}\sigma(u_2)n_2\cdot v_2\ \mathrm{d}\Gamma = 0,$$
$$\int_{\Gamma_{1,C}}\frac{1}{\gamma}[[\![u\cdot n]\!]-g-\gamma\sigma_n(u)]_+(v_1-v_2\circ\Pi\cdot n)\ \mathrm{d}\Gamma - \int_{\Gamma_{1,C}}\sigma_n(u_1)n\cdot v_1\ \mathrm{d}\Gamma$$
$$+\int_{\Gamma_{1,C}}\sigma_n(u_2\circ\Pi)n\cdot v_2\circ\Pi\ \ |\det(J_\Pi)|\ \ \mathrm{d}\Gamma = 0.$$

Hence

$$\int_{\Gamma_{1,C}}(\frac{1}{\gamma}[[\![u\cdot n]\!]-g-\gamma\sigma_n(u)]_+ + \sigma_n(u_1))v_1\cdot n\ \mathrm{d}\Gamma = 0\ \forall v_1\in H^2(\Omega_1),$$

and

$$\int_{\Gamma_{1,C}}(\frac{1}{\gamma}[[\![u\cdot n]\!]-g-\gamma\sigma_n(u)]_+ - \sigma_n(u_2\circ\Pi)\ \ |\det(J_\Pi)|\,)v_2\circ\Pi\cdot n\ \mathrm{d}\Gamma = 0\ \forall v_2\in H^2(\Omega_2).$$

Hence

$$\frac{1}{\gamma}[[\![u\cdot n]\!]-g-\gamma\sigma_n(u)]_+ = -\sigma_n(u_1) \qquad \text{a. e. on } \Omega_1,$$

which is a formulation equivalent to (2.3). Arguing in the same way as above the Neumann and Dirichlet conditions are recovered. □

3.4. **Stabilization method**

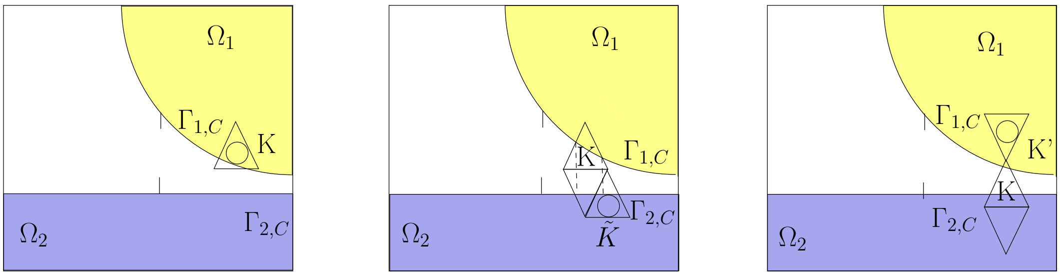
A stabilization technique is necessary to control the possible bad quality of $\sigma_n(u^h)$ on elements having very small intersection with the real domains. The stabilization used is the one proposed in [16] which consists in using extension of the normal stress on a neighbor element having a sufficiently large intersection with the real domain. The advantage of this stabilization technique is the absence of parameter to fit, except the threshold under which an intersection is considered to be too small.

Note that other stabilization techniques are available, such as the so-called ghost penalty stabilization considered in [4].

For a given small radius $1 > \hat{\rho} > 0$, let $R_{\hat{\rho}}$ (respectively $\overline{R}_{\hat{\rho}}$) be an operator of approximation of the normal stress of displacements $\sigma_n(u^h)$ (respectively $\sigma(u_i^h)$) which we define thereafter. For $K \in T_h$ such that $K \cap \Gamma_{1,C}$, we note $S_K = \{K' \in T_h \mid K' \cap \Pi(K) \neq \emptyset\}$. We note also E_K, the polynomial extrapolation of an element $v^h \in V^h$ define from K to Ω.

We distinguish three cases to define the stabilized operator $R_{\hat{\rho}}$. Let $K \in T_h$ and $K \cap \Gamma_{1,C} \neq \emptyset$ then:

- if the intersection between K and Ω_1 is sufficiently large i.e. it exists $\hat{y}_K > 0$ such that $B(\hat{y}_K, \hat{\rho}) \subset T_K^{-1}(K \cap \Omega_1)$ (see Figure 3.1 *a*)), then $R_{\hat{\rho}}(v^h)|_K = \sigma_n(v_1^h|_K)$,
- otherwise, if it exists $\widetilde{K} \in S_K$ intersecting Ω_2 such that it exists $\hat{y}_{\widetilde{K}} > 0$ with $B(\hat{y}_{\widetilde{K}}, \hat{\rho}) \subset T_{\widetilde{K}}^{-1}(\widetilde{K} \cap \Omega_2)$ (see Figure 3.1 *b*)), then $R_{\hat{\rho}}(v^h)|_K = \sigma_n(E_{\widetilde{K}}(v_2^h) \circ \Pi)\, |\det(J_\Pi)|$,
- otherwise, we suppose that it exits a neighbor element K' of K such that it exists $\hat{y}_{K'} > 0$ with $B(\hat{y}_{K'}, \hat{\rho}) \subset T_{K'}^{-1}(K' \cap \Omega_1)$ (see Figure 3.1 *c*)), then $R_{\hat{\rho}}(v^h)|_K = \sigma_n(E_{K'}(v_1^h))$.



a) If $\Omega_1 \cap K$ is sufficiently large   b) If $\exists \tilde{K} \in S_K$ such that $\Omega_2 \cap \tilde{K}$ is sufficiently large   c) Otherwise

Figure 3.1. The different cases for the definition of $R_{\hat{\rho}}$.

In the same way, we define the operator $\overline{R}_{\hat{\rho}}$ on $\Gamma_{i,D}$ for $i = 1, 2$:

$$\overline{R}_{\hat{\rho}} : \begin{array}{ccl} V_i^h & \to & L^2(\Gamma_{i,D})^d \\ v_i & \mapsto & \overline{R}_{\hat{\rho}}(v_i^h) = \begin{cases} \sigma(v_i^h) n_i & \exists\ \hat{y}_K > 0 \text{ such that } B(\hat{y}_K, \hat{\rho}) \subset T_K^{-1}(K \cap \Omega_1) \\ \sigma(E_{K'}(v_i^h)) n_i & \text{otherwise.} \end{cases} \end{array}$$

Let us introduce the stabilized discrete linear operators:

$$P_\tau^{h,\hat{\rho}} : \begin{array}{ccl} V_1^h \times V_2^h & \to & L^2(\Gamma_{1,C}) \\ v & \mapsto & [\![v \cdot n]\!] - \tau R_{\hat{\rho}}(v), \end{array}$$

$$\overline{P}_{i,\tau}^{h,\hat{\rho}} : \begin{array}{ccl} V_i^h & \to & L^2(\Gamma_{i,D})^d \\ v_i & \mapsto & v_i - \tau \overline{R}_{\hat{\rho}}(v_i). \end{array}$$

We define the discrete form of $A_{\theta\gamma}(.,.)$ as follows:

$$A_{\theta\gamma}^h(u^h, v^h) = a(u^h, v^h) - \int_{\Gamma_{1,C}} \theta\gamma R_{\hat{\rho}}(u^h) R_{\hat{\rho}}(v^h)\ \mathrm{d}\Gamma - \sum_{i=1,2} \int_{\Gamma_{i,D}} \theta\gamma \overline{R}_{\hat{\rho}}(u_i^h) \overline{R}_{\hat{\rho}}(v_i^h)\ \mathrm{d}\Gamma.$$

The stabilized version of our approximation (3.6) reads:

$$\begin{cases} \text{Find } u^h \in V^h \text{ such that} \\ A^h_{\theta\gamma}(u^h, v^h) + \displaystyle\int_{\Gamma_{1,C}} \frac{1}{\gamma}[P^{h,\hat\rho}_{\gamma}(u^h) - g]_+ P^{h,\hat\rho}_{\theta\gamma}(v^h)\ d\Gamma \\ + \displaystyle\sum_{i=1,2} \int_{\Gamma_{i,D}} \frac{1}{\gamma}(\overline{P}^{h,\hat\rho}_{i,\gamma}(u^h_i) - u_{i,D}) \cdot \overline{P}^{h,\hat\rho}_{i,\gamma\theta}(v^h_i)\ d\Gamma = L(v^h) \qquad \forall v^h \in V^h. \end{cases} \tag{3.7}$$

Note that strict consistency of this stabilized discrete problem do not occur. However, we have the following result.

Theorem 3.6. *Let u be the solution to Problem* (2.1)-(2.3)*. Assume u is sufficiently regular (typically, $(u_1, u_2) \in H^{2+\nu}(\Omega_1)^d \times H^{2+\nu}(\Omega_2)^d$ for $\nu > 0$), then u is also a solution to the following problem:*

$$\begin{cases} a(u, v^h) - \displaystyle\int_{\Gamma_{1,C}} \theta\gamma\sigma_n(u) R_{\hat\rho}(v^h)\ d\Gamma - \sum_{i=1,2}\int_{\Gamma_{i,D}} \theta\gamma\sigma(u_i)n_i \cdot \overline{R}_{\hat\rho}(v^h_i)\ d\Gamma \\ + \displaystyle\int_{\Gamma_{1,C}} \frac{1}{\gamma}[P^h_\gamma(u) - g]_+ P^{h,\hat\rho}_{\theta\gamma}(v^h)\ d\Gamma \\ + \displaystyle\sum_{i=1,2}\int_{\Gamma_{i,D}} \frac{1}{\gamma}(\overline{P}^h_{i,\gamma}(u_i) - u_{i,D}) \cdot \overline{P}^{h,\hat\rho}_{i,\gamma\theta}(v^h_i)\ d\Gamma = L(v^h) \qquad \forall v^h \in V^h. \end{cases} \tag{3.8}$$

Proof. The proof can be straightforwardly deduced from the one of Theorem 3.4.

4. Analysis of the Nitsche-based approximation

4.1. Existence and uniqueness results

Theorem 4.1. *Let $\gamma = \gamma_0 h_K$. It exists a unique solution $v^h \in V^h$ to the discrete problem* (3.7)*, for all $\gamma_0 > 0$ if $\theta = -1$ and for $\gamma_0 > 0$ sufficiently small if $\theta \neq -1$.*

Proof. The proof is adapted from [7]. The main adaptations concern the fictitious domain framework and in particular the stabilization term, the consideration of two elastic solids and the semi-coercivity of the bilinear form due to the fact that Dirichlet conditions are taken into account with Nitsche's method. We begin by providing some stability and approximation property for operators $R_{\hat\rho}$ and $\overline{R}_{\hat\rho}$ in Lemmas 4.2, 4.5 and 4.6. Then a coercivity property is proved in Lemma 4.7. Finally, the existence and uniqueness result is deduce from the hemi-continuity of the non-linear operator which corresponds to (3.7).

Lemma 4.2. *Let $v^h \in V^h$, there exists a constant $C > 0$ independent of h such that*

$$\left\| R_{\hat\rho}(v^h) \right\|^2_{0,\Gamma_{1,C}} \leqslant Ch^{-1}(\left\| v^h_1 \right\|^2_{1,\Omega_1} + \left\| v^h_2 \right\|^2_{1,\Omega_2}) \qquad \forall v^h \in V^h. \tag{4.1}$$

The proof of this lemma is detailed in the appendix.

Remark 4.3. The following more general operator $R_{\hat\rho}$ could be considered:

$$R_{\hat\rho}(u^h)|_K = (1-t)\sigma_n(E_{K'}(u^h_2 \circ \Pi))\,|\det(J_\Pi)| + t\sigma_n(E_{K''}(u^h_1)),$$

with $t \in [0,1]$, the element K' being K itself or a neighbor element such as the intersection between K' and Ω_2 is large enough and the element K'' being K itself or a neighbor element such as the intersection between K'' and Ω_1 is large enough. Lemma 4.2 can be easily extended to this operator. When the elastic coefficients in Ω_1 and Ω_2 are equal, a proposed optimum choice is given by (see [1]):

$$t_K = \frac{\text{mes}(\Omega_1 \cap K)}{\text{mes}(\Omega_1 \cap K) + \text{mes}(\Omega_2 \cap K)}.$$

Remark 4.4. When the initial gap between the two bodies vanishes, for $\hat{\rho}$ sufficiently small either $K \cap \Omega_1$ or $K \cap \Omega_2$ is sufficiently large and thus it is not necessary to consider any neighbor element.

Lemma 4.5. *Let $u^h \in V^h$, $\Gamma_{i,D}$ be Lipschitz continuous then it exits a constant $C > 0$ independent of h such that*

$$\left\|\overline{R}_{\hat{\rho}}(u_1^h)\right\|_{0,\Gamma_{1,D}}^2 \leqslant Ch^{-1}\left\|u_1^h\right\|_{1,\Omega_1}^2,$$

and

$$\left\|\overline{R}_{\hat{\rho}}(u_2^h)\right\|_{0,\Gamma_{2,D}}^2 \leqslant Ch^{-1}\left\|u_2^h\right\|_{1,\Omega_2}^2.$$

The proof of this lemma can be straightforwardly deduced from the one of Lemma 4.2.

Now, Let $u^h,\ v^h \in V^h$ and $\gamma = h_K\gamma_0$ and using Lemma 4.2, it holds:

$$\left\|\gamma^{\frac{1}{2}}R_{\hat{\rho}}(u^h - v^h)\right\|_{0,\Gamma_{1,C}}^2 \leqslant C\gamma_0 \sum_{i=1,2}\left\|u_i^h - v_i^h\right\|_{k+1,\Omega_i}^2,$$

$$\left\|\gamma^{\frac{1}{2}}\overline{R}_{\hat{\rho}}(u_i^h - v_i^h)\right\|_{0,\Gamma_{i,D}}^2 \leqslant C\gamma_0\left\|u_i^h - v_i^h\right\|_{1,\Omega_i}^2.$$

Due to the know approximation properties of the stabilized operators on regular and quasi-uniform families of meshes (see [16]), one obtains the following lemma:

Lemma 4.6. *For any $v \in H^{k+1}(\Omega_1)^d \times H^{k+1}(\Omega_2)^d$*

$$\left\|R_{\hat{\rho}}(\Pi^h(v)) - \sigma_n(v)\right\|_{0,\Gamma_{1,C}}^2 \leqslant Ch^{2k-1}\sum_{i=1,2}\|v\|_{1,\Omega_i}^2,$$

and

$$\sum_{i=1,2}\left\|\overline{R}_{\hat{\rho}}(\Pi_i^h(v_i)) - \sigma(v_i)n_i\right\|_{0,\Gamma_{i,D}}^2 \leqslant Ch^{2k-1}\sum_{i=1,2}\|v\|_{k+1,\Omega_i}^2.$$

The following coercivity property can be stated :

Lemma 4.7. *For $M > 0$ fixed, it exists $\alpha > 0$ such that for all γ with $M \geqslant \gamma > 0$ the following coercivity property holds:*

$$a(v,v) + \frac{1}{2}\sum_{i=1,2}\int_{\Gamma_{i,D}}\gamma^{-1}v_i^2\ d\Gamma \geqslant \alpha\sum_{i=1,2}\|v_i\|_{1,\Omega_i}^2 \qquad \forall v \in V. \tag{4.2}$$

The proof of this lemma is detailed in the appendix. Now, by defining the following operator B^h from V^h to V^h:

$$(B^h u^h, v^h)_{1,\Omega} = A_{\theta\gamma}^h(u^h, v^h) + \sum_{i=1,2}\int_{\Gamma_{i,D}}\frac{1}{\gamma}(\overline{P}_{i,\gamma}^{h,\hat{\rho}}(u_i^h) - u_{i,D})\cdot\overline{P}_{i,\gamma\theta}^{h,\hat{\rho}}(v_i^h)\ \mathrm{d}\Gamma$$
$$+\int_{\Gamma_{1,C}}\frac{1}{\gamma}[P_\gamma^{h,\hat{\rho}}(u^h) - g]_+ P_{\theta\gamma}^{h,\hat{\rho}}(v^h)\ \mathrm{d}\Gamma \qquad \forall u^h, v^h \in V^h, \tag{4.3}$$

it is sufficient to prove that B^h is hemi-continuous (see the Corollary 15 p. 126 of [3]) to end the proof of Theorem 4.1. The proof of hemi-continuity of B^h is detailed in the appendix. □

4.2. **A priori Error analysis**

In this section some optimal *a priori* error estimates are proved for the problem under consideration. The rate of convergence is the same as for standard finite element methods.

Theorem 4.8. *Let u be a solution of the stabilized problem (2.4) belonging to $H^{\frac{3}{2}+\nu}(\Omega_1)^d \times H^{\frac{3}{2}+\nu}(\Omega_2)^d$ with $\nu > 0$.*

(1) *If $\theta \neq -1$, we suppose $\gamma_0 > 0$ is sufficiently small. The solution u^h of the stabilized problem (3.7) satisfies the following error estimate:*

$$\sum_{i=1,2} \left\| u_i - u_i^h \right\|_{1,\Omega_i}^2 + \left\| \gamma^{\frac{1}{2}}(\sigma_n(u) + \frac{1}{\gamma}[P_\gamma^{h,\hat\rho}(u^h) - g]_+) \right\|_{0,\Gamma_{1,C}}^2 + \sum_{i=1,2} \left\| \gamma^{-\frac{1}{2}}(\overline{P}_{i,\gamma}^h(u_i) - \overline{P}_{i,\gamma}^{h,\hat\rho}(u_i)) \right\|_{0,\Gamma_{i,D}}^2$$
$$\leqslant C \inf_{v^h \in V^h} \Big(\sum_{i=1,2} \left\| u_i - v_i^h \right\|_{1,\Omega_i}^2 + \left\| \gamma^{-\frac{1}{2}}(u - v^h) \right\|_{0,\Gamma_{1,C}}^2 + \left\| \gamma^{\frac{1}{2}}(\sigma_n(u) - R_{\hat\rho}(v^h)) \right\|_{0,\Gamma_{1,C}}^2$$
$$+ \sum_{i=1,2} \left\| \gamma^{-\frac{1}{2}}(u_i - v_i^h) \right\|_{0,\Gamma_{i,D}}^2 + \sum_{i=1,2} \left\| \gamma^{\frac{1}{2}}(\sigma(u_i) - \overline{R}_{\hat\rho}(v_i^h)) \right\|_{0,\Gamma_{i,D}}^2 \Big) \quad (4.4)$$

with $C > 0$ a constant independent of h, u and γ_0.

(2) *If $\theta = -1$, then for all $\gamma_0 > 0$, the solution u^h of the stabilized problem (3.7) satisfies the error estimate (4.4) with $C > 0$ a constant independent of h and u.*

Proof. The proof is also an adaptation to our fictitious domain framework of the one in [7]. Let $v^h \in V^h$, using the coercivity inequality (4.2) and continuity of the form $a(.,.) + \frac{1}{2} \sum_{i=1,2} \int_{\Gamma_{i,D}} \gamma^{-1}(.)^2 \, d\Gamma$ and Young's inequality, it holds:

$$\begin{aligned}
\alpha \sum_{i=1,2} \left\| u_i - u_i^h \right\|_{1,\Omega_i}^2 &\leqslant a(u - u^h, u - u^h) + \frac{1}{2} \sum_{i=1,2} \int_{\Gamma_{i,D}} \gamma^{-1}(u_i - u_i^h)^2 \, d\Gamma, \\
&= a(u - u^h, (u - v^h) + (v^h - u^h)) + \frac{1}{2} \sum_{i=1,2} \left\| \gamma^{-\frac{1}{2}}(u_i - u_i^h) \right\|_{0,\Gamma_{i,D}}^2, \\
&\leqslant C \sum_{i=1,2} \left\| u_i - u_i^h \right\|_{1,\Omega_i} \left\| u_i - v_i^h \right\|_{1,\Omega_i} + a(u - u^h, v^h - u^h) \\
&\quad + \frac{1}{2} \sum_{i=1,2} \left\| \gamma^{-\frac{1}{2}}(u_i - u_i^h) \right\|_{0,\Gamma_{i,D}}^2, \\
&\leqslant \frac{\alpha}{2} \sum_{i=1,2} \left\| u_i - u_i^h \right\|_{1,\Omega_i}^2 + \frac{C^2}{2\alpha} \sum_{i=1,2} \left\| u_i - v_i^h \right\|_{1,\Omega_i}^2 \\
&\quad + \frac{1}{2} \sum_{i=1,2} \left\| \gamma^{-\frac{1}{2}}(u_i - u_i^h) \right\|_{0,\Gamma_{i,D}}^2 + a(u, v^h - u^h) - a(u^h, v^h - u^h).
\end{aligned}$$

Hence

$$\frac{\alpha}{2} \sum_{i=1,2} \left\| u_i - u_i^h \right\|_{1,\Omega_i}^2 \leqslant \frac{C^2}{2\alpha} \sum_{i=1,2} \left\| u_i - v_i^h \right\|_{1,\Omega_i}^2 + \frac{1}{2} \sum_{i=1,2} \left\| \gamma^{-\frac{1}{2}}(u_i - u_i^h) \right\|_{0,\Gamma_{i,D}}^2$$
$$+ a(u, v^h - u^h) - a(u^h, v^h - u^h). \quad (4.5)$$

Let u be the solution to (2.4), it verifies the stabilized formulation (3.8), thus we have:

$$\begin{aligned} a(u, v^h - u^h) - a(u^h, v^h - u^h) = &\int_{\Gamma_{1,C}} \theta\gamma(\sigma_n(u) - R_{\hat\rho}(u^h)) R_{\hat\rho}(v^h - u^h) \, \mathrm{d}\Gamma \\ &- \sum_{i=1,2} \int_{\Gamma_{i,D}} \frac{1}{\gamma}(\overline{P}^h_{i,\gamma}(u_i) - \overline{P}^{h,\hat\rho}_{i,\gamma}(u_i^h)) \cdot \overline{P}^{h,\hat\rho}_{i,\gamma\theta}(v_i^h - u_i^h) \, \mathrm{d}\Gamma \\ &+ \sum_{i=1,2} \int_{\Gamma_{i,D}} \theta\gamma(\sigma(u_i)n_i - \overline{R}_{\hat\rho}(u_i^h)) \cdot \overline{R}_{\hat\rho}(v_i^h - u_i^h) \, \mathrm{d}\Gamma \\ &+ \int_{\Gamma_{1,C}} \frac{1}{\gamma}([P^{h,\hat\rho}_\gamma(u^h) - g]_+ - [P^h_\gamma(u) - g]_+) P^{h,\hat\rho}_{\theta\gamma}(v^h - u^h) \, \mathrm{d}\Gamma. \end{aligned} \tag{4.6}$$

First, using Cauchy-Schwarz and Young's inequalities for $\beta_1 > 0$, it holds:

$$\begin{aligned} &\int_{\Gamma_{1,C}} \theta\gamma(\sigma_n(u) - R_{\hat\rho}(u^h)) R_{\hat\rho}(v^h - u^h) \, \mathrm{d}\Gamma \\ &\qquad \leqslant \frac{\theta^2}{2\beta_1} \left\| \gamma^{\frac{1}{2}}(\sigma_n(u) - R_{\hat\rho}(v^h)) \right\|^2_{0,\Gamma_{1,C}} + (\theta + \frac{\beta_1}{2}) \left\| \gamma^{\frac{1}{2}} R_{\hat\rho}(v^h - u^h) \right\|^2_{0,\Gamma_{1,C}}. \end{aligned} \tag{4.7}$$

For all $a, b \in \mathbb{R}$, we have the following estimate:

$$([a]_+ - [b]_+)(b - a) \leqslant -([a]_+ - [b]_+)^2. \tag{4.8}$$

Then, set:

$$\begin{aligned} \tau_1 &= \int_{\Gamma_{1,C}} \frac{1}{\gamma}([P^{h,\hat\rho}_\gamma(u^h) - g]_+ - [P^h_\gamma(u) - g]_+) P^{h,\hat\rho}_{\theta\gamma}(v^h - u^h) \, \mathrm{d}\Gamma, \\ &= \int_{\Gamma_{1,C}} (\sigma_n(u) + \frac{1}{\gamma}[P^{h,\hat\rho}_\gamma(u^h) - g]_+) P^{h,\hat\rho}_{\theta\gamma}(v^h - u^h) \, \mathrm{d}\Gamma. \end{aligned}$$

Using Cauchy-Schwarz and Young's inequalities for $\beta_2 > 0$ and $\beta_3 > 0$ and (4.8), it holds:

$$\begin{aligned} \tau_1 \leqslant &\Big(-1 + \frac{1}{2\beta_2} + \frac{|1-\theta|}{2\beta_3} \Big) \left\| \gamma^{\frac{1}{2}}(\sigma_n(u) + \frac{1}{\gamma}[P^{h,\hat\rho}_\gamma(u^h) - g]_+) \right\|^2_{0,\Gamma_{1,C}} \\ &+ \frac{\beta_2}{2} \left\| \gamma^{-\frac{1}{2}}(P^{h,\hat\rho}_\gamma(v^h) - P^h_\gamma(u)) \right\|^2_{0,\Gamma_{1,C}} + \frac{|1-\theta|\,\beta_3}{2} \left\| \gamma^{\frac{1}{2}} R_{\hat\rho}(v^h - u^h) \right\|^2_{0,\Gamma_{1,C}}. \end{aligned} \tag{4.9}$$

Moreover, set:

$$\begin{aligned} \tau_2 = &- \sum_{i=1,2} \left\| \gamma^{-\frac{1}{2}}(u_i - u_i^h) \right\|^2_{0,\Gamma_{i,D}} - \sum_{i=1,2} \int_{\Gamma_{i,D}} \gamma^{-1}(v_i^h - u_i) \cdot (\overline{P}^h_{i,\gamma}(u_i) - \overline{P}^{h,\rho}_{i,\gamma}(u_i^h)) \, \mathrm{d}\Gamma \\ &+ \sum_{i=1,2} \int_{\Gamma_{i,D}} (u_i - u_i^h) \cdot (\sigma(u_i)n_i - \overline{R}_{\hat\rho}(v_i^h)) \, \mathrm{d}\Gamma \\ &+ (1+\theta) \sum_{i=1,2} \int_{\Gamma_{i,D}} (u_i - u_i^h) \cdot (\overline{R}_{\hat\rho}(v_i^h - u_i^h)) \, \mathrm{d}\Gamma. \end{aligned} \tag{4.10}$$

Using Young's inequality for $\beta_4 > 0$, it holds:

$$
\begin{aligned}
-\sum_{i=1,2}\int_{\Gamma_{i,D}} & \gamma^{-1}(v_i^h - u_i)\cdot(\overline{P}_{i,\gamma}^h(u_i) - \overline{P}_{i,\gamma}^{h,\rho}(u_i^h))\ \mathrm{d}\Gamma, \\
&\leqslant \frac{\beta_4}{2}\sum_{i=1,2}\left\|\gamma^{-\frac{1}{2}}(v_i^h - u_i)\right\|_{0,\Gamma_{i,D}}^2 + \frac{1}{2\beta_4}\sum_{i=1,2}\left\|\gamma^{-\frac{1}{2}}(\overline{P}_{i,\gamma}^h(u_i) - \overline{P}_{i,\gamma}^{h,\rho}(u_i^h))\right\|_{0,\Gamma_{i,D}}^2, \\
&\leqslant \frac{\beta_4}{2}\sum_{i=1,2}\left\|\gamma^{-\frac{1}{2}}(v_i^h - u_i)\right\|_{0,\Gamma_{i,D}}^2 + \frac{1}{\beta_4}\sum_{i=1,2}\left\|\gamma^{-\frac{1}{2}}(u_i - u_i^h)\right\|_{0,\Gamma_{i,D}}^2 \\
&\quad + \frac{1}{\beta_4}\sum_{i=1,2}\left\|\gamma^{\frac{1}{2}}(\sigma(u_i)n_i - \overline{R}_{\hat{\rho}}(u_i^h))\right\|_{0,\Gamma_{i,D}}^2,
\end{aligned}
\tag{4.11}
$$

and for $\beta_5 > 0$

$$
\begin{aligned}
\sum_{i=1,2}\int_{\Gamma_{i,D}} & (u_i - u_i^h)\cdot(\sigma(u_i)n_i - \overline{R}_{\hat{\rho}}(v_i^h))\ \mathrm{d}\Gamma \\
&\leqslant \frac{1}{2\beta_5}\sum_{i=1,2}\left\|\gamma^{-\frac{1}{2}}(u_i - u_i^h)\right\|_{0,\Gamma_{i,D}}^2 + \frac{\beta_5}{2}\sum_{i=1,2}\left\|\gamma^{\frac{1}{2}}(\sigma(u_i)n_i - \overline{R}_{\hat{\rho}}(v_i^h))\right\|_{0,\Gamma_{i,D}}^2,
\end{aligned}
\tag{4.12}
$$

and for $\beta_6 > 0$

$$
\begin{aligned}
(1+\theta)\sum_{i=1,2}\int_{\Gamma_{i,D}} & (u_i - u_i^h)\cdot(\overline{R}_{\hat{\rho}}(v_i^h - u_i^h))\ \mathrm{d}\Gamma \\
&\leqslant \frac{|1+\theta|}{2\beta_6}\sum_{i=1,2}\left\|\gamma^{-\frac{1}{2}}(u_i - u_i^h)\right\|_{0,\Gamma_{i,D}}^2 + \frac{\beta_6\,|1+\theta|}{2}\sum_{i=1,2}\left\|\gamma^{\frac{1}{2}}(\overline{R}_{\hat{\rho}}(v_i^h - u_i^h))\right\|_{0,\Gamma_{i,D}}^2.
\end{aligned}
\tag{4.13}
$$

Using inequalities (4.11), (4.12) and (4.13) in $\tau_2 + \frac{1}{2}\sum_{i=1,2}\left\|\gamma^{-\frac{1}{2}}(u_i - u_i^h)\right\|_{0,\Gamma_{i,D}}^2$, it holds:

$$
\begin{aligned}
\tau_2 + \frac{1}{2}\sum_{i=1,2} & \left\|\gamma^{-\frac{1}{2}}(u_i - u_i^h)\right\|_{0,\Gamma_{i,D}}^2 \\
&\leqslant \Big(-\frac{1}{2} + \frac{1}{\beta_4} + \frac{1}{2\beta_5} + \frac{|1+\theta|}{2\beta_6}\Big)\sum_{i=1,2}\left\|\gamma^{-\frac{1}{2}}(u_i - u_i^h)\right\|_{0,\Gamma_{i,D}}^2 \\
&\quad + \frac{\beta_4}{2}\sum_{i=1,2}\left\|\gamma^{-\frac{1}{2}}(v_i^h - u_i)\right\|_{0,\Gamma_{i,D}}^2 + \frac{1}{\beta_4}\sum_{i=1,2}\left\|\gamma^{\frac{1}{2}}(\sigma(u_i)n_i - \overline{R}_{\hat{\rho}}(u_i^h))\right\|_{0,\Gamma_{i,D}}^2 \\
&\quad + \frac{\beta_5}{2}\sum_{i=1,2}\left\|\gamma^{\frac{1}{2}}(\sigma(u_i)n_i - \overline{R}_{\hat{\rho}}(v_i^h))\right\|_{0,\Gamma_{i,D}}^2 + \frac{\beta_6\,|1+\theta|}{2}\sum_{i=1,2}\left\|\gamma^{\frac{1}{2}}(\overline{R}_{\hat{\rho}}(v_i^h - u_i^h))\right\|_{0,\Gamma_{i,D}}^2.
\end{aligned}
\tag{4.14}
$$

Gathering (4.6), (4.7), (4.9) and (4.14) in (4.5), it holds:

$$
\begin{aligned}
\frac{\alpha}{2}\sum_{i=1,2}\left\|u_i-u_i^h\right\|^2_{1,\Omega_i} \leqslant\ & \frac{C^2}{2\alpha}\sum_{i=1,2}\left\|u_i-v_i^h\right\|^2_{1,\Omega_i}\\
&+\frac{\theta^2}{2\beta_1}\left\|\gamma^{\frac{1}{2}}(\sigma_n(u)-R_{\hat\rho}(v^h))\right\|^2_{0,\Gamma_{1,C}}\\
&+(\theta+\frac{\beta_1}{2}\frac{|1-\theta|\,\beta_3}{2})\left\|\gamma^{\frac{1}{2}}R_{\hat\rho}(v^h-u^h)\right\|^2_{0,\Gamma_{1,C}}\\
&+\Big(-1+\frac{1}{2\beta_2}+\frac{|1-\theta|}{2\beta_3}\Big)\left\|\gamma^{\frac{1}{2}}(\sigma_n(u)+\frac{1}{\gamma}[P_\gamma^{h,\rho}(u^h)-g]_+)\right\|^2_{0,\Gamma_{1,C}}\\
&+\frac{\beta_2}{2}\left\|\gamma^{-\frac{1}{2}}(P_\gamma^{h,\rho}(v^h)-P_\gamma^h(u))\right\|^2_{0,\Gamma_{1,C}}\\
&+\Big(-\frac{1}{2}+\frac{1}{\beta_4}+\frac{1}{2\beta_5}+\frac{|1+\theta|}{2\beta_6}\Big)\sum_{i=1,2}\left\|\gamma^{-\frac{1}{2}}(u_i-u_i^h)\right\|^2_{0,\Gamma_{i,D}}\\
&+\frac{\beta_4}{2}\sum_{i=1,2}\left\|\gamma^{-\frac{1}{2}}(v_i^h-u_i)\right\|^2_{0,\Gamma_{i,D}}\\
&+\frac{1}{\beta_4}\sum_{i=1,2}\left\|\gamma^{\frac{1}{2}}(\sigma(u_i)n_i-\overline{R}_{\hat\rho}(u_i^h))\right\|^2_{0,\Gamma_{i,D}}\\
&+\frac{\beta_5}{2}\sum_{i=1,2}\left\|\gamma^{\frac{1}{2}}(\sigma(u_i)n_i-\overline{R}_{\hat\rho}(v_i^h))\right\|^2_{0,\Gamma_{i,D}}\\
&+\frac{\beta_6\,|1+\theta|}{2}\sum_{i=1,2}\left\|\gamma^{\frac{1}{2}}(\overline{R}_{\hat\rho}(v_i^h-u_i^h))\right\|^2_{0,\Gamma_{i,D}}.
\end{aligned}
\tag{4.15}
$$

Using Lemmas 4.2 and 4.5, we obtain:

$$
\begin{aligned}
\left\|\gamma^{\frac{1}{2}}R_{\hat\rho}(v^h-u^h)\right\|^2_{0,\Gamma_{1,C}} &\leqslant C_1\gamma_0\sum_{i=1,2}\left\|v_i^h-u_i^h\right\|^2_{1,\Omega_i},\\
&\leqslant 2C_1\gamma_0(\sum_{i=1,2}\left\|v_i^h-u_i\right\|^2_{1,\Omega_i}+\sum_{i=1,2}\left\|u_i-u_i^h\right\|^2_{1,\Omega_i}),\\
\sum_{i=1,2}\left\|\gamma^{\frac{1}{2}}R_{\hat\rho}(v_i^h-u_i^h)\right\|^2_{0,\Gamma_{i,D}} &\leqslant C_2\gamma_0\sum_{i=1,2}\left\|v_i^h-u_i^h\right\|^2_{1,\Omega_i},\\
&\leqslant 2C_2\gamma_0(\sum_{i=1,2}\left\|v_i^h-u_i\right\|^2_{1,\Omega_i}+\sum_{i=1,2}\left\|u_i-u_i^h\right\|^2_{1,\Omega_i}),
\end{aligned}
$$

and we know

$$
\left\|\gamma^{-\frac{1}{2}}P_\gamma^{h,\rho}(v^h)-P_\gamma^h(u)\right\|^2_{0,\Gamma_{1,C}}\leqslant 2\left\|\gamma^{-\frac{1}{2}}(v^h-u)\right\|^2_{0,\Gamma_{1,C}}+2\left\|\gamma^{\frac{1}{2}}(R_{\hat\rho}(v^h)-\sigma_n(u))\right\|^2_{0,\Gamma_{1,C}},
$$

and using Young's inequality for $\beta>0$, it holds:

$$
\begin{aligned}
-\sum_{i=1,2}\left\|\gamma^{-\frac{1}{2}}(u_i-u_i^h)\right\|^2_{0,\Gamma_{i,D}} \leqslant &-(1-\frac{1}{\beta})\sum_{i=1,2}\left\|\gamma^{\frac{1}{2}}\sigma(u_i)n_i-R_{\hat\rho}(u_i^h)\right\|^2_{0,\Gamma_{i,D}}\\
&-(1-\beta)\sum_{i=1,2}\left\|\gamma^{-\frac{1}{2}}\overline{P}_\gamma^h(u)-\overline{P}_\gamma^{h,\rho}(u^h)\right\|^2_{0,\Gamma_{i,D}}.
\end{aligned}
\tag{4.16}
$$

Let $\theta\in\mathbb{R}$ be fixed, if β_2, β_3, β_4, β_5 and β_6 are chosen sufficiently large such that:

$$
-1+\frac{1}{2\beta_2}+\frac{|1-\theta|}{2\beta_3}<-\frac{1}{2},
$$

and

$$-\frac{1}{2}+\frac{1}{\beta_4}+\frac{1}{2\beta_5}+\frac{|1+\theta|}{2\beta_6}<-\frac{1}{4}.$$

And if γ_0 is sufficiently small and $\beta<1$, we get the inequality (4.4).

In the case $\theta=-1$, thanks to (4.15), it holds:

$$\begin{aligned}\frac{\alpha}{2}\sum_{i=1,2}\left\|u_i-u_i^h\right\|_{1,\Omega_i}^2 \leqslant{}& \frac{C^2}{2\alpha}\sum_{i=1,2}\left\|u_i-v_i^h\right\|_{1,\Omega_i}^2\\
&+\frac{1}{2\beta_1}\left\|\gamma^{\frac{1}{2}}(\sigma_n(u)-R_{\hat{\rho}}(v^h))\right\|_{0,\Gamma_{1,C}}^2\\
&+(-1+\frac{\beta_1}{2}+\beta_3)\left\|\gamma^{\frac{1}{2}}R_{\hat{\rho}}(v^h-u^h)\right\|_{0,\Gamma_{1,C}}^2\\
&+\Big(-1+\frac{1}{2\beta_2}+\frac{1}{\beta_3}\Big)\left\|\gamma^{\frac{1}{2}}(\sigma_n(u)+\frac{1}{\gamma}[P_\gamma(u^h)-g]_+)\right\|_{0,\Gamma_{1,C}}^2\\
&+\frac{\beta_2}{2}\left\|\gamma^{-\frac{1}{2}}P_\gamma^{h,\rho}(v^h)-P_\gamma^h(u)\right\|_{0,\Gamma_{1,C}}^2\\
&+\Big(-\frac{1}{2}+\frac{1}{\beta_4}+\frac{1}{2\beta_5}\Big)\sum_{i=1,2}\left\|\gamma^{-\frac{1}{2}}(u_i-u_i^h)\right\|_{0,\Gamma_{i,D}}^2\\
&+\frac{\beta_4}{2}\sum_{i=1,2}\left\|\gamma^{-\frac{1}{2}}(v_i^h-u_i)\right\|_{0,\Gamma_{i,D}}^2+\frac{1}{\beta_4}\sum_{i=1,2}\left\|\gamma^{\frac{1}{2}}(\sigma(u_i)n_i-\overline{R}_{\hat{\rho}}(u_i^h))\right\|_{0,\Gamma_{i,D}}^2\\
&+\frac{\beta_5}{2}\sum_{i=1,2}\left\|\gamma^{\frac{1}{2}}(\sigma(u_i)n_i-\overline{R}_{\hat{\rho}}(v_i^h))\right\|_{0,\Gamma_{i,D}}^2.\end{aligned}\tag{4.17}$$

Let $\eta_1>0$ and $\eta_2>0$, we take $\beta_1=2\eta_1$, $\beta_2=1+1/\eta_1$, $\beta_3=1+\eta_1$, $\beta_4=2(1+\eta_2)$, $\beta_5=2(1+1/\eta_2)$, then it holds:

$$\begin{aligned}\frac{\alpha}{2}\sum_{i=1,2}\left\|u_i-u_i^h\right\|_{1,\Omega_i}^2 \leqslant{}& \frac{C^2}{2\alpha}\sum_{i=1,2}\left\|u_i-v_i^h\right\|_{1,\Omega_i}^2\\
&+\frac{1}{4\eta_1}\left\|\gamma^{\frac{1}{2}}(\sigma_n(u)-R_{\hat{\rho}}(v^h))\right\|_{0,\Gamma_{1,C}}^2+2\eta_1\left\|\gamma^{\frac{1}{2}}R_{\hat{\rho}}(v^h-u^h)\right\|_{0,\Gamma_{1,C}}^2\\
&-\frac{\eta_1}{2(1+\eta_1)}\left\|\gamma^{\frac{1}{2}}(\sigma_n(u)+\frac{1}{\gamma}[P_\gamma(u^h)-g]_+)\right\|_{0,\Gamma_{1,C}}^2\\
&+\frac{1+\eta_1}{2\eta_1}\left\|\gamma^{-\frac{1}{2}}P_\gamma^{h,\rho}(v^h)-P_\gamma^h(u)\right\|_{0,\Gamma_{1,C}}^2\\
&-\frac{\eta_2}{4(1+\eta_2)}\sum_{i=1,2}\left\|\gamma^{-\frac{1}{2}}(u_i-u_i^h)\right\|_{0,\Gamma_{i,D}}^2\\
&+(1+\eta_2)\sum_{i=1,2}\left\|\gamma^{-\frac{1}{2}}(v_i^h-u_i)\right\|_{0,\Gamma_{i,D}}^2\\
&+\frac{1}{2(1+\eta_2)}\sum_{i=1,2}\left\|\gamma^{\frac{1}{2}}(\sigma(u_i)n_i-\overline{R}_{\hat{\rho}}(u_i^h))\right\|_{0,\Gamma_{i,D}}^2\\
&+(1+1/\eta_2)\sum_{i=1,2}\left\|\gamma^{\frac{1}{2}}(\sigma(u_i)n_i-\overline{R}_{\hat{\rho}}(v_i^h))\right\|_{0,\Gamma_{i,D}}^2.\end{aligned}$$

Using (4.16) and $\beta = \dfrac{\eta_2}{2+\eta_2} < 1$, we have:

$$-\frac{\eta_2}{4(1+\eta_2)}\sum_{i=1,2}\left\|\gamma^{-\frac{1}{2}}(u_i - u_i^h)\right\|_{0,\Gamma_{i,D}}^2 \leqslant \frac{1}{2(1+\eta_2)}\sum_{i=1,2}\left\|\gamma^{\frac{1}{2}}\sigma(u_i)n_i - R_{\hat{\rho}}(u_i^h)\right\|_{0,\Gamma_{i,D}}^2 - \frac{\eta_2}{2(1+\eta_2)(2+\eta_2)}\sum_{i=1,2}\left\|\gamma^{-\frac{1}{2}}\overline{P}_{\gamma}^{h}(u) - \overline{P}_{\gamma}^{h,\rho}(u^h)\right\|_{0,\Gamma_{i,D}}^2,$$

and

$$\sum_{i=1,2}\left\|\gamma^{\frac{1}{2}}\sigma(u_i)n_i - R_{\hat{\rho}}(u_i^h)\right\|_{0,\Gamma_{i,D}}^2 \leqslant 2\sum_{i=1,2}\left\|\gamma^{\frac{1}{2}}\sigma(u_i)n_i - R_{\hat{\rho}}(v_i^h)\right\|_{0,\Gamma_{i,D}}^2 + 2\sum_{i=1,2}\left\|\gamma^{\frac{1}{2}}\sigma(R_{\hat{\rho}}(v_i^h - u_i^h)\right\|_{0,\Gamma_{i,D}}^2.$$

Let γ_0 be positive. If we take $\eta_1 = \alpha/(32C_1\gamma_0)$ and $\eta_2 = C_2\gamma_0/(32\alpha)$, then we get the inequality (4.4). This ends the proof of Theorem 4.8 □

Theorem 4.9. *Let u be a solution of the variational problem* (2.4)*. Suppose that u belongs to $(H^{\frac{3}{2}+\nu}(\Omega_1))^d \times (H^{\frac{3}{2}+\nu}(\Omega_2))^d$ with $1/2 \geqslant \nu > 0$ if $k=1$ and with $1 > \nu > 0$ if $k=2$. Then, if additionally $\gamma_0 > 0$ is sufficiently small when $\theta \neq -1$, the solution u^h of the stabilize problem* (3.6) *satisfies the following a priori error estimate:*

$$\sum_{i=1,2}\left\|u_i - u_i^h\right\|_{1,\Omega_i}^2 + \left\|\gamma^{\frac{1}{2}}(\sigma_n(u) + \frac{1}{\gamma}[P_{\gamma}^{h,\hat{\rho}}(u^h) - g]_+)\right\|_{0,\Gamma_{1,C}}^2 + \sum_{i=1,2}\left\|\gamma^{-\frac{1}{2}}(\overline{P}_{i,\gamma}^{h,\hat{\rho}}(u_i^h) - \overline{P}_{i,\gamma}^{h}(u_i))\right\|_{0,\Gamma_{i,D}}^2 \leqslant Ch^{1+2\nu}\sum_{i=1,2}\|u\|_{\frac{3}{2}+\nu,\Omega_i}^2 \quad (4.18)$$

with $C > 0$ a constant independent of h and u.

Proof. Now let us establish the inequality (4.18). Set $v_i^h = \Pi_i^h(u_i)$, we have the following estimates:

$$\left\|u_i - \Pi_i^h(u_i)\right\|_{m,\Omega_i} \leqslant Ch^{k+1-m}\|u_i\|_{k+1,\Omega_i},$$

$$\left\|R_{\hat{\rho}}(\Pi^h(u)) - \sigma_n(u)\right\|_{0,\Gamma_{1,C}}^2 \leqslant Ch^{2k-1}\sum_{i=1,2}\|u\|_{k+1,\Omega_i}^2,$$

and

$$\sum_{i=1,2}\left\|\overline{R}_{\hat{\rho}}(\Pi_i^h(u_i)) - \sigma(u_i)n_i\right\|_{0,\Gamma_{i,D}}^2 \leqslant Ch^{2k-1}\sum_{i=1,2}\|u\|_{k+1,\Omega_i}^2.$$

If we replace v_i^h by $\Pi_i^h(u_i)$ in (4.4), $\gamma = \gamma_0 h$ and we use the previous inequalities, we get (4.18). We can write:

$$\sum_{i=1,2}\left\|u_i - u_i^h\right\|_{1,\Omega_i} + \left\|\gamma^{\frac{1}{2}}(\sigma_n(u) + \frac{1}{\gamma}[P_{\gamma}^{h,\hat{\rho}}(u^h) - g]_+)\right\|_{0,\Gamma_{1,C}} + \sum_{i=1,2}\left\|\gamma^{-\frac{1}{2}}(\overline{P}_{i,\gamma}^{h,\hat{\rho}}(u_i^h) - \overline{P}_{i,\gamma}^{h}(u_i))\right\|_{0,\Gamma_{i,D}} \leqslant Ch^{1/2+\nu}\sum_{i=1,2}\|u\|_{\frac{3}{2}+\nu,\Omega_i} \quad (4.19)$$

□

5. Numerical study

This section is dedicated to some numerical experiments with isoparametric Lagrange $P1$ or $P2$ finite element methods. The accuracy of the method is discussed for the different cases with respect to the finite element used, the mesh size and the value of the parameter γ_0. Note that the following results are obtained without the stabilization introduced in Section 3.4. From a numerical viewpoint, the stabilization seems not strictly necessary to obtain an optimal rate of convergence. This has already been observed in a linear case in [16]. The numerical tests in two dimensions (resp. three dimensions) are performed on a fictitious domain $\Omega =]-0.5, 0.5[^2$ (resp. $\Omega =]-0.5, 0.5[^3$ which contains the first body Ω_1, a circle of radius 0.25 and center $(0, 0)$ (resp. a sphere of radius 0.25 and center $(0, 0, 0)$), and the second $\Omega_2 =]-0.5, 0.5[\times]-0.5, -0.25[$ (resp. $\Omega_2 =]-0.5, 0.5[^2\times]-0.5, -0.25[$). A Dirichlet condition is prescribed on the bottom of the rectangle (resp. parallelepiped).

The projector Π is defined from the lower part of the boundary of Ω_1 (i.e. for $\Gamma_{1,C} = \{x \in \partial\Omega_1 : x_d \leq 0\}$) onto its projection on the top boundary of Ω_2. All remaining parts of the boundaries of Ω_1 and Ω_2 are considered traction free.

Since no Dirichlet condition is applied on Ω_1, the problem is only semi-coercive. In order to recover the uniqueness of the solution, it is needed to prescribe the horizontal rigid translation in 2D and two horizontal translations and one rotation in 3D. This is done by prescribing the displacement on some given convenient points.

We use a generalized Newton's method to solve the discrete problem (3.6) (see [27] for more details) and our finite element library GetFEM++[1]. The tool for fictitious domain methods of GetFEM++ has been used which provides cut integration methods. The geometries are described with zero level sets of some signed distances to the domain boundaries. The distance functions are approximated by quadratic Lagrange finite elements. In order to build cut integration methods, each element of the mesh which crosses a domain boundary is cut into a set of sub-elements conforming to this boundary. Then, an integration method is produced on each sub-element lying on the interior of a domain and on each sub-element boundary lying on a domain boundary. In order to obtain a convenient order for the produced integration methods and for the approximation of the domain boundaries, curved sub-elements are used.

Moreover, no specific treatment have been considered for the fact that boundary terms for the contact condition approximated by Nitsche's method is non-regular (due to the positive part). We used an order four numerical integration method on each sub-element and we noted no improvement of the accuracy with higher order or refined numerical integration method.

For simplicity, we consider a dimensionless situation with Lamé coefficients $\lambda = 1$ and $\mu = 1$ and a vertical volume density of force -0.1.

The situation studied is not strictly speaking of Hertz type due to the fact that Ω_2 is bounded. The expression of the exact solution being unknown, the convergence is studied with respect to a reference solution computed with a P_2 isoparametric element on a very fine mesh ($h = 1/200$ in 2D and $h = 1/30$ in 3D) with the skew-symmetric method $\theta = -1$ (see Figures 5.1 and 5.2).

5.1. Numerical convergence in the two dimensional case

We perform a numerical convergence study on the three methods $\theta = 1$, $\theta = 0$ and $\theta = -1$ for a fixed parameter $\gamma_0 = 1/200$ (chosen small in order to have the convergence for the three cases). On Figures 5.3, 5.4 and 5.5, the relative error in percentage in L^2 and H^1-norms on each bodies for P_1 Lagrange finite elements are plotted. As expected the optimal convergence is obtained in H^1-norm for all methods in good accordance with Theorem 4.9. The rate of convergence in L^2-norm is slightly sub-optimal on Ω_2 if one refers to Aubin-Nitsche lemma in the linear case. However, such a result is

[1] see `http://download.gna.org/getfem/html/homepage/`

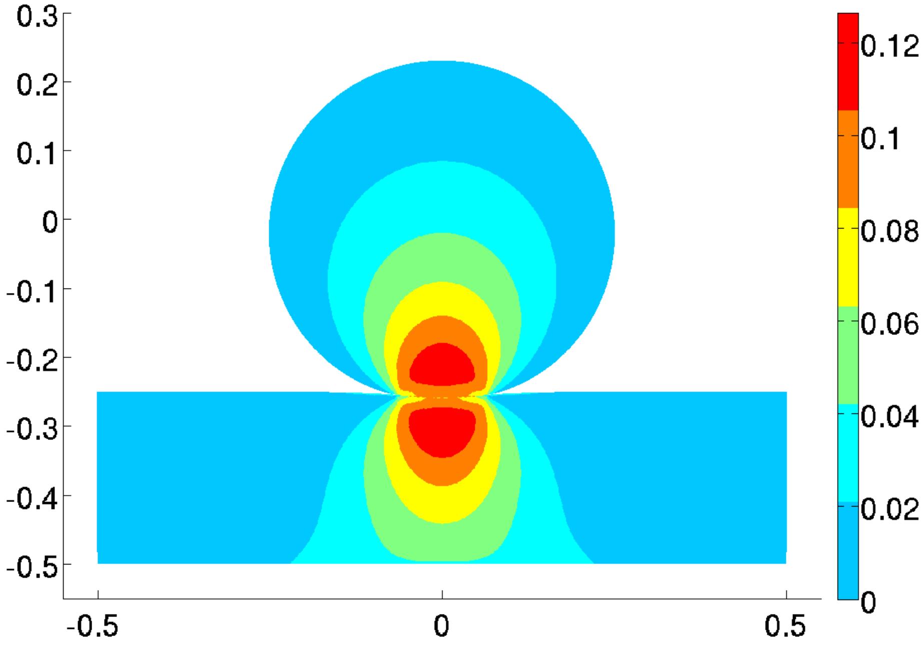


FIGURE 5.1. 2D numerical reference solution with contour plot of Von Mises stress. Parameters $h = 1/400$, $\gamma_0 = 1/200$, $\theta = -1$ and P_2 elements.

not available for the nonlinear contact problem. Moreover, this slight sub-optimal convergence may be caused by the Neumann-Dirichlet transition at the bottom of Ω_2.

On Figures 5.6, 5.7 and 5.8, the same experiments are reported but for P_2 isoparametric Lagrange finite elements. The convergence rate for the three cases is close to 1.6 on Ω_1 and 1.3 on Ω_2. This is also close to optimality if one takes into account that the expected maximal regularity of the displacement next to the transition between contact and non-contact should be $H^{5/2-\eta}$ for any $\eta > 0$ (However, this result has only been proved in a scalar case in [22]). Accordingly, one could expect that the convergence rate in the L^2-norm would be close to 2.5. This is approximately the case with again some sub-optimal rates which may due to the nonlinear characteristic of the contact condition or to the presence of non-regularities on the transition between the Dirichlet and the Neumann condition.

5.2. **Influence of the parameter** γ_0

The influence of γ_0 on the H^1-norm of the error is plotted in Figure 5.9 for P_1 elements and on Figure 5.10 for P_2 elements. The most affected method is the one for $\theta = 1$. Indeed, it converges only for γ_0 very small. The large oscillation in the error norm comes from the fact that Newton's algorithm do not fully converge for all numerical experiments probably because there is no solution to the discrete problem in some cases. The method for $\theta = 0$ gives a more regular error with respect to γ_0. It is still important to have γ_0 small to keep a good solution but a larger value is allowed. Accordingly to the theoretical result of Theorem 4.9, the influence of γ_0 on the method $\theta = -1$ is more limited. There is only a slight increase of the error for large values of γ_0. Note that the nonlinear discrete system (3.6) becomes very stiff when γ_0 is very small. Thus, the possibility to have a large γ_0 is an advantage.

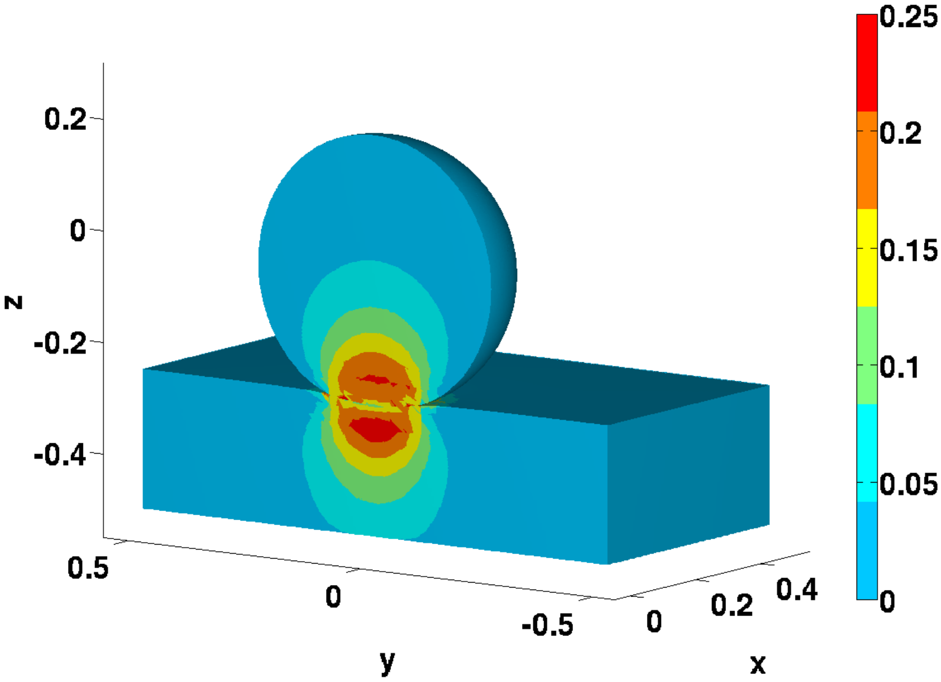


Figure 5.2. Cross-section of 3D numerical reference solution with contour plot of Von Mises stress. Parameters $h = 1/30$, $\gamma_0 = 1/200$, $\theta = -1$ and P_2 elements.

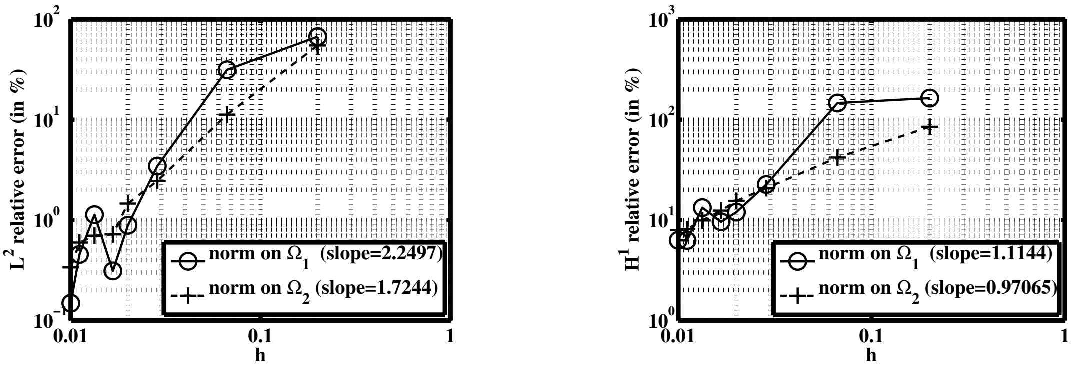


Figure 5.3. Convergence curves in 2D for the method $\theta = 1$, with $\gamma_0 = 1/200$ and $P1$ finite elements for the relative L^2-norm of the error (on the left) and the relative H^1-norm of the error (on the right).

5.3. **Numerical experiments in the 3D case**

Due to the high number of degrees of freedom in 3D, it obviously has not been possible to produce convergence curves with a mesh size as small as in 2D. The convergence curves for 3D are shown in Figures 5.11, 5.12 and 5.13 only for P_1 elements. Although we also made some tests with P_2 elements and on the influence of γ_0, we do not reproduce them for brevity of the paper. Indeed, the conclusions that can be drawn are were very similar to the 2D case.

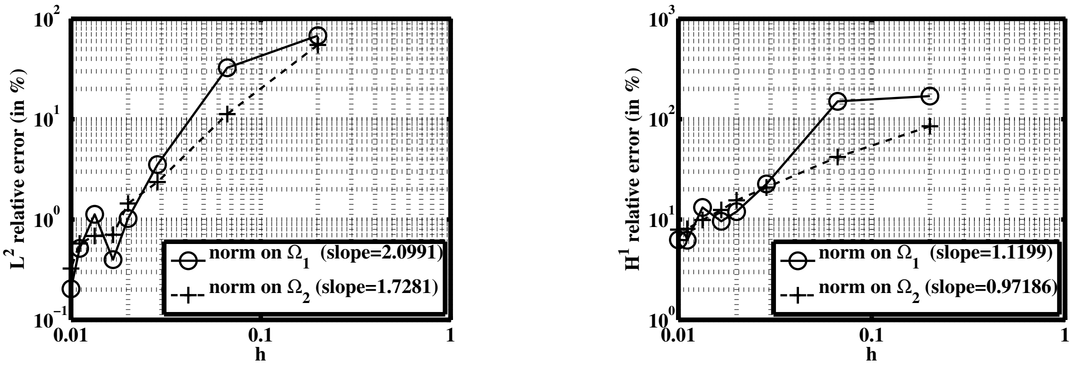


Figure 5.4. Convergence curves in 2D for the method $\theta = 0$, with $\gamma_0 = 1/200$ and $P1$ finite elements for the relative L^2-norm of the error (on the left) and the relative H^1-norm of the error (on the right).

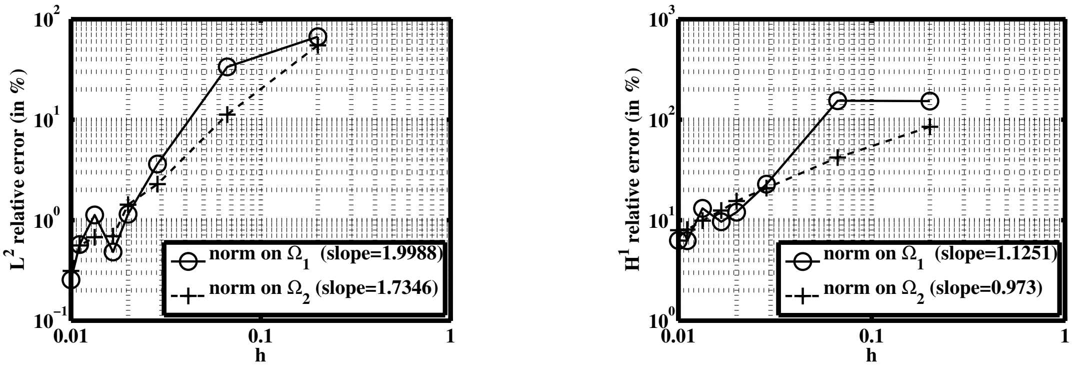


Figure 5.5. Convergence curves in 2D for the method $\theta = -1$, with $\gamma_0 = 1/200$ and $P1$ finite elements for the relative L^2-norm of the error (on the left) and the relative H^1-norm of the error (on the right).

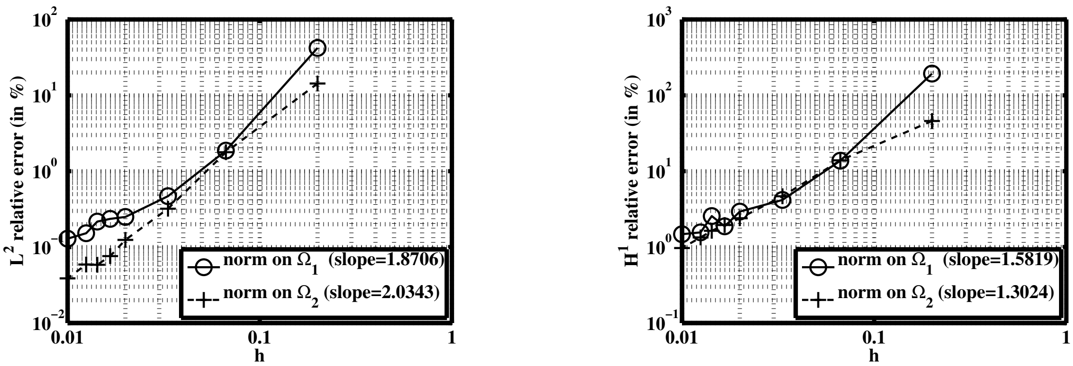


Figure 5.6. Convergence curves in 2D for the method $\theta = 1$, with $\gamma_0 = 1/200$ and $P2$ finite elements for the relative L^2-norm of the error (on the left) and the relative H^1-norm of the error (on the right).

Conclusion

In this paper, we developed a fictitious domain approach for the approximation in small deformations of the frictionless contact with nonzero initial gap of two elastic bodies. The main ingredients are the

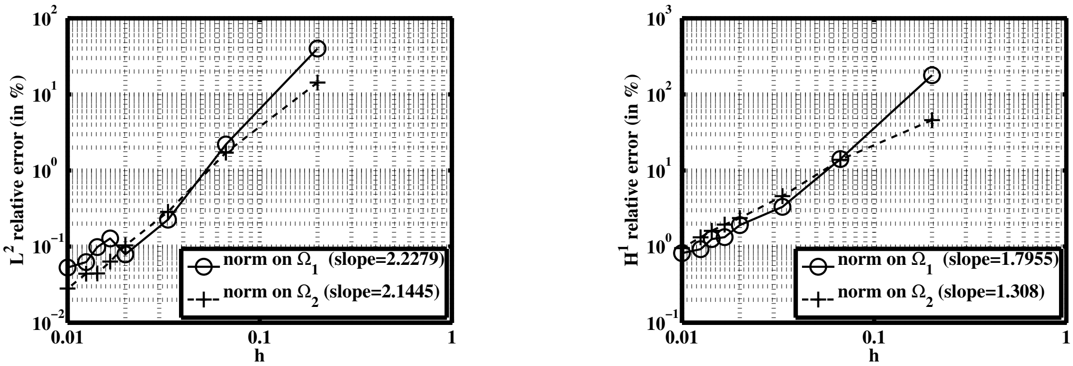


FIGURE 5.7. Convergence curves in 2D for the method $\theta = 0$, with $\gamma_0 = 1/200$ and $P2$ finite elements for the relative L^2-norm of the error (on the left) and the relative H^1-norm of the error (on the right).

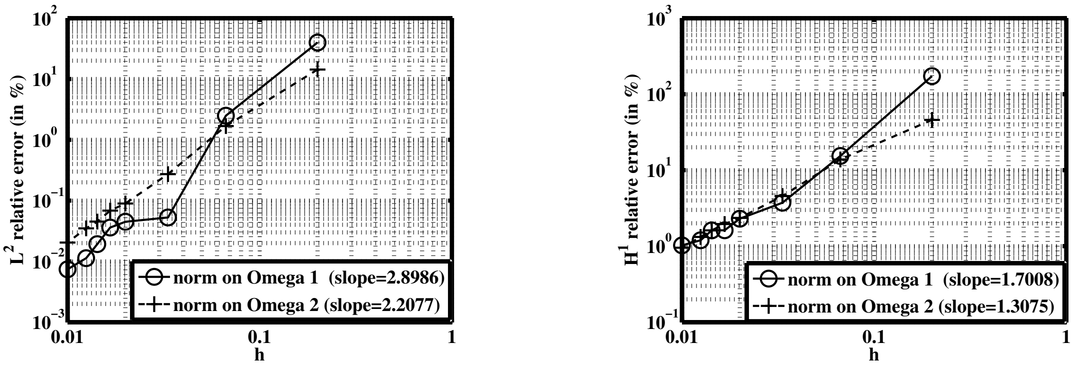


FIGURE 5.8. Convergence curves in 2D for the method $\theta = -1$, with $\gamma_0 = 1/200$ and $P2$ finite elements for the relative L^2-norm of the error (on the left) and the relative H^1-norm of the error (on the right).

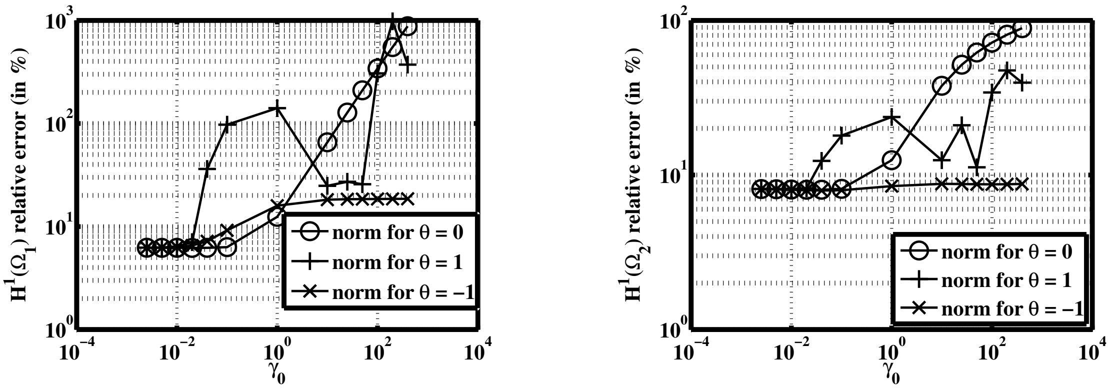


FIGURE 5.9. Influence of γ_0 on the relative H^1-norm of the error on Ω_1 (on the left) and on Ω_2 (on the right) in 2D for h = 1/90 and P_1 elements.

adaptation of Nitsche's method for the contact condition introduced in [6, 7] and the fictitious domain

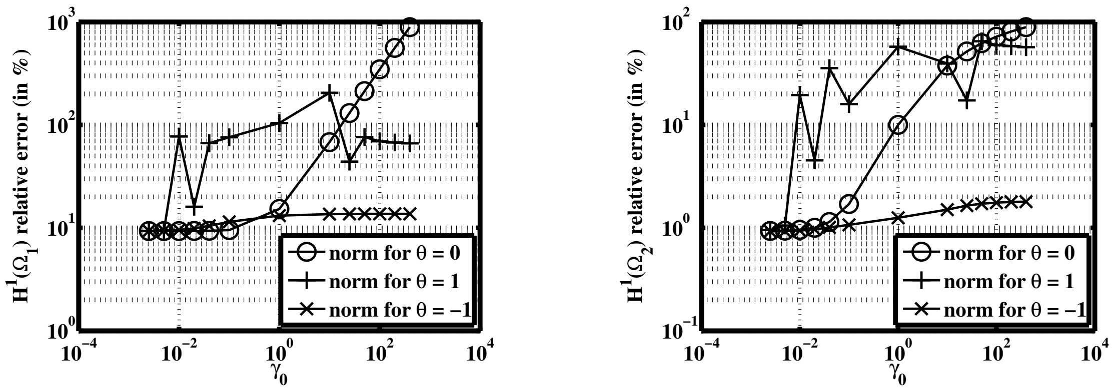


FIGURE 5.10. Influence of γ_0 on the relative H^1-norm of the error on Ω_1 (on the left) and on Ω_2 (on the right) in 2D for $h = 1/90$ and P_2 elements.

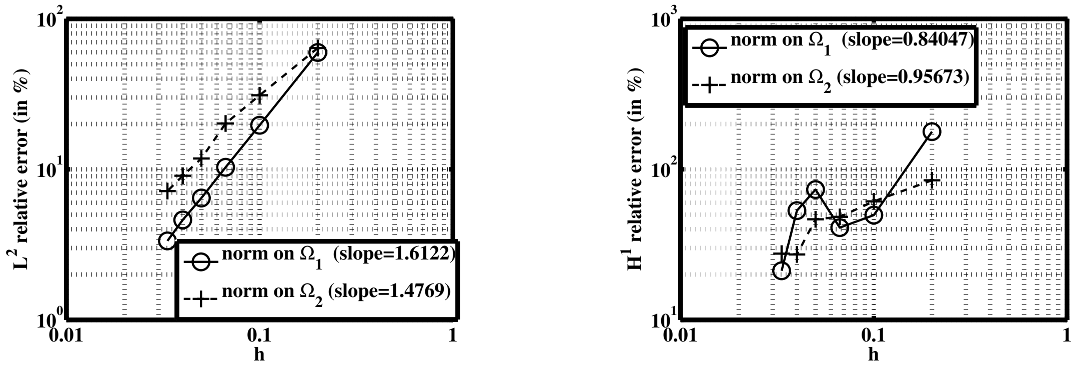


FIGURE 5.11. Convergence curves in 3D for the method $\theta = 1$, with $\gamma_0 = 1/100$ and $P1$ finite elements for the relative L^2-norm of the error (on the left) and the relative H^1-norm of the error (on the right).

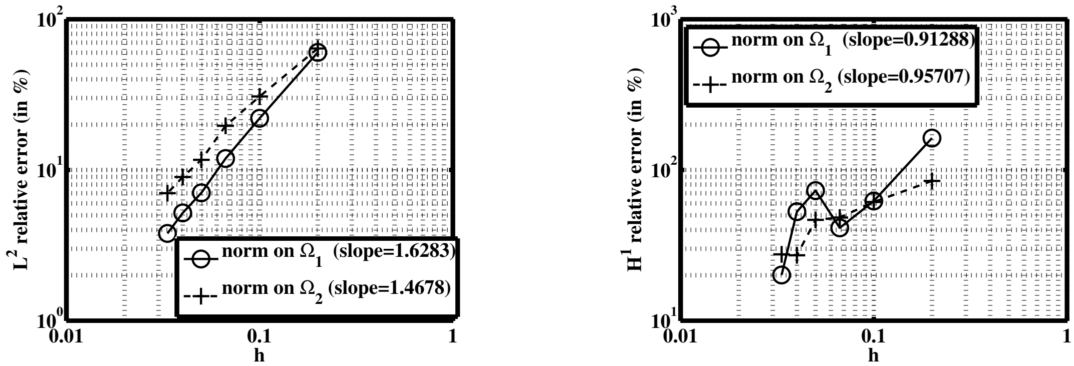


FIGURE 5.12. Convergence curves in 3D for the method $\theta = 0$, with $\gamma_0 = 1/100$ and $P1$ finite elements for the relative L^2-norm of the error (on the left) and the relative H^1-norm of the error (on the right).

method (inspired by the X-fem) developed in [16] including the stabilization proposed for the elements having a small intersection with the real domains.

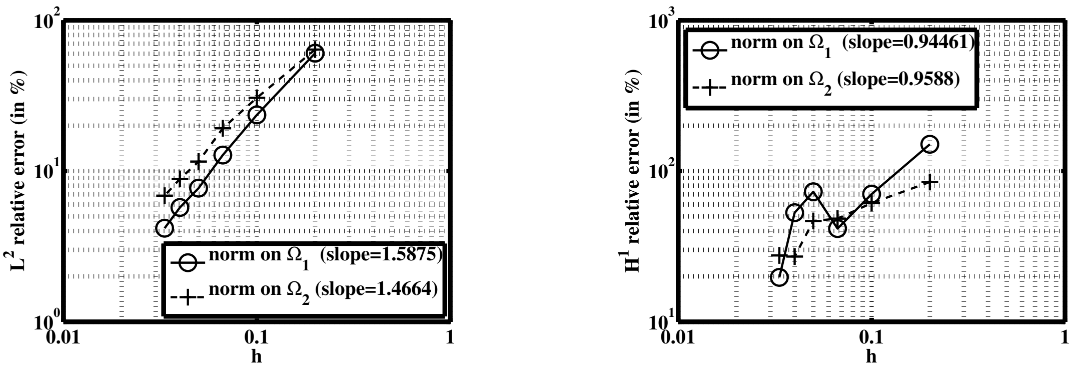


Figure 5.13. Convergence curves in 3D for the method $\theta = -1$, with $\gamma_0 = 1/100$ and $P1$ finite elements for the relative L^2-norm of the error (on the left) and the relative H^1-norm of the error (on the right).

Perspective of this works would be to weakened the conditions on the projection operator Π to include for instance non regular situations such as the one illustrated in Figure 5.14 where Π is only piecewise regular. Another possibility would be to consider a non-orthogonal projection.

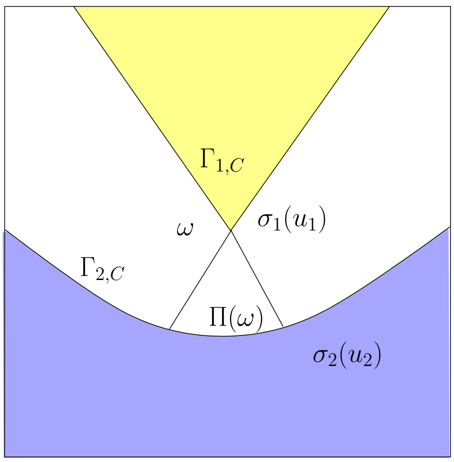


Figure 5.14. Example of non regular situation on $\Gamma_{1,C}$.

As already mentioned, the analysis can be easily adapted to Tresca friction similarly as it has been done in [5] for the non-fictitious domain situation.

From this study we conclude that the presented method allow an optimal approximation of unilateral contact problems for affine and quadratic finite element methods. The method for $\theta = 1$ is symmetric which can be an advantage for the numerical solving but requires a very small parameter γ_0 which may lead to a very stiff discrete problem (3.6). The method for $\theta = 0$ has the advantage of the simplicity and allows the use of a moderate γ_0. Finally, the skew-symmetric method $\theta = -1$ allows the use of larger value of γ_0 which can be a real advantage for the solving of the discrete problem.

Appendix

Proof of Lemma 4.2. First, we define the following matrix norms:

$$|||A|||_{\infty,\hat{K}} = \sup_{x\in\hat{K}}(|||A(x)|||_F) \quad \text{and} \quad |||A|||^2_{2,\hat{K}} = \int_{\hat{K}} |||A(x)|||^2_F \ \mathrm{d}x,$$

where $|||.|||_F$ is Frobenius' norm. If v is a fixed vector, we define the translation of a vector u, by $t_v(u) = u + v$. In the following, the constant C may vary from a line to another but is independent of h. In order to prove (4.1), we distinguish the three different cases from the definition of $R_{\hat{\rho}}$. First, by using the geometric transformation, the integral is expressed on the reference element. Then by using the equivalence of the infinity norm with the 2-norm located on a ball, we are able to deal with the 2-norm located on the current element. Finally by using the definition of the stress tensor, we obtain the result.

- If K satisfies $\exists\ \hat{y}_K > 0$ such that $B(\hat{y}_K, \hat{\rho}) \subset T_K^{-1}(K \cap \Omega_1)$, then $R_{\hat{\rho}}(u^h)|_K = \sigma_n(u_1^h|_K)$ and it holds:

$$\left\|R_{\hat{\rho}}(u^h)\right\|^2_{0,\Gamma_{1,C}\cap K} = \int_{\Gamma_{1,C}\cap K} \sigma_n(u_1^h)^2 \ \mathrm{d}\Gamma.$$

We define $\hat{\Gamma}_1 = T_K^{-1}(\Gamma_{1,C} \cap K)$ and $\hat{\sigma}(u_1) = \sigma(u_1^h) \circ T_K$ and $\hat{n}_1$ a unit normal vector on $\hat{\Gamma}_{1,C}$.

$$\begin{aligned}\int_{\Gamma_{1,C}\cap K} \sigma_n(u_1^h)^2 \ \mathrm{d}\Gamma &= \int_{\hat{\Gamma}_1} \hat{\sigma}_n(u_1)^2 \left|\det(J_K)\right| \left\|J_K^{-1}\hat{n}_1\right\| \ \mathrm{d}\hat{\Gamma}, \\ &= \int_{\hat{\Gamma}_1} \left|\hat{\sigma}(u_1)n \cdot n\right|^2 \left|\det(J_K)\right| \left\|J_K^{-1}\hat{n}_1\right\| \ \mathrm{d}\hat{\Gamma}, \\ &\leqslant Ch_K^{d-1} \, |||\hat{\sigma}(u_1)|||^2_{\infty,\hat{K}} \left|\hat{\Gamma}_1\right|.\end{aligned} \tag{5.1}$$

because

$$\left|\hat{\sigma}(u_1)n \cdot n\right| \leqslant |||\hat{\sigma}(u_1)|||_F \, \|n\|_2^2 = |||\hat{\sigma}(u_1)|||_F \, .$$

Moreover, $\left|\hat{\Gamma}\right|_1$ is bounded, indeed the operator T_K is a continuous one to one correspondence. Now using the equivalence of norms in $P^k(\hat{K})^d$, we have:

$$\begin{aligned}|||\hat{\sigma}(u_1)|||^2_{\infty,\hat{K}} &\leqslant |||\hat{\sigma}(u_1)|||^2_{\infty,B(\hat{y}_K,2)} = |||\hat{\sigma}(u_1) \circ t_{-\hat{y}_K}|||^2_{\infty,B(0,2)} \, , \\ &\leqslant C \, |||\hat{\sigma}(u_1) \circ t_{-\hat{y}_K}|||^2_{2,B(0,\hat{\rho})} = C \, |||\hat{\sigma}(u_1)|||^2_{2,B(\hat{y}_K,\hat{\rho})} \, , \\ &\leqslant C \, |||\hat{\sigma}(u_1)|||^2_{2,T_K^{-1}(\Omega_1\cap K))} = C \int_{T_K^{-1}(\Omega_1\cap K)} |||\hat{\sigma}(u_1)|||^2_F \ \mathrm{d}\hat{x}.\end{aligned} \tag{5.2}$$

Using the upper bound of $\left|\hat{\Gamma}_1\right|$ and the previous inequalities, it holds:

$$\begin{aligned}\int_{\Gamma_{1,C}\cap K} R_{\hat{\rho}}(u^h)^2 \ \mathrm{d}\Gamma &\leqslant C\frac{h_K^{d-1}}{h_K^d} \int_{T_K^{-1}(\Omega_1\cap K)} |||\hat{\sigma}(u_1)|||^2_F \left|\det(J_K)\right| \ \mathrm{d}\hat{x}, \\ &\leqslant Ch_K^{-1} \int_{\Omega_1\cap K} \left|\left|\left|\sigma(u_1^h)\right|\right|\right|^2_F \ \mathrm{d}x, \\ &\leqslant Ch_K^{-1} \int_{\Omega_1\cap K} \left|\left|\left|A\nabla u_1^h\right|\right|\right|^2_F \ \mathrm{d}x, \\ &\leqslant Ch_K^{-1} \int_{\Omega_1\cap K} \left\|\nabla u_1^h\right\|^2_2 \ \mathrm{d}x.\end{aligned}$$

- Otherwise, if $\exists \widetilde{K} \in S_K$ such as $\exists\ \hat{y}_{\widetilde{K}} > 0$ such that $B(\hat{y}_{\widetilde{K}}, \hat{\rho}) \subset T_{\widetilde{K}}^{-1}(\widetilde{K} \cap \Omega_2)$, then $R_{\hat{\rho}}(v^h)|_K = \sigma_n(E_{\widetilde{K}}(v_2^h) \circ \Pi)\,|\det(\nabla Pi)|$ and using the continuous of J_Π i.e. $|\det(J_\Pi)| \leqslant C$, it holds:
$$\begin{aligned}\left\| R_{\hat{\rho}}(u^h) \right\|^2_{0,\Gamma_{1,C}\cap K} &= \int_{\Gamma_{1,C}\cap K} \sigma_n(E_{\widetilde{K}}(u_2^h) \circ \Pi)^2\ |\det(J_\Pi)|^2\ \mathrm{d}\Gamma,\\ &\leqslant C \int_{\Gamma_{1,C}\cap K} \sigma_n(E_{\widetilde{K}}(u_2^h) \circ \Pi)^2\ |\det(J_\Pi)|\ \mathrm{d}\Gamma,\\ &\leqslant C \int_{\Pi(\Gamma_{1,C}\cap K)} \sigma_n(E_{\widetilde{K}}(u_2^h))^2\ \mathrm{d}\Gamma,\\ &\leqslant C \int_{\bigcup_{\overline{K}\in S_K} \Gamma_{2,C}\cap\overline{K}} \sigma_n(E_{\widetilde{K}}(u_2^h))^2\ \mathrm{d}\Gamma.\end{aligned}$$
We define $\hat{\Gamma}_2 = T_{\widetilde{K}}^{-1}(\bigcup_{\overline{K}\in S_K} \Gamma_{2,C} \cap \overline{K})$ and $\hat{\sigma}(u_2) = \sigma(u_2^h) \circ T_{\widetilde{K}}$ and $\hat{n}_2$ a unit normal vector on $\hat{\Gamma}_{2,C}$. As previously, we have $\left|\hat{\Gamma}_2\right|$ bounded. In the same way as in (5.1), we have:
$$\int_{\bigcup_{\overline{K}\in S_K} \Gamma_{2,C}\cap\overline{K}} \sigma_n(E_{\widetilde{K}}(u_2^h))^2\ \mathrm{d}\Gamma \leqslant C h_{\widetilde{K}}^{d-1}\, |||\sigma(\hat{u}_2)|||^2_{\infty,\hat{\Gamma}_2} \left|\hat{\Gamma}_2\right|.$$
Now using the equivalence of norms in $P^k(\hat{K})^d$ and in the same way as in (5.2), we obtain:
$$\begin{aligned}|||\sigma(\hat{u}_2)|||^2_{\infty,\hat{\Gamma}_2} &\leqslant |||\sigma(\hat{u}_2)|||^2_{\infty,B(\hat{y}_{\widetilde{K}},2\ ||\Pi||\ ||T_{\widetilde{K}}^{-1}||)},\\ &\leqslant C \int_{T_{\widetilde{K}}^{-1}(\Omega_1\cap\widetilde{K})} |||\sigma(\hat{u}_2)|||^2_F\ \mathrm{d}\hat{x}.\end{aligned}$$
Hence, using the previous inequalities, it holds:
$$\begin{aligned}\int_{\Gamma_{1,C}\cap K} R_{\hat{\rho}}(u^h)^2\ \mathrm{d}\Gamma &\leqslant C \frac{h_{\widetilde{K}}^{d-1}}{h_{\widetilde{K}}^{d}} \int_{T_{\widetilde{K}}^{-1}(\Omega_1\cap\widetilde{K})} |||\hat{\sigma}(u_1)|||^2_F \left|\det(J_{\widetilde{K}})\right|\ \mathrm{d}\hat{x},\\ &\leqslant C h_{\widetilde{K}}^{-1} \int_{\Omega_1\cap\widetilde{K}} \left\|\nabla u_2^h\right\|^2_2\ \mathrm{d}x.\end{aligned}$$

- Otherwise, we suppose it exits an neighbor element K' of K such that $\exists\ \hat{y}_{K'} > 0$ such that $B(\hat{y}_{K'}, \hat{\rho}) \subset T_{K'}^{-1}(K' \cap \Omega_1)$, then $R_{\hat{\rho}}(v^h)|_K = \sigma_n(E_{K'}(v_1^h))$. Then, it holds:
$$\left\| R_{\hat{\rho}}(u^h) \right\|^2_{0,\Gamma_{1,C}\cap K} = \int_{\Gamma_{1,C}\cap K} \sigma_n(E_{K'}(u_1^h))^2\ \mathrm{d}\Gamma.$$
We define by $\hat{\Gamma}'_1 = T_{K'}^{-1}(\Gamma_{1,C} \cap K)$ and $\hat{\sigma}'(u_1) = \sigma(u_1^h) \circ T_{K'}$ and by $\hat{n}_1$ a unit normal vector on $\hat{\Gamma}_{1,C}$. As previously, we have $\left|\hat{\Gamma}'_1\right|$ bounded. In the same way as in (5.1), we have:
$$\begin{aligned}\int_{\Gamma_{1,C}\cap K} \sigma_n(E_{K'}(u_1^h))^2\ \mathrm{d}\Gamma &= \int_{\hat{\Gamma}'_1} \hat{\sigma}'(u_1)^2\, |\det(J_{K'})| \left\| J_{K'}^{-1}\hat{n}_1 \right\|\ \mathrm{d}\hat{\Gamma},\\ &\leqslant C h_{K'}^{d-1}\, |||\hat{\sigma}'(u_1)|||^2_{\infty,T_{K'}^{-1}(K)} \left|\hat{\Gamma}'_1\right|.\end{aligned}$$
Now using the equivalence of norms in $P^k(\hat{K})^d$ and in the same way as in (5.2), we have:
$$\begin{aligned}|||\sigma(\hat{u}'_1)|||^2_{\infty,T_{K'}^{-1}(K)} &\leqslant |||\sigma(\hat{u}'_1)|||^2_{\infty,B(\hat{y}_{K'},4)},\\ &\leqslant C \int_{T_{K'}^{-1}(\Omega_1\cap K')} |||\sigma(\hat{u}'_1)|||^2_F\ \mathrm{d}\hat{x}.\end{aligned}$$

Hence, using the previous inequalities, it holds:

$$\int_{\Gamma_{1,C}\cap K} R_{\hat{\rho}}(u^h)^2 \ \mathrm{d}\Gamma \leqslant C\frac{h_{K'}^{d-1}}{h_{K'}^{d}} \int_{T_{K'}^{-1}(\Omega_1\cap K')} |||\sigma(\hat{u}_1')|||_F^2 \,|\det(J_{K'})| \ \mathrm{d}\hat{x},$$
$$\leqslant Ch_{K'}^{-1} \int_{\Omega_1\cap K'} \left\|\nabla u_1^h\right\|_2^2 \ \mathrm{d}x.$$

Finally, by iterating on all the elements K intersecting $\Gamma_{1,C}$ and using the quasi uniformity of the mesh, we obtain (4.1). □

Proof of Lemma 4.7. We argue by contradiction. It is sufficient to prove the result for $\gamma = M$. Suppose there exists $(v_n)_{n\in\mathbb{N}} \subset V$ such that $\sum_{i=1,2} \|v_{i,n}\|_{1,\Omega_i}^2 = 1$, for $n \in \mathbb{N}$, which satisfies

$$a(v_n, v_n) + \frac{1}{2} \sum_{i=1,2} \int_{\Gamma_{i,D}} M^{-1} v_{i,n}^2 \ \mathrm{d}\Gamma \leqslant \frac{1}{n}.$$

Hence, it holds $\displaystyle\lim_{n\to+\infty} \sum_{i=1,2} \int_{\Gamma_{i,D}} v_{i,n}^2 \ \mathrm{d}\Gamma = 0$ and $\displaystyle\lim_{n\to+\infty} a(v_n, v_n) = 0$. From the weak sequential compactness of the unit ball of V, there exits $v \in V$ and a subsequence still denoted by v_n which weakly converges to v. The compact injection of H^1 into L^2 implies that up to a subsequence, v_n converges to v strongly in $L^2(\Omega_1)^d \times L^2(\Omega_2)^d$. First, we show that $v = 0$ and then that v_n converges to v strongly in $H^1(\Omega_1)^d \times H^1(\Omega_2)^d$. By using the lower semi-continuity of $v \mapsto \sum_{i=1,2} \int_{\Gamma_{i,D}} v_{i,n}^2 \ \mathrm{d}\Gamma$, we have $\displaystyle\sum_{i=1,2} \int_{\Gamma_{i,D}} v_i^2 \ \mathrm{d}\Gamma = 0$ with $\mathrm{mes}(\Gamma_{i,D}) \neq 0$. Furthermore, due to the L^2-convergence, one has $\displaystyle\lim_{n\to+\infty} \sum_{i=1,2} \|v_{i,n}\|_{0,\Omega_i} = \sum_{i=1,2} \|v_i\|_{0,\Omega_i}$. Similarly by using the weak lower semi-continuity of $a(.,.)$, we deduce $a(v, v) = 0$ and using the property of the fourth order tensor A, it holds:

$$0 = \int_{\Omega_i} \sigma(v_i) : \varepsilon(v_i) \ \mathrm{d}\Omega = \int_{\Omega_i} \varepsilon(v_i) : \varepsilon(v_i) \ \mathrm{d}\Omega = \|\varepsilon(v_i)\|_{0,\Omega_i}.$$

Let us finally show that $v = 0$. Since, the tensor A is uniformly elliptic, it holds:

$$a(v_n, v_n) \geqslant C \sum_{i=1,2} \int_{\Omega_i} \varepsilon(v_{i,n}) : \varepsilon(v_{i,n}) \ \mathrm{d}\Omega = C \sum_{i=1,2} \|\varepsilon(v_{i,n})\|_{0,\Omega_i} \qquad \text{and} \ \lim_{n\to+\infty} a(v_n, v_n) = 0.$$

Hence

$$\lim_{n\to+\infty} \sum_{i=1,2} \|\varepsilon(v_{i,n})\|_{0,\Omega_i} = \sum_{i=1,2} \|\varepsilon(v_i)\|_{1,\Omega_i} = 0.$$

Moreover, thanks to Korn's inequality (see [9]), it holds:

$$\sum_{i=1,2} \|\varepsilon(v_{i,n})\|_{0,\Omega_i} + \sum_{i=1,2} \|v_{i,n}\|_{0,\Gamma_{i,D}} \geqslant C \sum_{i=1,2} \|v_{i,n}\|_{0,\Omega_i}.$$

We deduce:

$$\lim_{n\to+\infty} \sum_{i=1,2} \|v_{i,n}\|_{1,\Omega_i} = \sum_{i=1,2} \|v_i\|_{1,\Omega_i} = 0$$

which contradicts $\sum_{i=1,2} \|v_{i,n}\|_{1,\Omega_i}^2 = 1$. □

Proof for operator B^h to be hemi-continuous (for the proof of Theorem 4.1).

First, we need to prove B^h is coercive which is a consequence of the previous lemmas. Then we establish an estimate which will imply the hemi-continuity. Let u^h, $v^h \in V^h$, it holds:

$$(B^h u^h - B^h v^h, u^h - v^h)_{1,\Omega} = I + II + III \tag{5.3}$$

with

$$
\begin{aligned}
I &= a(u^h - v^h, u^h - v^h) - \int_{\Gamma_{1,C}} \theta\gamma R_{\hat\rho}(u^h - v^h) R_{\hat\rho}(u^h - v^h)\ \mathrm{d}\Gamma \\
&\qquad - \sum_{i=1,2} \int_{\Gamma_{i,D}} \theta\gamma \overline{R}_{\hat\rho}(u_i^h - v_i^h) \cdot \overline{R}_{\hat\rho}(u_i^h - v_i^h)\ \mathrm{d}\Gamma, \\
II &= \sum_{i=1,2} \int_{\Gamma_{i,D}} \frac{1}{\gamma} (\overline{P}^{h,\hat\rho}_{i,\gamma}(u_i^h) - \overline{P}^{h,\hat\rho}_{i,\gamma}(v_i^h)) \cdot (\overline{P}^{h,\hat\rho}_{i,\gamma\theta}(u_i^h) - \overline{P}^{h,\hat\rho}_{i,\gamma\theta}(v_i^h))\ \mathrm{d}\Gamma, \\
III &= \int_{\Gamma_{1,C}} \frac{1}{\gamma} ([P^{h,\hat\rho}_{\gamma}(u^h) - g]_+ - [P^{h,\hat\rho}_{\gamma}(v^h) - g]_+)(P^{h,\hat\rho}_{\theta\gamma}(u^h) - P^{h,\hat\rho}_{\theta\gamma}(v^h))\ \mathrm{d}\Gamma.
\end{aligned}
$$

Now, we need to bound I, II, III from below to prove the coercivity.

Using Young's inequality for $\beta > 0$, it holds:

$$
\begin{aligned}
III \geqslant (1 - \frac{|1-\theta|}{2\beta}) \left\| \gamma^{-\frac{1}{2}} ([P^{h,\hat\rho}_{\gamma}(u^h) - g]_+ - [P^{h,\hat\rho}_{\gamma}(v^h) - g]_+) \right\|^2_{0,\Gamma_{1,C}} \\
- \frac{|1-\theta|\,\beta}{2} \left\| \gamma^{\frac{1}{2}} R_{\hat\rho}(u^h - v^h) \right\|^2_{0,\Gamma_{1,C}}.
\end{aligned} \tag{5.4}
$$

Using Young's inequality for $\beta' > 0$:

$$
II \geqslant (1 - \frac{|1+\theta|}{2\beta'}) \sum_{i=1,2} \left\| \gamma^{-\frac{1}{2}} (u_i^h - v_i^h) \right\|^2_{\Gamma_{i,D}} + (\theta - \frac{|1+\theta|\,\beta'}{2}) \sum_{i=1,2} \left\| \gamma^{\frac{1}{2}} (\overline{R}_{\hat\rho}(u_i^h) - \overline{R}_{\hat\rho}(v_i^h)) \right\|^2_{\Gamma_{i,D}}. \tag{5.5}
$$

We deduced from the estimates of II and III that:

$$
\begin{aligned}
&(B^h u^h - B^h v^h, u^h - v^h)_{1,\Omega} \\
&\quad \geqslant a(u^h - v^h, u^h - v^h) - \theta \left\| \gamma^{\frac{1}{2}} R_{\hat\rho}(u^h - v^h) \right\|^2_{0,\Gamma_{1,C}} - \theta \sum_{i=1,2} \left\| \gamma^{\frac{1}{2}} (\overline{R}_{\hat\rho}(u_i^h) - \overline{R}_{\hat\rho}(v_i^h)) \right\|^2_{\Gamma_{i,D}} \\
&\qquad + (1 - \frac{|1-\theta|}{2\beta}) \left\| \gamma^{-\frac{1}{2}} ([P^{h,\hat\rho}_{\gamma}(u^{h,\hat\rho}) - g]_+ - [P^{h,\hat\rho}_{\gamma}(v^h) - g]_+) \right\|^2_{0,\Gamma_{1,C}} \\
&\qquad - \frac{|1-\theta|\,\beta}{2} \left\| \gamma^{\frac{1}{2}} R_{\hat\rho}(u^h - v^h) \right\|^2_{0,\Gamma_{1,C}} \\
&\qquad + (1 - \frac{|1+\theta|}{2\beta'}) \sum_{i=1,2} \left\| \gamma^{-\frac{1}{2}} (u_i^h - v_i^h) \right\|^2_{\Gamma_{i,D}} + (\theta - \frac{|1+\theta|\,\beta'}{2}) \sum_{i=1,2} \left\| \gamma^{\frac{1}{2}} (\overline{R}_{\hat\rho}(u_i^h) - \overline{R}_{\hat\rho}(v_i^h)) \right\|^2_{\Gamma_{i,D}}, \\
&\quad \geqslant a(u^h - v^h, u^h - v^h) + (1 - \frac{|1+\theta|}{2\beta'}) \sum_{i=1,2} \left\| \gamma^{-\frac{1}{2}} (u_i^h - v_i^h) \right\|^2_{\Gamma_{i,D}} \\
&\qquad - \frac{|1+\theta|\,\beta'}{2} \sum_{i=1,2} \left\| \gamma^{\frac{1}{2}} (\overline{R}_{\hat\rho}(u_i^h) - \overline{R}_{\hat\rho}(v_i^h)) \right\|^2_{\Gamma_{i,D}} - (\theta + \frac{|1-\theta|\,\beta}{2}) \left\| \gamma^{\frac{1}{2}} R_{\hat\rho}(u^h - v^h) \right\|^2_{0,\Gamma_{1,C}} \\
&\qquad + (1 - \frac{|1-\theta|}{2\beta}) \left\| \gamma^{-\frac{1}{2}} ([P^{h,\hat\rho}_{\gamma}(u^h) - g]_+ - [P^{h,\hat\rho}_{\gamma}(v^h) - g]_+) \right\|^2_{0,\Gamma_{1,C}}.
\end{aligned}
$$

If $\theta = 1$ and $\beta' = 2$, we have:

$$(B^h u^h - B^h v^h, u^h - v^h)_{1,\Omega}$$
$$\geqslant a(u^h - v^h, u^h - v^h) + \frac{1}{2}\sum_{i=1,2}\left\|\gamma^{-\frac{1}{2}}(u_i^h - v_i^h)\right\|^2_{\Gamma_{i,D}} - 2\sum_{i=1,2}\left\|\gamma^{\frac{1}{2}}(\overline{R}_{\hat\rho}(u_i^h) - \overline{R}_{\hat\rho}(v_i^h))\right\|^2_{\Gamma_{i,D}}$$
$$+ \left\|\gamma^{-\frac{1}{2}}([P_\gamma^{h,\hat\rho}(u^h) - g]_+ - [P_\gamma^{h,\hat\rho}(v^h) - g]_+)\right\|^2_{0,\Gamma_{1,C}} - \left\|\gamma^{\frac{1}{2}} R_{\hat\rho}(u^h - v^h)\right\|^2_{0,\Gamma_{1,C}}.$$

Thus, if γ_0 is sufficiently small and using the coercivity (4.2) for I and the previous Lemma 4.7:

$$(B^h u^h - B^h v^h, u^h - v^h)_{1,\Omega} \geqslant C\sum_{i=1,2}\left\|u_i^h - v_i^h\right\|^2_{1,\Omega_i}.$$

If $\theta = -1$, choose $\beta = \frac{|1-\theta|}{2}$, it holds:

$$(B^h u^h - B^h v^h, u^h - v^h)_{1,\Omega} \geqslant a(u^h - v^h, u^h - v^h) + \sum_{i=1,2}\left\|\gamma^{-\frac{1}{2}}(u_i^h - v_i^h)\right\|^2_{\Gamma_{i,D}},$$

and from the coercivity (4.2) for I, we obtain:

$$(B^h u^h - B^h v^h, u^h - v^h)_{1,\Omega} \geqslant C\sum_{i=1,2}\left\|u_i^h - v_i^h\right\|^2_{1,\Omega_i}.$$

If $\theta \neq -1$, we take $\beta = \frac{|1-\theta|}{2}$ and $\beta' = |1+\theta|$, it holds:

$$(B^h u^h - B^h v^h, u^h - v^h)_{1,\Omega}$$
$$\geqslant a(u^h - v^h, u^h - v^h) + \frac{1}{2}\sum_{i=1,2}\left\|\gamma^{-\frac{1}{2}}(u_i^h - v_i^h)\right\|^2_{\Gamma_{i,D}} - \frac{(1+\theta)^2}{2}\sum_{i=1,2}\left\|\gamma^{\frac{1}{2}}(\overline{R}_{\hat\rho}(u_i^h) - \overline{R}_{\hat\rho}(v_i^h))\right\|^2_{\Gamma_{i,D}}$$
$$- \frac{1}{4}(1+\theta)^2\left\|\gamma^{\frac{1}{2}} R_{\hat\rho}(u^h - v^h)\right\|^2_{0,\Gamma_{1,C}}.$$

So, using γ_0 sufficiently small and using the coercivity (4.2) for I and previous Lemma 4.7, it holds:

$$(B^h u^h - B^h v^h, u^h - v^h)_{1,\Omega} \geqslant C\sum_{i=1,2}\left\|u_i^h - v_i^h\right\|^2_{1,\Omega_i}.$$

Now, we prove the hemi-continuity of B^h. Let $t, s \in [0,1]$ and $u^h, v^h \in V^h$, we have:

$$\begin{aligned}
&\left|(B^h(u^h - tv^h) - B^h(u^h - sv^h), v^h)_{1,\Omega}\right| \\
&\leqslant |s-t|\, a(v^h, v^h) + |s-t|\,|\theta|\left\|\gamma^{\frac{1}{2}} R_{\hat\rho}(v^h)\right\|^2_{\Gamma_{1,C}} + |s-t|\,|\theta|\sum_{i=1,2}\left\|\gamma^{\frac{1}{2}}\overline{R}_{\hat\rho}(v_i^h)\right\|^2_{\Gamma_{i,D}} \\
&\quad + |s-t|\sum_{i=1,2}\int_{\Gamma_{i,D}}\frac{1}{\gamma}\left|\overline{P}^{h,\hat\rho}_{i,\gamma}(v_i^h)\right|\cdot\left|\overline{P}^{h,\hat\rho}_{i,\gamma\theta}(v_i^h)\right|\,\mathrm{d}\Gamma \\
&\quad + \int_{\Gamma_{1,C}}\frac{1}{\gamma}\left|[P_\gamma^{h,\hat\rho}(u^h - tv^h) - g]_+ - [P_\gamma^{h,\hat\rho}(u^h - sv^h) - g]_+\right|\left|P^{h,\hat\rho}_{\theta\gamma}(v^h)\right|\,\mathrm{d}\Gamma.
\end{aligned} \tag{5.6}$$

For all a, b in $\mathbb{R}$, we have the following estimate:

$$|[a]_+ - [b]_+| \leqslant |a-b|.$$

So we deduce that

$$int_{\Gamma_{1,C}} \frac{1}{\gamma} \left| [P_{\gamma}^{h,\hat{\rho}}(u^h - tv^h) - g]_+ - [P_{\gamma}^{h,\hat{\rho}}(u^h - sv^h) - g]_+ \right| \left| P_{\theta\gamma}^{h,\hat{\rho}}(v^h) \right| \ \mathrm{d}\Gamma$$
$$\leqslant |s-t| \int_{\Gamma_{1,C}} \frac{1}{\gamma} \left| [\![v^h \cdot n]\!] - \gamma R_{\hat{\rho}}(v^h) \right| \left| P_{\theta\gamma}^{h,\hat{\rho}}(v^h) \right| \ \mathrm{d}\Gamma. \quad (5.7)$$

Hence

$$\begin{aligned} &\left| (B^h(u^h - tv^h) - B^h(u^h - sv^h), v^h)_{1,\Omega} \right| \\ &\quad \leqslant |s-t| \Big(a(v^h, v^h) + |\theta| \left\| \gamma^{\frac{1}{2}} R_{\hat{\rho}}(v^h) \right\|^2_{\Gamma_{1,C}} + |\theta| \sum_{i=1,2} \left\| \gamma^{\frac{1}{2}} \overline{R}_{\hat{\rho}}(v_i^h) \right\|^2_{\Gamma_{i,D}} \\ &\quad + \sum_{i=1,2} \int_{\Gamma_{i,D}} \frac{1}{\gamma} \left| \overline{P}_{i,\gamma}^{h,\hat{\rho}}(v_i^h) \right| \cdot \left| \overline{P}_{i,\gamma\theta}^{h,\hat{\rho}}(v_i^h) \right| \ \mathrm{d}\Gamma + \int_{\Gamma_{1,C}} \frac{1}{\gamma} \left| P_{\gamma}^{h,\hat{\rho}}(v^h) \right| \left| P_{\theta\gamma}^{h,\hat{\rho}}(v^h) \right| \ \mathrm{d}\Gamma \Big). \end{aligned}$$

Hence B^h is hemi-continuous. □

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