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ON THE GROMOV HYPERBOLICITY OF DOMAINS IN $\mathbb{C}^n$

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Abstract. — We present several recent results dealing with the metric properties of domains in the complex Euclidean space $\mathbb{C}^n$. We provide with examples of domains, endowed with Finsler or Kähler metrics, that are (or are not) Gromov hyperbolic. We also present how different notions, such as the Gromov hyperbolicity, the holomorphic bisectional curvature or the d'Angelo type may be related for smooth bounded domains in $\mathbb{C}^n$. We also present several open questions in the core of the paper.

Introduction

The Gromov hyperbolicity theory deals with "large scale" curvature properties of metric spaces. The links between the Gromov hyperbolicity of a manifold and different notions of curvature one may consider on the manifold have been intensively studied. It is for instance standard that if M is a simply connected complete Riemannian manifold, with sectional curvature bounded from above by a negative constant k, then it is a $CAT(k)$ space and, almost by definition, it is Gromov hyperbolic.

The study of the Gromov hyperbolicity of a complex manifold M, endowed with some specific metric d_M, was initiated by Z. Balogh–M. Bonk [1] and the links between different classical invariants of complex geometry or CR geometry of M and the Gromov hyperbolicity of (M, d_M) is the core of recent developments, with the aim not only to provide with new examples of Gromov hyperbolic spaces, but mainly to search for applications of that property.

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Consider a bounded domain D in $\mathbb{C}^n$. One may endow D with some complex Finsler metric and study how geodesics behave near the boundary ∂D of D, meaning whether the corresponding metric space might be hyperbolic in the sense of Gromov. One may also endow D with some Kähler metric g and study the geometric properties of D, depending on the asymptotic behaviour of the holomorphic (bi)sectional curvature of g. In case D has a smooth boundary, one may finally consider classical Cauchy–Riemann invariants of ∂D and inquire how properties of the invariants impact the asymptotic behaviour of invariant metrics on D. The aim of the paper is to present recent results related to that considerations.

First we give some examples of (non) Gromov hyperbolic domains for different invariant metrics. Then we exhibit, on examples, possible links between the asymptotic behaviour of the holomorphic bisectional curvature of a complete Kähler metric g on a domain $D \subset \mathbb{C}^n$, the properties of the d'Angelo type of ∂D, and the Gromov hyperbolicity of the metric space (D, d_g), where d_g is the distance induced by the Kähler metric considered on D. We also study the links with some classical invariant distances on D, such as the Kobayashi distance.

For the question "Which complex manifolds are Gromov hyperbolic?" to be consistent, one needs to specify some distance. This may be a Kähler distance, such as the Bergman distance, the Kähler–Einstein distance or a non invariant Kähler distance, or a complex Finsler distance, such as the Carathéodory distance or the Kobayashi distance. Notice that in all the situations we will consider, the (pseudo)distances will be actual distances. It is sufficient for the Bergman, the Carathéodory or the Kobayashi pseudodistances to be distances to consider a bounded domain in $\mathbb{C}^n$, or a convex domain not containing complex lines. For the existence of a complete Kähler–Einstein metric, it is enough to consider a bounded pseudoconvex domain. The paper deals essentially with bounded domains and we do not address here the question of the hyperbolicity, in the sense of Kobayashi, of unbounded domains in $\mathbb{C}^n$.

1. Gromov hyperbolicity of some distances for domains in $\mathbb{C}^n$

We start here with the case of complex dimension one. In this situation, complete results concerning the Gromov hyperbolicity of domains of hyperbolic type were proved recently.

1.1. Domains of hyperbolic type

We recall that the Poincaré metric is the Hermitian metric on the unit disc $\mathbb{D} \subset \mathbb{C}$, with constant (Gauss) curvature equal to -1. It is defined on $\mathbb{D}$ by

$$ds^2_{\mathbb{D}}(\zeta) := \frac{4d\zeta \otimes d\bar{\zeta}}{\left(1-\zeta\bar{\zeta}\right)^2}.$$

More generally, if S is a Riemann surface with a Hermitian metric ds^2, we denote by λ_S the density of the metric. In local holomorphic coordinates, we have

$$ds^2 = \lambda_S^2 d\zeta \otimes d\bar{\zeta}.$$

For a hyperbolic Riemann surface S, that is a Riemann surface whose universal cover is the unit disc $\mathbb{D}$, we denote by h_S the density of the hyperbolic metric H_S on S. We recall that h_S is (uniquely) defined by projecting on S the Poincaré metric by a universal covering map (which is a local isometry); the curvature of h_S is identically equal to -1. Moreover, the hyperbolic metric decreases under the action of holomorphic maps: if S, S' are two Riemann surfaces and $f : S \to S'$ is holomorphic, then $f^*(h_{S'}) := (h_{S'} \circ f) \cdot |f'| \leqslant h_S$. In particular, if $S \subset S'$, then $h_{S'} \leqslant h_S$.

We recall that a domain contained in $\mathbb{C}$ is of hyperbolic type, meaning that its universal cover is the unit disc, if it has at least two finite boundary points. Finally, the quasihyperbolic metric on a domain $D \subsetneq \mathbb{C}$ is the Hermitian metric with density $1/\delta_D$, where δ_D is the Euclidean distance to $\mathbb{C}\backslash D$. We denote by Q_D the associated quasihyperbolic distance on D.

The links between the hyperbolic and the quasihyperbolic metrics in domains of hyperbolic type are provided by the following

PROPOSITION 1.1. — *Let $D \subset \mathbb{C}$ be a domain of hyperbolic type. Then*

(i) *For every $\zeta \in D$, $h_D(\zeta) \leqslant \frac{2}{\delta_D(\zeta)}$.*

(ii) *If D is simply connected, then for every $\zeta \in D$, $\frac{1}{2\delta_D(\zeta)} \leqslant h_D(\zeta)$.*

In particular, the hyperbolic metric and the quasihyperbolic metric are bi-Lipschitz on simply connected domains of hyperbolic type.

The proof of Proposition 1.1 is standard. Point (i) uses the fact that for $\zeta \in D$, the disc $D(\zeta, \delta_D(\zeta))$ is contained in D. Hence $h_D(\zeta) \leqslant h_{D(\zeta,\,\delta_D(\zeta))}(\zeta)$ $= 2/\delta_D(\zeta)$. Point (ii) uses the fact that if $f : \mathbb{D} \to D$ is a conformal map with $f(0) = \zeta$, then $h_D(\zeta) = 2/|f'(0)|$, then from the Koebe 1/4-Theorem, we have $D(\zeta, |f'(0)|/4) \subset D$. This implies $\delta_D(\zeta) \geqslant |f'(0)]/4 = 1/(2h_D(\zeta))$.

The lower bound for h_D in (ii) is not satisfied for arbitrary domains of hyperbolic type. The precise link between h_D and δ_D was obtained by A. Beardon and C. Pommerenke [2]. Let

$$\beta_D(\zeta) := \inf\left\{\left|\log\frac{|z-a]}{|b-a|}\right|, a \in \partial D,\ b \in \partial D,\ |\zeta - a| = d(\zeta, \partial D)\right\}.$$

THEOREM 1.2. — *There exist (universal) constants $c \geqslant 1$, $k \geqslant 0$, such that for every domain D of hyperbolic type and every $\zeta \in D$, we have*

$$\frac{1}{C} \leqslant h_D(\zeta)\delta_D(\zeta)\left(\beta_D(\zeta) + k\right) \leqslant C.$$

Using Theorem 1.2, it is not difficult to construct some domain D of hyperbolic type in $\mathbb{C}$ for which the quantity $h_D\delta_D$ is not bounded from below by a positive constant on D.

The unit disc, endowed with the Poincaré distance, is Gromov hyperbolic; we will give an elementary proof of this in Corollary 1.8. Then it is clear from Proposition 1.1 that (D, Q_D) is Gromov hyperbolic on every simply connected domain D of hyperbolic type. The following result due to S. Buckley and D. Herron [8] is much more surprising, according to Theorem 1.2:

THEOREM 1.3. — *Let $D \subset \mathbb{C}$ be a domain of hyperbolic type. Then (D, H_D) is Gromov hyperbolic if and only if (D, Q_D) is Gromov hyperbolic.*

The authors also prove that geodesiscs and quasi-geodesics "are the same" for both the hyperbolic and the quasihyperbolic distances. More precisely (see [8])

THEOREM 1.4. — *There exist $H_0 > 1$ and $K_0 > 1$ such that for every hyperbolic plane domain D, every pair of points $a, b \in D$, every hyperbolic geodesic $[a, b]_h$ and every quasihyperbolic geodesic $[a, b]_k$, $l_k([a, b]_h) \leqslant K_0 k(a, b)$ and $l_h([a, b]_k) \leqslant H_0 h(a, b)$.*

Moreover, for each $\lambda \geqslant 1$, there are explicit constants H and K that depend only on λ such that for every hyperbolic $(\lambda, 0)$-quasi-geodesic γ_h and every quasihyperbolic $(\lambda, 0)$-quasi-geodesic γ_k both with endpoints a and b, $l_k(\gamma_h) \leqslant Kk(a, b)$ and $l_h(\gamma_k) \leqslant Hh(a, b)$.

One may wonder if an analog of Theorem 1.3 might be true for classical invariant distances in complex manifolds in any complex dimension. We study this question for domains in $\mathbb{C}^n$ in the next Subsection 1.2.

1.2. The Carathéodory and the Kobayashi metrics

The Poincaré metric has an intrinsic definition from a purely complex point of view. Indeed, it follows from the Schwarz–Pick Lemma that for every holomorphic self map of the unit disc and every $\zeta \in \mathbb{D}$,

$$\frac{|f'(\zeta|}{1-|f(\zeta|^2} \leqslant \frac{1}{1-|\zeta|^2}$$

with equality at some point $\zeta_0 \in \mathbb{D}$ if and only if $f \in Aut(\mathbb{D})$. This means that holomorphic self maps of $\mathbb{D}$ decrease the Poincaré metric and automorphisms of $\mathbb{D}$ are isometries for the Poincaré metric. Using the form of elements of $Aut(\mathbb{D})$, we may provide with the two following dual definitions of the Poincaré metric:

$$\begin{aligned}\forall\, \zeta \in \mathbb{D},\, \forall\, v \in \mathbb{C},\, & h_{\mathbb{D}}(\zeta; v) \\ & = \sup\{|f'(\zeta).v|,\, f : \mathbb{D} \to \mathbb{D},\, f \text{ holomorphic},\, f(\zeta) = 0\} \\ & = \inf\{v/|f'(0)|,\, f : \mathbb{D} \to \mathbb{D},\, f \text{ holomorphic},\, f(0) = \zeta\}\,.\end{aligned}$$

One sees that the quantities defined by the two equalities may be generalized to any (almost) complex manifold M, replacing the source domain $\mathbb{D}$ in the first equality and the image domain $\mathbb{D}$ in the second one by M. This drives to two well-defined infinitesimal pseudometrics on any complex manifolds.

Definition 1.5. — *Let M be a complex manifold, let $p \in M,\ v \in T_pM$. Then:*

(i) *The Carathéodory infinitesimal metric $c_M(p; v)$ is defined by*

$$c_M(p; v) := \sup\{|f'(p).v|,\, f : M \to \mathbb{D},\, f \text{ holomorphic},\, f(p) = 0\}\,,$$

(ii) *The Kobayashi–Royden infinitesimal metric $k_M(p; v)$ is defined by*

$$\inf\{v/|f'(0)|,\, f : \mathbb{D} \to M,\, f \text{ holomorphic},\, f(0) = p\}\,.$$

We refer to [22] and references therein for the different notions about invariant metrics. The Carathéodory and the Kobayashi infinitesimal pseudometrics are complex Finsler mectrics that may be only continuous, depending on M. One can then define the integrated Carathéodory pseudodistance C_M^i and the integrated Kobayashi pseudodistance K_M^i on a connected complex manifold M by taking the infimum of the lengths of piecewise C^1 curves in M. We may notice that the integrated Kobayashi pseudodistance is equal to the classical Kobayashi pseudodistance K_M

on M defined by chains. However, the integrated Carathéodory pseudodistance may be larger than the classical Carathéodory pseudodistance C_M defined by

DEFINITION 1.6. — *Let M be a complex manifold. The Carathéodory pseudodistance is defined on M by*

$$\forall\, p, q \in M,\ C_M(p,q) = \sup\left\{H_{\mathbb{D}}\left(f(p), f(q)\right),\ f : M \to \mathbb{D},\ f \text{ holomorphic}\right\}.$$

Then C_M (respectively K_M) is the smallest (respectively the largest) pseudodistance on M that decreases under the action of holomorphic maps.

The Carathéodory and the Kobayashi distances may not be equivalent as can be seen considering the punctured disc. Indeed, the Kobayashi distance is equal to the hyperbolic metric with density $h_{\mathbb{D}\setminus\{0\}}(\zeta) = -1/(2|z|\log|z|)$ up to a multiplicative constant, and the space $(\mathbb{D}\setminus\{0\}, K_{\mathbb{D}\setminus\{0\}})$ is complete. Notice also that the Kobayashi distance is not equivalent to the quasi-hyperbolic distance on $\mathbb{D}\setminus\{0\}$. It follows from the Hopf–Rinow Theorem that $(\mathbb{D}\setminus\{0\}, K_{\mathbb{D}\setminus\{0\}})$ is a geodesic space, meaning that any two points may be joined by a geodesic segment. Although $(\mathbb{D}\setminus\{0\}, K_{\mathbb{D}\setminus\{0\}})$ is a complete Riemannian manifold with negative sectional curvature, it is not uniquely geodesic since there are two geodesic segments joining opposite points in $\mathbb{D}\setminus\{0\}$. The topological obstruction for the non uniqueness of geodesics is the non simply connectedness of $\mathbb{D}\setminus\{0\}$.

On the contrary, the Carathéodory and the integrated Carathéodory distances coincide on $\mathbb{D}\setminus\{0\}$ and are the restriction to $\mathbb{D}\setminus\{0\}$ of the Poincaré distance of $\mathbb{D}$. In particular, $(\mathbb{D}\setminus\{0\}, C_{\mathbb{D}\setminus\{0\}})$ is not complete and not geodesic since one cannot join opposite points by a geodesic segment.

One can show that both $(\mathbb{D}\setminus\{0\}, C_{\mathbb{D}\setminus\{0\}})$ and $(\mathbb{D}\setminus\{0\}, K_{\mathbb{D}\setminus\{0\}})$ are Gromov hyperbolic. It is direct for $C_{\mathbb{D}\setminus\{0\}}$ and based on the study of geodesics for $(\mathbb{D}\setminus\{0\}, K_{\mathbb{D}\setminus\{0\}})$. Notice however that the universal cover N of a complex manifold M may be Gromov hyperbolic, when endowed with its Kobayashi metric, although (M, K_M) might not be Gromov hyperbolic. The Denjoy domains, domains of hyperbolic domains whose finite boundary is contained in the real axis, provide with such examples, see [20, Theorem 4.3].

It is well known that the Carathéodory distance and the Kobayashi distance coincide on $\mathbb{B}^n$ and are equal, up to some multipicative constant, to the Bergman distance on $\mathbb{B}^n$ viewed as a representation of the hyperbolic complex space. For every $n \geqslant 1$, the metric space $(\mathbb{B}^n, K_{\mathbb{B}^n})$ is Gromov hyperbolic. There exist several proofs of these facts. For $\mathbb{D}$, they are mainly based on the Gauss–Bonnet formula or on the estimate of the insize of

geodesic triangles in $\mathbb{D}$. The fact that $(\mathbb{B}^n, K_{\mathbb{B}^n})$ is Gromov hyperbolic relies in general on the fact that $(\mathbb{B}^n, K_{\mathbb{B}^n})$ is a complete simply connected Riemannian manifold with Riemann sectional curvature pinched between -1 and $-1/4$. Hence, it is a Cat(k) space, with $k < 0$, and so Gromov hyperbolic. Using the transitivity of the automorphism groups of $\mathbb{D}$ and $\mathbb{B}^n$, we may also provide with a different easy proof of this fact.

PROPOSITION 1.7. — *Let D be a bounded domain in $\mathbb{C}^n$. Assume that:*

(i) *There exists $0 < \varepsilon_0 << 1$ such that for every $z \in D$, there is an injective holomorphic map $\Phi_z : D \to \mathbb{B}^n$ satisfying $\Phi_z(z) = 0$ and $(1-\varepsilon_0)\mathbb{B}^n \subset \Phi_z(D)$,*

(ii) *For every $z \in D$, for every $s, t > 0$ and for every geodesic segment $\gamma : [-s, t] \to \Phi_z(D)$, with $\gamma(0) = 0$, if $dist(\gamma(-s), \partial\mathbb{B}^n) < 2\varepsilon_0$ and $dist(\gamma(t), \partial\mathbb{B}^n) < 2\varepsilon_0$, then $|\gamma(-s) - \gamma(t)| > 1/4$,*

(iii) *There is $0 < r_0 < 1 - 2\varepsilon_0$ such that for every $z \in D$ and for every $p, q \in \Phi_z(D)$,*

$$|p - q| > 1/4 \Rightarrow [p, q]_{\Phi_z(D)} \cap B(0, r_0) \neq \emptyset,$$

where $[p, q]_{\Phi_z(D)}$ denotes any geodesic segment in $\Phi_z(D)$ joining p to q.

Then (D, K_D) is Gromov hyperbolic.

Proof. — We argue by contradiction. Let $(T_\nu = [x^\nu, y^\nu, z^\nu])_\nu$ be a sequence of geodesic triangles in D and let $w^\nu \in [x^\nu, y^\nu]$ be such that

$$K_D\left(w^\nu, [x^\nu, z^\nu] \cup [y^\nu, z^\nu]\right) \to_{\nu\to\infty} \infty.$$

Let $\Phi_\nu := \Phi_{w^\nu}$ and $D_\nu := \Phi_\nu(D)$. Then $\Phi_\nu(w^\nu) = 0$. We keep the notations T_ν, x^ν, y^ν and z^ν for $\Phi_\nu(T_\nu)$, $\Phi_\nu(x^\nu)$, $\Phi_\nu(y^\nu)$ and $\Phi_\nu(z^\nu)$, meaning that T_ν is now a geodesic triangle in D_ν and

$$\lim_{\nu\to\infty} K_{D_\nu}\left(0, [x^\nu, z^\nu] \cup [y^\nu, z^\nu]\right) = \infty. \tag{1.1}$$

Since for every $\nu \geqslant 1$ and for every $z \in \overline{B(0, 1-2\varepsilon_0)}$, $K_{D_\nu}(0, z) \leqslant K_{B(0,1-\varepsilon)}(0, z)$, we have for sufficiently large ν,

$$d(x^\nu, \partial\mathbb{B}^n) < 2\varepsilon_0,\ d(y^\nu, \partial\mathbb{B}^n) < 2\varepsilon_0,\ d(z^\nu, \partial\mathbb{B}^n) < 2\varepsilon_0.$$

It follows from Condition (ii) that $|x^\nu - y^\nu| > 1/4$ and, if ε_0 is chosen such that for every three points $x, y, z \in \mathbb{B}^n \backslash (1-2\varepsilon_0)\mathbb{B}^n$,

$$|x - y| > 1/4 \Rightarrow max(|x - z|, |y - z|) > 1/4,$$

we may assume that for sufficiently large ν, $|x^\nu - z^\nu| > 1/4$.

It follows now from Condition (iii) that $[x^\nu, z^\nu]_{D_\nu} \cap B(0, r_0) \neq \emptyset$. This implies, by Condition (i),

$$K_{D_\nu}(0, [x^\nu, z^\nu]_{D_\nu}) \leqslant K_{D_\nu}(0, \partial B(0, r_0)) \leqslant K_{B(1-\varepsilon_0)}(0, \partial B(0, r_0)).$$

The last quantity being independent of ν, this contradicts (1.1). □

If $D = \mathbb{D}$ or $D = \mathbb{B}^n$, then $Aut(D)$ is transitive and if $z \in D$, we just consider for Φ_z an automorphism of D that sends z to the origin. Hence, as a direct application of Proposition 1.7, we have

COROLLARY 1.8. — *The metric spaces* $(\mathbb{D}, d_{Poinc})$ *and* $(\mathbb{B}^n, K_{\mathbb{B}^n})$ *are complete, geodesic, Gromov hyperbolic spaces.*

The same approach as in Proposition 1.7, by contradiction, was considered in [27] for the case of $\mathbb{B}^n$. However, in the case of the unit disc $\mathbb{D} \subset \mathbb{C}$ and the unit ball $\mathbb{B}^n$, that approach provides with an upper bound of the constant of Gromov hyperbolicity. We explain this for the unit disc case.

Let $T = [x, y, z]$ be a triangle in $\mathbb{D}$ and let $w \in [x, y]$. Since automorphisms of $\mathbb{D}$ are isometries for the Poincaré metric and the automorphism group of $\mathbb{D}$ is transitive, we may assume that $w = 0$. Moreover, up to a rotation, we may assume that $x \in \mathbb{D} \cap \{\zeta \in \mathbb{C} / \ Re(\zeta) < 0, \ Im(\zeta) = 0\}$ and $y \in \mathbb{D} \cap \{\zeta \in \mathbb{C} / \ Re(\zeta) > 0, \ Im(\zeta) = 0\}$. Finally, we may assume, without loss of generality, that $Im(z) > 0, \ Re(z) \leqslant 0$. The image of the (unique) geodesic joining two distinct points ζ, $\zeta' \in \overline{\mathbb{D}}$ being the arc of the circle passing through ζ, ζ' and orthogonal to $\partial\mathbb{D}$, one can see that

$$K_{\mathbb{D}}(w, [x, z] \cup [y, z]) \leqslant K_{\mathbb{D}}(w, [y, z]) \leqslant K_{\mathbb{D}}(0, Im(\gamma)) = K_{\mathbb{D}}\left(0, \left(\sqrt{2} - 1\right)(1 + i)\right),$$

where γ is the union geodesic line joining the two points 1 and i in $\partial\mathbb{D}$. The second inequality relies on the fact that the largest value of $K_{\mathbb{D}}(w, [y, z])$ is obtained when y tends to point $1 \in \partial\mathbb{D}$ and z tends to point $i \in \partial\mathbb{D}$.

We finally obtain

$$K_{\mathbb{D}}(w, [x, z] \cup [y, z]) \leqslant -\frac{1}{2} \ln\left(\sqrt{2} - 1\right).$$

1.3. Comparison of the distances

Although convexity is not a notion invariant by biholomorphisms, convex domains share particular geometric properties that really simplify their

study from a complex point of view. If D is a convex domain in $\mathbb{C}^n$, it is well known that these two metrics satisfy the following estimates:

$$\forall\, z \in D,\ \forall\, v \in \mathbb{C}^n \backslash \{0\},\ \frac{1}{2\delta_D(z;v)} \leqslant c_D(z;v) = k_D(z;v) \leqslant \frac{1}{\delta_D(z;v)},$$

where $\delta_D(z;v)$ denotes the Euclidean distance from z to $(\mathbb{C}^n \backslash D) \cap (z + \mathbb{C}v)$.

One can then show that (D, K_D) is complete (or equivalently D is complete hyperbolic in the sense of Kobayashi) if and only if D does not contain any entire curve and that this is also equivalent to D being biholomorphic to a bounded domain. Notice that it is unknown whether convex complete hyperbolic domains always admit bounded convex representations.

It is also classical but much involved that if D is a bounded strictly pseudoconvex domain in $\mathbb{C}^n$, then the Carathéodory distance, the Bergman distance, the Kähler–Einstein distance and the Kobayashi distance are quasi-isometric in D. In particular, it follows from the work of [1] that D, endowed with any of these distances, is Gromov hyperbolic.

Finally, it is well known that if M and M' are two complex manifolds, then

$$K_{M\times M'} = \sup\left(K_M, K_{M'}\right), \quad C_{M\times M'} = \sup\left(C_M, C_{M'}\right).$$

In particular, the Kobayashi distance and the Carathéodory distance are equivalent on the product of two bounded strictly pseudoconvex domains. It follows then from [1] that the product of two strictly bounded domains is not Gromov hyperbolic, when endowed with either the Kobayashi distance or the Carathéodory distance. The proof in [1] works perfectly in the case of product of two complete complex manifolds (for either the Carathéodory or the Kobayashi distance):

PROPOSITION 1.9. — *Let M and M' be two complex manifolds. Assume that (M, C_M) and $(M', C_{M'})$ are C-hyperbolic and complete (resp. that (M, K_M) and $(M', K_{M'})$ are complete). Then $(M \times M', C_{M\times M'})$ (resp. $(M \times M', K_{M\times M'})$) is not Gromov hyperbolic.*

We need to be careful with the notions involved here. The proof in [1] relies on the fact that the closed balls for the Kobayashi distance $K_{M\times M'}$ are compact when M and M' are strictly pseudoconvex domains. This is still true as soon as (M, K_M) and $(M', K_{M'})$ are complete (and in that case $K_{M\times M'}$ defines the usual topology on $M \times M'$). However, the closed balls of $C_{M\times M'}$ are known to be compact when $(M, C_{M\times M'})$ is complete and $C_{M\times M'}$ defines the usual product topology on $M \times M'$. This is achieved when (M, C_M) and $(M', C_{M'})$ are *C-hyperbolic*, meaning that the distances

C_M and $C_{M'}$ are complete on M and M' and that they define the usual topologies on M and M'.

There exist different examples of bounded pseudoconvex domains in $\mathbb{C}^n$ for which the Carathéodory distance and the Kobayashi distance are not equivalent. We may expect in particular that for a given domain D, one of the corresponding metric spaces might be Gromov hyperbolic whereas the other one might not. Based on the example of the punctured disc, we may construct an analog example in higher dimension.

PROPOSITION 1.10. — *Let $D := \mathbb{B}^2 \backslash (\mathbb{D} \times \{0\}) \subset \mathbb{C}^2$. Then (D, C_D) is Gromov hyperbolic whereas (D, K_D) is not Gromov hyperbolic.*

Proof. — Since C_D is the restriction to D of the Carathéodory distance on $\mathbb{B}^2$, the Gromov hyperbolicity of (D, C_D) is guaranteed. The non Gromov hyperbolicity of (D, K_D) is based on the Geodesic Stability Theorem which states that if (X, d) is a proper geodesic metric space, then (X, d) is Gromov hyperbolic if and only if for any real numbers A, B, with $A \geqslant 1$, $B \geqslant 0$, there exists $M(D, \delta, A, B) > 0$ such that for every $p, q \in X$, for every geodesic segment γ joining p and q and for every (A, B)-quasigeodesic segment φ joining p and q, the Hausdorff distance between $Im(\gamma)$ and $Im(\varphi)$ is bounded from above by M. The equivalence of the two previous assertions is proved in [5]. Notice that in any geodesic metric space, we have the implication

$$Im(\varphi) \subset \mathcal{N}_K\left(Im(\gamma)\right) \Rightarrow Im(\gamma) \subset \mathcal{N}_{2K}\left(Im(\varphi)\right),$$

where, for $Y \subset X$, $\mathcal{N}_K(Y) := \{x \in X / \ d(x, Y) < K\}$.

Then the proof of Proposition 1.10 consists in constructing, for some fixed $A \geqslant 1$, $B \geqslant 0$, a sequence $(T_\nu)_{\nu \in \mathbb{N}}$ of (A, B)-quasigeodesic triangles such that for every $\nu \geqslant 0$, T_ν does not satisfy the Rips condition with constant ν, meaning that T_ν is not ν-slim. For $D = \mathbb{B}^2 \backslash (\mathbb{D} \times \{0\})$, one may consider $T_\nu = [x_\nu, y_\nu] \cup [y_\nu, z_\nu] \cup [x_\nu, z_\nu]$, where for every $\nu \geqslant 1$, $x_\nu = (0, 1/2)$, $y_\nu = (0, 1/\nu)$ and $z_\nu = (1 - t_\nu, 1/\nu)$, for some choice of t_ν, and $[x_\nu, y_\nu]$, $[y_\nu, z_\nu]$, $[x_\nu, z_\nu]$ are the Euclidean segments joining them. One first needs to prove that these Euclidean segments are uniform (A, B) quasigeodesics and then that T_ν is not ν-slim for sufficient large ν. This was proved by A. Chrih-F. Haggui [18] for $\Omega \backslash H$, where Ω is any bounded convex domain in $\mathbb{C}^n$ and H is a complex hyperplane in $\mathbb{C}^n$ such that $\Omega \cap H \neq \emptyset$. □

This drives us to the following question, motivated by the situation in $\mathbb{C}$:

Question 1. — Are there (geometric) conditions on $D \subset\subset \mathbb{C}^n$ ensuring that (D, C_D) is Gromov hyperbolic if and only if (D, K_D) is Gromov hyperbolic?

We may for instance ask whether for every $D \subset\subset \mathbb{C}^n$ such that C_D, and consequently K_D, are complete on D, then (D, C_D) is Gromov hyperbolic if and only if (D, K_D) is Gromov hyperbolic.

Based on the consideration in complex dimension one, we may also study the relation between the Kobayashi distance (or the Carathéodory distance) and some analog of the quasihyperbolic distance in higher dimension.

Let $D \subset\subset \mathbb{C}^n$. Let f_D be the continuous complex Finsler metric defined for every $z \in D$, $v \in \mathbb{C}^n \backslash \{0\}$, by

$$f_D(z; v) := \frac{|v|}{\delta_D(z; v)}.$$

We define the *complex slice distance* F_D as the integrated distance associated with f_D. For instance, as mentioned above, the *complex slice distance* is equivalent to the Carathéodory and the Kobayashi distances on any convex domain. In contrast, it is equivalent neither to the Carathéodory diatance, nor to the Kobayashi distance, on the punctured disc. There exist domains, in any dimension, such that the Kobayashi metric and the *complex slice distance* are not equivalent. Such examples may be constructed as products of a ball with the punctured disk or with some Denjoy domains. After the work of S. Buckley and D. Herron [8], we may consider the following

Question 2. — Let $D \subset \mathbb{C}^n$ be a hyperbolic domain, meaning that K_D is a distance on D inducing the Euclidean topology on D. Is (D, K_D) Gromov hyperbolic if and only if (D, F_D) is Gromov hyperbolic?

Notice that the same question replacing K_D with C_D has a negative answer, considering $D = \mathbb{B}^2 \backslash (\mathbb{D} \times \{0\})$. However, the question is relevant for domains D with C_D complete.

2. Links with CR geometry

Cauchy–Riemann (CR) invariants were introduced in the purpose of classifying complex manifolds under the action of biholomorphisms. These are obstructions for two complex manifolds, having different invariants, to be biholomorphic. The most achieved theory in that vein is due to S.S. Chern–J. Moser [11]: they provide with a complete list of invariants that ensure

the local equivalence of two strictly pseudoconvex manifolds sharing the same invariants. In $\mathbb{C}^n$, the strategy to prove the non equivalence of two domains might be the following. Consider a biholomorphism between two domains D and D'. If one can prove that f extends to ∂D, then one might hope to compare CR invariants of ∂D and $\partial D'$. Such an extension result for strictly pseudoconvex domains was the core of the celebrated extension Theorem by C. Fefferman [13]; different simpler proofs were provided later on. Fundamental questions consist first in defining invariants for more general hypersurfaces than the strictly pseudoconvex ones (we focus on the d'Angelo type here) and second in studying the extension of biholomorphisms between domains.

We recall the following definition of the d'Angelo type for smooth hypersurfaces in $\mathbb{C}^n$. Let Γ be a smooth real hypersurface in $\mathbb{C}^n$ and let r be a local defining function for Γ. For $p \in \mathbb{C}^n$, let $C^*(0,p)$ denote the set of germs of non constant holomorphic maps z from $\mathbb{C}$ to $\mathbb{C}^n$, such that $z(0) = p$. If g is a smooth function defined in a neighborhood of $0 \in \mathbb{C}^n$, we denote by $\nu(g)$ the order of vanishing of the function $g - g(0)$ at the origin. Following [12], the *type* $\tau(\Gamma, p)$ of M at $p \in \Gamma$ is defined by

$$\tau(\Gamma, p) := \sup_{z \in C^*(0,\, p)} \frac{\nu(r \circ z)}{\nu(z)}.$$

Then the hypersurface Γ is of *finite type* (in the sense of D'Angelo) if $\tau(\Gamma, p) < \infty$ for every $p \in \Gamma$.

One might expect that some curvature condition on the boundary of a domain in $\mathbb{C}^n$, given for instance by the d'Angelo type, might force the Gromov hyperbolicity of the domain endowed with its Kobayashi distance. Z. Balogh–M. Bonk [1] proved for instance that this is true for bounded strictly pseudoconvex domains. The following very nice result, proved by A.Zimmer [26], gives the precise relation between geometric (metric) properties and CR properties of bounded smooth convex domains.

THEOREM 2.1. — *Let D be a bounded convex domain in $\mathbb{C}^n$, with ∂D of class $\mathcal{C}^\infty$. Then (D, K_D) is Gromov hyperbolic if and only if ∂D is of finite d'Angelo type.*

The general situation is open and is far from being reachable at the moment, even in complex dimension two:

Question 3 2.2. — Let D be a smooth bounded complete hyperbolic domain in $\mathbb{C}^n$. Is (D, K_D) Gromov hyperbolic of and only if ∂D is of finite d'Angelo type?

2.1. Gromov hyperbolicity and extension of quasi-isometries

Under finite type d'Angelo conditions, in the aim of generalizing the Fefferman extension Theorem, several authors studied smooth extension of biholomorphisms between domains in $\mathbb{C}^n$. It is reasonable to study how the Gromov hyperbolicity of domains in $\mathbb{C}^n$, endowed with their Kobayashi distance, impacts the extension of biholomorphisms. It is clear that an isometric bijection between two Gromov hyperbolic spaces extends to their Gromov boundary. Since we are interested here in the extension to the Euclidean boundary, it is then necessary to understand the precise link between the Gromov boundary $\partial_G D$ and the Euclidean boundary ∂D of a domain $D \subsetneq \mathbb{C}^n$ and to study the relation between the Gromov closure $\overline{D}^G$ and some compactification of $\overline{D}$, both endowed with their natural topologies. Although Gromov hyperbolicity deals with metric considerations and consequently seems adapted to prove topological extension, the recent work by L. Capogna – E. Le Donne [9] allows to recover the Fefferman extension Theorem from only metric arguments and Gromov hyperbolicity tools. In a recent work with F. Bracci and A. Zimmer [7], we investigated the extension of quasi-isometries between non smooth convex unbounded domains endowed with their Kobayashi distances. The Gromov hyperbolicity theory seems well adapted here since, due to the non smoothness of the boundaries, we may not hope for more than a homeomorphic extension. We proved the following

THEOREM 2.3. — *Let D be a convex domain in $\mathbb{C}^n$. If (D, K_D) is Gromov hyperbolic, then the identity map $id : D \to D$ extends to a homeomorphism $\overline{id} : \overline{D}^\star \to \overline{D}^G$.*

Here $\overline{D}^\star$ denotes the closure $\overline{D}$ of D if D is bounded, the one point compactification of $\overline{D}$ if D has one end and the two points compactification of $\overline{D}$ if D has two ends. Using the previous observation, we obtained the following extension result for quasi-isometries [7]:

COROLLARY 2.4. — *Let D and Ω be domains in $\mathbb{C}^n$. We assume:*

(1) *D is either a bounded, $\mathcal{C}^2$-smooth strongly pseudoconvex domain, or a convex $\mathbb{C}$-proper domain, such that (D, K_D) is Gromov hyperbolic,*
(2) *Ω is convex.*

Then every quasi-isometric homeomorphism $F : (D, K_D) \to (\Omega, K_\Omega)$ extends as a homeomorphism $\overline{F} : \overline{D}^\star \to \overline{\Omega}^\star$. In particular, every biholomorphism $F : D \to \Omega$ extends as a homeomorphism $\overline{F} : \overline{D}^\star \to \overline{\Omega}^\star$.

Although the convexity assumption in Theorem 2.3 might seem artificial, the existence of a homeomorphism between the Gromov boundary $\partial_G D$ and the Euclidean boundary ∂D of a bounded domain $D \subset \mathbb{C}^n$ such that (D, K_D) is Gromov hyperbolic, proper and geodesic, is not guaranteed. For instance, consider the domain

$$D := \{\zeta \in \mathbb{C} / \ -1 < Re(\zeta) < 1, \ -1 < Im(\zeta) < 1\} \setminus \cup_{n \geqslant 2} \left(\left\{ -1 + \frac{1}{n} \right\} \times]-1, 0] \right).$$

Then (D, K_D) is Gromov hyperbolic by the Riemann mapping Theorem. However, the two points $(-1, -3/4)$ and $(-1, -1)$ correspond to the same Gromov point in $\partial_G D$.

Notice that D is not locally connected and, in contrast with Corollary 2.4, the Riemann map from $\mathbb{D}$ to D does not extend continuously to $\overline{\mathbb{D}}$.

Following Theorem 2.3 and the previous remark, one may complete the previous Question 3 as follows.

Question 4 2.5. — Let D be a bounded domain in $\mathbb{C}^2$, with ∂D smooth of class $\mathcal{C}^\infty$. Assume that D is pseudoconvex of finite d'Angelo type. Are the following true?

- (D, K_D) is Gromov hyperbolic,
- There is a homeomorphism between $D \cup \partial_G D$ and $\overline{D}$.

2.2. Finite type condition and holomorphic bisectional curvature

Let (M, J) be a complex manifold with Kähler metric g and let $R(g)$ denote the curvature tensor of (M, g). Then the *holomorphic bisectional curvature* of non-zero $X, Y \in T_p M$ is given by

$$B(g)(X, Y) = \frac{R(g)(X, JX, Y, JY)}{g(X, X)g(Y, Y)}.$$

The existence of analytic varieties in the boundary of a domain D in $\mathbb{C}^n$ is an obstruction for the existence on D of a complete Kähler metric with negative pinched curvature, that is such that there exist $0 < b < a$ satisfying

$$-a \leqslant B(g)(X, Y) \leqslant -b$$

for all $p \in D$ and non-zero vectors $X, Y \in T_p D$.

For instance, P.Yang [24] proved that there is no complete Kähler metric on the bidisc $\mathbb{D}^2 \subset \mathbb{C}^2$ with negative pinched holomorphic bisectional curvature. Notice that the holomorphic sectional curvature does not play any role: the Bergman metric on the bidisc has negative pinched holomorphic sectional curvature. P. Yang's result was generalized to different context, for instance to the product of two complex manifolds by H. Seshadri – F. Zheng [23].

There are also very few results on the positive side and constructing, for a given domain, a complete Kähler metric with negative pinched sectional curvature seems quite difficult. S.Y. Cheng – S.T. Yau [10] proved the existence of a complete Kähler–Einstein metric on every strictly pseudoconvex domain and proved that it has pinched negative holomorphic bisectional curvature in a neighborhood of the boundary of the domain. J. Bland [4] proved that this is also the case for the Kähler–Einstein metric on Thüllen domains. More recently, S. Gontard [16] considered the case of tube domains $\{(w,z) \in \mathbb{C}^2 / \ Re(w) + (Re(z))^{2m} < 0\}$, where $m \in \mathbb{N}\backslash\{0\}$. He proved that the holomorphic bisectional curvature of the Kähler–Einstein metric is negative pinched in a neighborhood of the boundary, at least for nontangential approach.

Finally, in the recent paper with F. Bracci and A. Zimmer [6], we studied the link between holomorphic bisectional curvature of Kähler metrics, the d'Angelo type and the Goromov hyperbolicity. We proved the following

THEOREM 2.6. — *Let D be a bounded convex domain in $\mathbb{C}^n$, with ∂D of class $\mathcal{C}^\infty$. If there exists on D a complete Kähler metric g, with negative pinched holomorphic bisectional curvature outside a compact subset of D, then ∂D is of finite d'Angelo type. Moreover, (D, d_g) is Gromov hyperbolic.*

Here d_g denotes the integrated distance associated with the Kähler metric g. Notice that since D is a smooth bounded convex domain of finite type, then (D, K_D) is Gromov hyperbolic according to [26]. Moreover, under the assumptions of Theorem 2.6, it follows from the Yau–Schwarz Lemma that, up to multiplicative constants, d_g is bounded on D from below by the Carathéodory distance C_D and from above by the Kobayashi distance K_D. Since D is convex, $C_D = K_D$: d_g and K_D are bi-Lipschitz. This proves that (D, d_g) is Gromov hyperbolic, as stated in Theorem 1.7.

If D is a pseudoconvex domain of finite d'Angelo type, we may propose the following

CONJECTURE 2.7. — *Let D be a smooth bounded pseudoconvex in $\mathbb{C}^n$. The following conditions are equivalent:*

(i) *There exists on D a complete Kähler metric with pinched negative holomorphic bisectional curvature outside a compact subset of D,*
(ii) *There exists on D a complete Kähler metric g such that the corresponding metric space (D, d_g) is Gromov hyperbolic,*
(iii) *∂D is of finite type,*
(iv) *(D, K_D) is Gromov hyperbolic.*

Notice that it seems reasonable to replace K_D either with C_D, with the Bergman distance on D, or the Kähler–Einstein distance on D in Condition (iv).

3. Uniform squeezing property

Let D be a bounded domain in $\mathbb{C}^n$, $n \geqslant 1$. Fix some point $z \in D$. Then for every injective holomorphic map $\Phi : D \to \mathbb{B}^n$ such that $\Phi(z) = 0$, let

$$r_D(\Phi, z) := \sup \{0 \leqslant r \leqslant 1 / \ r\mathbb{B}^n \subset \Phi(D)\}$$

and

$$r_D(z) := \sup \{r_D(\Phi, z)\}.$$

Then D is uniformly squeezing if $\inf_{z \in D} r_D(z) > 0$. The uniform squeezing property is in some sense an analog of the quasi-bounded geometry condition introduced by S.Y. Cheng – S.T. Yau in [10]. The uniform squeezing condition was introduced by S.K. Yeung [25] and several nice properties of domains satisfying a uniform squeezing property are proved there.

If $D = \mathbb{B}^n$, then $r_D \equiv 1$. It is also quite direct that if D is a bounded convex domain in $\mathbb{C}^n$, then $\inf_{z \in D} r_D(z) > 0$, as proved by K.T. Kim – L. Zhang in [21].

S.K. Yeung proved in [25, Theorem 1] that the integrated Carathéodory distance, the Kobayashi distance, the Bergman distance and the Kähler–Einstein distance are all equivalent and complete on a domain D satisfying a uniform squeezing property and that D is pseudoconvex. These facts may be obtained quite direclty and also include the Carathéodory distance.

Indeed, let D be a domain in $\mathbb{C}^n$, satisfying a uniform squeezing property. There exists $r_0 > 0$ such that for every $z \in D$, there is a biholomorphism Φ from D to $\Phi(D)$, $\Phi(z) = 0$, $r_0\mathbb{B}^n \subset \Phi(D)$. In particular, there exists $c_0 > 1$ such that if we denote by d either the Carathéodory distance, the Kobayashi distance, or the Kähler–Einstein distance, we have for every $w \in \Phi(D)$:

$$d_{\mathbb{B}^n}(0, w) \leqslant d_{\Phi(D)}(0, w). \tag{3.1}$$

In particular, if $c_0 = K_{\mathbb{B}^n}(0, \partial B(0, r_0/2))$, where

$$B(0,r) := \{z \in \mathbb{C}^n / \ |z| < r\},$$

we have

$$B_{\Phi(D),d}(0,c_0) := \{w \in \Phi(D) / \ d(0,w) < c_0\} \subset B(0,r_0/2) \subset\subset \Phi(D).$$

Hence, for every $z \in D$,

$$B_{D,d}(z,c_0) := \{w \in D / \ d(z,w) < c_0\} \subset\subset D,$$

which implies directly that all these distances are complete on D and D is pseudoconvex.

Moreover, there exists $C_0 > 1$ such that for every $w \in B(0, r_0/2)$

$$d_{\mathbb{B}^n}(0,w) \leqslant d_{\Phi(D)}(0,\, w) \leqslant d_{B(0,\, r_0)}(0,w) \leqslant C_0 d_{\mathbb{B}^n}(0,w). \tag{3.2}$$

It follows from (3.2) that there exists $c \geqslant 1$ such that if d and d' denote any of the distances mentioned above, then for every $z \in D$, we have

$$B_{D,\,d}(z, 1/c) \subset B_{D,\,d'}(z,1) \subset B_{D,\,d}(z,c).$$

This also implies that the Carathéodory, the Kähler–Einstein and the Kobayashi distances are equivalent on D.

For the Bergman distance, notice that there exist $\Omega \subset \Omega' \subset \mathbb{C}^n$ and points $z, w \in \Omega$ such that $B_\Omega(z,w) < B_{\Omega'}(z,w)$, where B_Ω denotes the Bergman distance on Ω.

If Ω is a bounded domain in $\mathbb{C}^n$, $w \in D$ and $v \in \mathbb{C}^n$, let $\beta_{\Phi(D)}(w;v)$ denote the Bergman length of v evaluated at w. Let also

$$b_0^\Omega(w) := \sup\left\{|f(w)|^2, \ f \in Hol(\Omega, \mathbb{C}), \ \int_\Omega |f|^2 \leqslant 1\right\}$$

and

$$b_1^\Omega(w;v) := \sup\left\{|f'(w)\cdot v|^2, \ f \in Hol(\Omega,\mathbb{C}), \ f(w) = 0, \ \int_\Omega |f|^2 \leqslant 1\right\}.$$

Then, according to [3], we have for every $w \in B(0, r_0/2)$ and every $v \in \mathbb{C}^n$,

$$\beta_{\Phi(D)}(w;v) = \frac{b_1^{\Phi(D)}(w;v)}{b_0^{\Phi(D)}(w)} \leqslant \frac{b_1^{B(0,\, r_0)}(w;v)}{b_0^{B(0,\,1)}(w)} = c(r_0).$$

Notice that $c(r_0)$ depends only on r_0, not on D.

Now, there exists $c'(r_0) > 0$ such that

$$\inf_{w \,\in\, B(0,\, r_0/2)} k_{\Phi(D)}(w;v) \geqslant \inf_{w \,\in\, B(0,\, r_0/2)} k_{B(0,\,1)}(w;v) \geqslant c'(r_0),$$

where $c'(r_0)$ depends only on r_0. Hence, we have for every $w \in B(0, r_0/2)$,

$$C_{\Phi(D)}(0, w) \leqslant B_{\Phi(D)}(0, w) \leqslant c''(r_0) K_{\Phi(D)}(0, w),$$

where $c''(r_0)$ depends only on r_0. The first inequality is called the Hahn–Lu comparison Theorem (see [19]).

These inequalities imply that C_D, B_D and K_D are equivalent on D and ends the proof.

There is a priori no relation between the uniform squeezing property and the Gromov hyperbolicity of a domain. Indeed, already in complex dimension one, the Denjoy domains provide with examples of domains that do not satisfy a uniform squeezing property (consider a Denjoy domain with non equivalent Carathéodory and Kobayashi distances) and for which the hyperbolic distance (or equivalently the Kobayashi distance) may or may not be Gromov hyperbolic. Moreover, the bidisc and the ball satisfy a uniform squeezing property, the bidisc endowed with the Carathéodory distance (or equivalently the Kobayashi distance) is not Gromov hyperbolic whereas the unit ball endowed with the Carathéodory distance (or equivalently the Kobayashi distance) is Gromov hyperbolic. We may notice that in Proposition 1.7, D satisfies a uniform squeezing property, with an extra condition, called Euclidean visibility.

J.E. Fornaees – E. Wold [15] proved that if D is a bounded strictly pseudoconvex domain in $\mathbb{C}^n$, then $\lim_{z \to \partial D} r_D(z) = 1$. J.E. Fornaess – F. Rong [14] also provided with examples of smooth pseudoconvex domains, not strictly pseudoconvex, for which the squeezing function is not uniformly bounded from above by a constant less than one.

If D is a bounded domain in $\mathbb{C}^n$ for which $\lim_{z \to \partial D} r_D(z) = 1$, one might consider the family $(\Phi_z)_{z \in D}$ providing $r_D(z)$ as a substitute to the non transitivity of the automorphism group of D. In particular, the following question is related to the proof of the Gromov hyperbolicity of $(\mathbb{B}^n, K_{\mathbb{B}^n})$ we proposed on Proposition 1.7:

Question 5 3.1. — Let D be a bounded domain in $\mathbb{C}^n$. Assume that $\lim_{z \to \partial D} r_D(z) = 1$. Is (D, K_D) Gromov hyperbolic?

Finally, one may also wonder whether the condition "$\lim_{z \to \partial D} r_D(z) = 1$" gives some information on the holomorphic bisectional curvature of the Kähler metric of a bounded pseudoconvex domain D. This was investigated by S. Gontard [17] who proved that under that condition, the holomorphic bisectional curvature of the complete Kähler–Einstein metric of D is pinched negative in a neighborhood of ∂D and converges at the

boundary to the holomorphic bisectional curvature of the Kähler–Einstein metric of the ball.

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