Winter Braids Lecture Notes

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Abstract

This article is based on the lectures in the Winter Braids V (Pau, February 2015). We introduce some studies of twisted Alexander polynomials to non-experts through many concrete examples. In this article we follow the definition of the twisted Alexander polynomial by Wada, which can be defined for a finitely presented group with an epimorphism onto a free abelian group. The main tool is FoxÕs free calculus. In the last two sections we discuss some applications on the fiberedness of a knot and the existence of epimorphisms between knot groups.

1. Introduction

This article is based on the lectures in the Winter Braids V (Pau, February 2015). One purpose of these lectures was to explain how to compute twisted Alexander polynomials for non-experts. For this purpose we treated only twisted Alexander polynomials for knots and discussed many concrete examples. It is also keeping in this article. The author intended to write concrete computations in this article to be self-contained.

There are two good survey papers [18, 44] on this subjects. Since this article is more elementary, then we recommend to read them for more advanced topics.

First we recall there are many definitions (many faces) of the classical Alexander polynomial:

- Seifert form on a Seifert surface.
- Fox's free differentials to a presentation of a knot group.
- an order of the Alexander module (an infinite cyclic covering).
- Reidemeister torsion.
- Burau representation of the braid group.
- Obstruction to deform an abelian representation into non commutative direction.
- Skein relation.
- Euler characteristic of the knot Floer homology.

We can generalize some of them to twisted Alexander polynomials.

- Lin defined twisted Alexander polynomial for a knot by using a Seifert surface.
- Wada also defined it for a finitely presentable group by using Fox's free differential.
- Jang and Wang generalized Lin's idea to other invariants.

- Kirk and Livingston organized each of these perspectives, in particular, an order of the Alexander module. This is also related with an infinite cyclic covering.
- Twisted Alexander polynomial of a knot can be described as the Reidemeister torsion of its knot exterior.

From each position of these studies we have slightly different invariants, but essentially the same one, which are called twisted Alexander polynomials. In this lecture note, we mainly follow the definition of the twisted Alexander polynomial by Wada. Twisted Alexander polynomial (Wada's invariant) can be defined for a finitely presentable group with an epimorphism onto a free abelian group. For simplicity, we discuss this invariant only for a knot group with the abelianization.

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2. Fox's free differentials

To define the Alexander polynomial we need one algebraic tool. It is the Fox's free differentials. See [14, 10] as a reference.

Definition 2.1. *An integral group ring of a group G is a ring given by*

$$\mathbb{Z}G = \{\text{a finite formal sum } \sum_{g \in G} n_g g \mid n_g \in \mathbb{Z}\}$$

as a set. Here finite means the number of $n_g \neq 0$ is finite. The two operations of a group ring are defined by the following;

- *sum:* $\displaystyle\sum_{g \in G} n_g g + \sum_{g \in G} m_g g = \sum_{g \in G} (n_g + m_g) g.$
- *multiplication:* $\displaystyle\sum_{g \in G} n_g g \cdot \sum_{g \in G} m_g g = \sum_{g \in G} \left(\sum_{h \in G} n_h \cdot m_{h^{-1}g} \right) g.$

Remark 2.2.

- The unit of $\mathbb{Z}G$ as a group ring is $1 = 1(\in \mathbb{Z}) \times 1(\in G)$.
- We can define a group ring of G over $\mathbb{Q}, \mathbb{R}, \mathbb{C}$, and write respectively $\mathbb{Q}G$, $\mathbb{R}G$ and $\mathbb{C}G$ for them.

Example 2.3. $\mathbb{Z} = \langle t \rangle$
For any element of $\mathbb{Z}\mathbb{Z} = \mathbb{Z}\langle t \rangle$, it is of the form $\displaystyle\sum_{k \in \mathbb{Z}} n_k t^k$. This can be considered as a Laurent polynomial of t. From now we always identify the group ring $\mathbb{Z}\mathbb{Z} = \mathbb{Z}\langle t \rangle$ with the Laurent polynomial ring $\mathbb{Z}[t, t^{-1}]$.

Let $F_n = \langle x_1, \cdots, x_n \rangle$ be the free group generated by $\{x_1, \cdots, x_n\}$. Fox's free differentials are algebraic derivations on $\mathbb{Z}F_n$.

Definition 2.4. *Fox's free differentials are maps*

$$\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n} : \mathbb{Z}F_n \to \mathbb{Z}F_n$$

satisfying the following conditions:

1. *They are linear over* $\mathbb{Z}$.

2. *For any* i, j, $\dfrac{\partial}{\partial x_j}(x_i) = \delta_{ij} = \begin{cases} 1 \ (i = j), \\ 0 \ (i \neq j). \end{cases}$

3. *For any* $g, g' \in F_n$, $\dfrac{\partial}{\partial x_j}(gg') = \dfrac{\partial}{\partial x_j}(g) + g\dfrac{\partial}{\partial x_j}(g')$.

Lemma 2.5. *The followings hold;*

- $\dfrac{\partial}{\partial x_j}(1) = 0$.

- $\dfrac{\partial}{\partial x_j}(g^{-1}) = -g^{-1}\dfrac{\partial}{\partial x_j}(g)$ *for any* $g \in F_n$.

- $\dfrac{\partial}{\partial x_j}(x_j^k) = \begin{cases} 1 + x_j + \cdots + x_j^{k-1} \ (k > 0), \\ -(x_j^{-1} + \cdots + x_j^k) \ \ (k < 0). \end{cases}$

- *For any* $g \in F_n$, $\dfrac{\partial}{\partial x_j}(g^k) = \begin{cases} \dfrac{g^k - 1}{g - 1}\dfrac{\partial}{\partial x_j}(g) \ \ (k > 0), \\ -\dfrac{g^k - 1}{g - 1}\dfrac{\partial}{\partial x_j}(g) \ \ (k < 0). \end{cases}$

For simplicity, we frequently write $\dfrac{\partial w}{\partial x_i}$ for $\dfrac{\partial}{\partial x_i}(w)$ for any $w \in \mathbb{Z}F_n$.
The following formula is the algebraic version of a linear approximation in the group ring of a free group.

Proposition 2.6 (Fundamental formula of free differentials)**.** *For any* $w \in \mathbb{Z}F_n$*, it holds that*

$$w - 1 = \sum_{j=1}^{n} \frac{\partial w}{\partial x_j}(x_j - 1).$$

Proof. We prove this formula by induction on the word length $l(w)$ of $w \in F_n$.
For the case of $l(w) = 0$, that is, $w = 1$, it is clear that $w - 1 = 0$ and $\sum_{j=1}^{n} \dfrac{\partial w}{\partial x_j}(x_j - 1) = 0$.
Assume it is true for any word w with $l(w) = k$. Take any $w \in F_n$ with $l(w) = k + 1$. We may assume $w = w_k x_i^{\pm 1}$ with $l(w_k) = k$. If $w = w_k x_i$, then one has

$$\begin{aligned} \sum_{j=1}^{n} \frac{\partial w}{\partial x_j}(x_j - 1) &= \sum_{j=1}^{n} \frac{\partial (w_k x_i)}{\partial x_j}(x_j - 1) \\ &= \sum_{j=1}^{n} \left(\frac{\partial w_k}{\partial x_j} + w_k \delta_{i,j} \right)(x_j - 1) \\ &= \sum_{j=1}^{n} \frac{\partial w_k}{\partial x_j}(x_j - 1) + w_k(x_i - 1). \end{aligned}$$

By the assumption on the induction,

$$\sum_{j=1}^{n} \frac{\partial w_k}{\partial x_j}(x_j - 1) = w_k - 1.$$

Hence we obtain

$$\begin{aligned}\sum_{j=1}^{n}\frac{\partial w}{\partial x_j}(x_j-1) &= \sum_{j=1}^{n}\frac{\partial w_k}{\partial x_j}(x_j-1)+w_k(x_i-1)\\ &= w_k-1+w_k(x_i-1)\\ &= w_kx_i-1\\ &= w-1.\end{aligned}$$

Similarly it can be proved for the case of $w=w_kx_i^{-1}$.
Further it can be done for any $w\in\mathbb{Z}F_n$ by using the linearity of free differentials.
This completes the proof. □

3. Alexander polynomials

In this section we apply the Fox's free differentials to get a knot invariant as follows. We put [5, 48] for terminologies and definitions of knot theory as references.

3.1. Definition

Let $K\subset S^3$ a knot in S^3 and $G(K)=\pi_1(S^3-K)$ the knot group of K. We take and fix a presentation of $G(K)$ as

$$G(K)=\langle x_1,\dots,x_n \mid r_1,\dots,r_{n-d}\rangle.$$

Now we do not assume it is a Wirtinger presentation.
For simplicity we explain first how to define the invariant for the case of $d=1$. Here the number d is called the deficiency of a finite presented group, which is defined by the number of generators minus the number of relators.
Let us take a presentation of deficiency one as

$$G(K)=\langle x_1,\dots,x_n \mid r_1,\dots,r_{n-1}\rangle.$$

By using the above fixed presentation, an epimorphism

$$F_n\ni x_i\mapsto x_i\in G(K)$$

is naturally defined. Further we consider a ring homomorphism

$$\mathbb{Z}F_n\to\mathbb{Z}G(K)$$

induced from this epimorphism $F_n\to G(K)$.
The abelianization of $G(K)$ is given as

$$\alpha: G(K)\to G(K)/[G(K),G(K)]\cong\mathbb{Z}=\langle t\rangle$$

and the induced map on group rings as

$$\alpha_*:\mathbb{Z}G(K)\to\mathbb{Z}\langle t\rangle=\mathbb{Z}[t,t^{-1}].$$

Definition 3.1. *The $(n-1)\times n$-matrix A defined by*

$$A=\left(\alpha_*\left(\frac{\partial r_i}{\partial x_j}\right)\right)\in M\big((n-1)\times n;\mathbb{Z}[t,t^{-1}]\big)\ (1\le i\le n-1,\ 1\le j\le n)$$

is called the Alexander matrix of $G(K)=\langle x_1,\dots,x_n \mid r_1,\dots,r_{n-1}\rangle$.

Let A_k be the $(n-1)\times(n-1)$-matrix obtained by removing the k-th column from A.

Lemma 3.2. *There exists an integer $k\in\{1,\cdots,n\}$ such that $\alpha_*(x_k)-1\neq 0\in\mathbb{Z}[t,t^{-1}]$.*

Proof. If $\alpha(x_k)=1$ for any k, then clearly $\alpha: G(K)\to\mathbb{Z}$ is the trivial homomorphism, not an epimorphism. It contradicts that α is an epimorphism. □

Lemma 3.3. *For any $k, l \in \{1, \cdots, n\}$,*

$$(\alpha_*(x_l) - 1)\det A_k = \pm(\alpha_*(x_k) - 1)\det A_l.$$

Proof. We may assume $k = 1, l = 2$ without loss of generality.
For any relator $r_i = 1 \in \mathbb{Z}G(K)$, by applying the fundamental formula and projection on $\mathbb{Z}G(K)$, it is seen that

$$0 = r_i - 1 = \sum_{j=1}^{n} \frac{\partial r_i}{\partial x_j}(x_j - 1).$$

By applying α_* to both sides, we obtain

$$\sum_{j=1}^{n} \alpha_*\left(\frac{\partial r_i}{\partial x_j}\right)(\alpha_*(x_j) - 1) = 0.$$

Hence one obtains

$$(\alpha_*(x_1) - 1)\alpha_*\left(\frac{\partial r_i}{\partial x_1}\right) = -\sum_{j=2}^{n} \alpha_*\left(\frac{\partial r_i}{\partial x_j}\right)(\alpha_*(x_j) - 1).$$

Here let A_2 be the matrix obtained by removing the second column from A and $\tilde{A}_2$ be the one obtained by replacing the first column $\alpha_*\left(\frac{\partial r_i}{\partial x_1}\right)$ by $(\alpha_*(x_1) - 1)\alpha_*\left(\frac{\partial r_i}{\partial x_1}\right)$ in A_2.
Take the determinant

$$\begin{aligned}
\det \tilde{A}_2 &= \begin{vmatrix} (\alpha_*(x_1) - 1)\,\alpha_*\left(\frac{\partial r_1}{\partial x_1}\right) & \alpha_*\left(\frac{\partial r_1}{\partial x_3}\right) & \dots & \alpha_*\left(\frac{\partial r_1}{\partial x_n}\right) \\ \vdots & \vdots & \dots & \vdots \\ (\alpha_*(x_1) - 1)\,\alpha_*\left(\frac{\partial r_{n-1}}{\partial x_1}\right) & \alpha_*\left(\frac{\partial r_{n-1}}{\partial x_3}\right) & \dots & \alpha_*\left(\frac{\partial r_{n-1}}{\partial x_n}\right) \end{vmatrix} \\
&= (\alpha_*(x_1) - 1)\det A_2.
\end{aligned}$$

On the other hand, replace $(\alpha_*(x_1) - 1)\,\alpha_*\left(\frac{\partial r_i}{\partial x_1}\right)$ by $-\sum_{j=2}^{n} \alpha_*\left(\frac{\partial r_i}{\partial x_j}\right)(\alpha_*(x_j) - 1)$, the same determinant is given by

$$\begin{aligned}
\det \tilde{A}_2 &= \begin{vmatrix} -\sum_{j=2}^{n} \alpha_*\left(\frac{\partial r_1}{\partial x_j}\right)(\alpha_*(x_j) - 1) & \dots & \alpha_*\left(\frac{\partial r_1}{\partial x_n}\right) \\ \vdots & \dots & \vdots \\ -\sum_{j=2}^{n} \alpha_*\left(\frac{\partial r_{n-1}}{\partial x_j}\right)(\alpha_*(x_j) - 1) & \dots & \alpha_*\left(\frac{\partial r_{n-1}}{\partial x_n}\right) \end{vmatrix} \\
&= -\sum_{j=2}^{n} (\alpha_*(x_j) - 1) \begin{vmatrix} \alpha_*\left(\frac{\partial r_1}{\partial x_j}\right) & \alpha_*\left(\frac{\partial r_1}{\partial x_3}\right) & \dots & \alpha_*\left(\frac{\partial r_1}{\partial x_n}\right) \\ \vdots & \dots\dots\dots\dots\dots & & \vdots \\ \alpha_*\left(\frac{\partial r_{n-1}}{\partial x_j}\right) & \alpha_*\left(\frac{\partial r_{n-1}}{\partial x_3}\right) & \dots & \alpha_*\left(\frac{\partial r_{n-1}}{\partial x_n}\right) \end{vmatrix} \\
&= -(\alpha_*(x_2) - 1) \begin{vmatrix} \alpha_*\left(\frac{\partial r_1}{\partial x_2}\right) & \alpha_*\left(\frac{\partial r_1}{\partial x_3}\right) & \dots & \alpha_*\left(\frac{\partial r_1}{\partial x_n}\right) \\ \vdots & \dots\dots\dots\dots\dots & & \vdots \\ \alpha_*\left(\frac{\partial r_{n-1}}{\partial x_2}\right) & \alpha_*\left(\frac{\partial r_{n-1}}{\partial x_3}\right) & \dots & \alpha_*\left(\frac{\partial r_{n-1}}{\partial x_n}\right) \end{vmatrix} \\
&= -(\alpha_*(x_2) - 1)\det A_1.
\end{aligned}$$

Therefore it holds that

$$(\alpha_*(x_1) - 1)\det A_2 = -(\alpha_*(x_2) - 1)\det A_1.$$

□

From these two lemmas, we can consider

$$\frac{\det A_k}{\alpha_*(x_k)-1}$$

as an invariant of $G(K)$ with a presentation with deficiency one.
Now we supposed that the deficiency of a presentation is one. To prove that this invariant is independent of choices of a presentation, up to $\pm t^s$ ($s \in \mathbb{Z}$), we define it for the case of higher deficiencies and apply the Tietze transformations to them.
We take and fix a presentation of $G(K)$ as

$$G(K) = \langle x_1, \dots, x_n \mid r_1, \dots, r_{n-d} \rangle$$

where $1 \leq d \leq n-1$.
The Alexander matrix associated to the above presentation is similarly defined by

$$A = \left(\alpha_* \left(\frac{\partial r_i}{\partial x_j} \right) \right) \in M\big((n-d) \times n; \mathbb{Z}[t, t^{-1}]\big) \ (1 \leq i \leq n-d, \ 1 \leq j \leq n).$$

Let A_k be the $(n-d) \times (n-1)$-matrix obtained by removing the k-th column from A. This is not a square matrix if $d \geq 2$.
Let A_k^I be the $(n-d) \times (n-d)$-matrix consisting of the columns whose indices belong to $I = (i_1, \dots, i_{n-d})$ $(1 \leq i_1 < \cdots < i_{n-d} \leq n)$.
By similar arguments as for the deficiency one case, we can also prove the following lemma.

Lemma 3.4. *For any $k, l \in \{1, \cdots, n\}$ and any choice of I such that $k, l \notin I$,*

$$(\alpha_*(x_l) - 1) \det A_k^I = \pm (\alpha_*(x_k) - 1) \det A_l^I.$$

Furthermore it is similarly seen that there exists an integer $k \in \{1, \cdots, n\}$ such that $\alpha_*(x_k) - 1 \neq 0 \in \mathbb{Z}[t, t^{-1}]$. Now we put Q_k to be the greatest common divisor of $\det A_k^I$ for all indices I. From the above, we can consider

$$\frac{Q_k}{\alpha_*(x_k)-1}$$

as an invariant of $G(K)$.

Remark 3.5. For the case of $d = 1$, we can chose the index set I as $I = (1, \dots, k-1, k+1, \dots, n)$. Hence the above definition gives the same one as in the case of deficiency one presentations.

Now we recall Tietze transformations as follows. See [37] for example.

Theorem 3.6 (Tietze)**.** *Any presentation $G = \langle x_1, \cdots, x_k \mid r_1, \cdots, r_l \rangle$ can be transformed to any other presentation of G by an application of a finite sequence of the following two type operations and their inverses:*

(I) To add a consequence r of the relators $r_1, \cdots, r_l$ to the set of relators. The resulting presentation is given by $\langle x_1, \cdots, x_k \mid r_1, \cdots, r_l, r \rangle$.

(II) To add a new generator x and a new relator xw^{-1} where w is any word in $x_1, \cdots, x_k$. The resulting presentation is given by $\langle x_1, \cdots, x_k, x \mid r_1, \cdots, r_l, xw^{-1} \rangle$.

We can prove the following.

Proposition 3.7. *Up to $\pm t^s$ ($s \in \mathbb{Z}$), the rational expression*

$$\frac{Q_k}{\alpha_*(x_k)-1}$$

is independent of the choice of a presentation of $G(K)$. Namely it is an invariant of the group $G(K)$ up to $\pm t^s$ ($s \in \mathbb{Z}$).

Proof. Take presentations as

$$P = \langle x_1, \dots, x_n \mid r_1, \dots, r_{n-d} \rangle$$

and

$$P' = \langle x_1, \dots, x_n \mid r_1, \dots, r_{n-d}, r \rangle$$

by applying the Tietze transformation (I). Now assume r has a form as

$$r = \prod_{k=1}^{p} w_k r_{i_k}^{\epsilon_k} w_k^{-1}.$$

where $1 \le i_k \le n-d$, $w_k \in F_n$ and $\epsilon_k = \pm 1$ for $1 \le k \le p$. By applying Fox's free differentials, one has

$$\begin{aligned} \frac{\partial r}{\partial x_j} &= \sum_{k=1}^{p} \left(\prod_{l=1}^{k-1} w_l r_{i_l}^{\epsilon_l} w_l^{-1} \right) \left(\frac{\partial w_k}{\partial x_j} + u_k \frac{\partial r_{i_k}}{\partial x_j} - w_k r_{i_k}^{\epsilon_k} w_k^{-1} \frac{\partial w_k}{\partial x_j} \right) \\ &= \sum_{k=1}^{p} \left(\prod_{l=1}^{k-1} w_l r_{i_l}^{\epsilon_l} w_l^{-1} \right) \left((1 - w_k r_{i_k}^{\epsilon_k} w_k^{-1}) \frac{\partial w_k}{\partial x_j} + u_k \frac{\partial r_{i_k}}{\partial x_j} \right). \end{aligned}$$

Here

$$u_k = \begin{cases} w_k \ (\epsilon_k = 1), \\ -w_k r_{i_k}^{-1} \ (\epsilon_k = -1). \end{cases}$$

Because $\alpha_*(r_i) = 1 \in \mathbb{Z}[t, t^{-1}]$, one obtains

$$\begin{aligned} \alpha_*\left(\frac{\partial r}{\partial x_j} \right) &= \sum_{k=1}^{p} \alpha_*(u_k) \alpha_*\left(\frac{\partial r_{i_k}}{\partial x_j} \right) \\ &= \sum_{k=1}^{p} \epsilon_k \alpha_*(w_k) \alpha_*\left(\frac{\partial r_{i_k}}{\partial x_j} \right). \end{aligned}$$

This shows that the last row of the Alexander matrix A' associated to P' is a linear combination of p rows of the Alexander matrix A associated to P. It is clear that the first $n-d$ rows of A' associated to P' are exactly same with the first $n-d$ rows of A associated to P. Therefore it is shown that the invariant $\dfrac{\det {A'}_k^I}{\alpha_*(x_k) - 1}$ is the same as the one computed by A.

Next take a presentation

$$P'' = \langle x_1, \dots, x_n, x (= x_{n+1}) \mid r_1, \dots, r_{n-d}, x w^{-1} \rangle$$

obtained from P by applying the Tietze transformation (II). By direct computations, we see that the Alexander matrix A'' associated to P'' has the form of

$$A'' = \begin{pmatrix} A & 0 \\ * & 1 \end{pmatrix}$$

where the last row is

$$\begin{aligned} &\left(-\alpha_*(x)\alpha_*(w)\alpha_*\left(\frac{\partial w}{\partial x_1} \right), \dots, -\alpha_*(x)\alpha_*(w)\alpha_*\left(\frac{\partial w}{\partial x_n} \right), 1 \right) \\ =&\left(-\alpha_*(x)\alpha_*(w)\alpha_*\left(\frac{\partial x_{n+1}}{\partial x_1} \right), \dots, -\alpha_*(x)\alpha_*(w)\alpha_*\left(\frac{\partial x_{n+1}}{\partial x_n} \right), 1 \right) \\ =&\,(0, \dots, 0, 1). \end{aligned}$$

Here suppose $\alpha_*(x_k) - 1 \neq 0$. Then the determinant of ${A''}_k^J$ for an index set $J = (j_1, \dots, j_{n-d+1})$ can be non-zero if and only if J has the form $J = (j_1, \dots, j_{n-d}, n+1)$. Then for $J = (j_1, \dots, j_{n-d}, n+1)$ and $I = (j_1, \dots, j_{n-d})$, it is seen that

$$\det {A''}_k^J = \det A_k^I.$$

Hence we have

$$\frac{Q_k(A'')}{\alpha(x_k) - 1} = \frac{Q_k(A)}{\alpha(x_k) - 1}.$$

This completes the proof. □

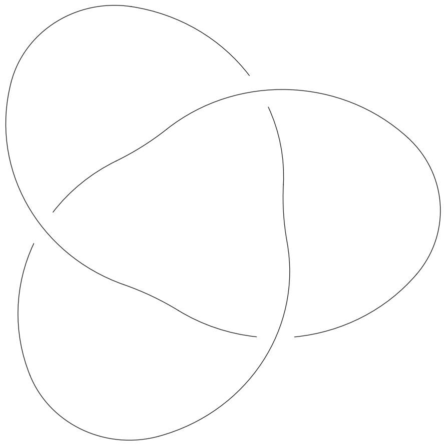
For any knot K, we can take some special presentation of $G(K)$, which is a Wirtinger presentation derived from a regular diagram on the plane. In this case, we may assume that $\alpha(x_1) = \cdots = \alpha(x_n) = t$. Hence the denominator is always $t-1$. Therefore the numerator itself is an invariant of $G(K)$ up to $\pm t^s$.

Definition 3.8. *This is called the Alexander polynomial $\Delta_K(t) = \det A_k$ of K.*

Remark 3.9. It is clear that the Alexander polynomial is well-defined up to $\pm t^s$.

3.2. Examples

Example 3.10. We consider the trefoil knot $3_1 = T(2,3)$ first.



$3_1 = T(2,3)$

Fix the following presentation

$$G(3_1) = \langle x, y \mid r = xyx(yxy)^{-1} \rangle.$$

By applying the abelianization α, the relator $r = xyx(yxy)^{-1}$ goes to

$$\begin{aligned}\alpha(r) &= \alpha(x)\alpha(y)\alpha(x)\alpha(y)^{-1}\alpha(x)^{-1}\alpha(y)^{-1}\\ &= \alpha(x)\alpha(y)^{-1} \in G(3_1)/[G(3_1), G(3_1)].\end{aligned}$$

Because $\alpha(r) = 1$, then we get

$$\alpha(x)\alpha(y)^{-1} = 1 \in G(3_1)/[G(3_1), G(3_1)].$$

Hence the abelianization can be given by

$$\alpha : G(3_1) \ni x, y \mapsto t \in \langle t \rangle.$$

By applying $\dfrac{\partial}{\partial x}$ to r and mapping it on $\mathbb{Z}G(3_1)$, we have

$$\begin{aligned}\frac{\partial}{\partial x}(r) &= \frac{\partial}{\partial x}(xyx(yxy)^{-1})\\ &= \frac{\partial}{\partial x}(xyx) - xyx(yxy)^{-1}\frac{\partial}{\partial x}(yxy)\\ &= \frac{\partial}{\partial x}(xyx) - r\frac{\partial}{\partial x}(yxy)\\ &= \frac{\partial}{\partial x}(xyx) - \frac{\partial}{\partial x}(yxy)\\ &= \frac{\partial}{\partial x}(xyx - yxy).\end{aligned}$$

Here we used the property $r=1$ in $\mathbb{Z}G(3_1)$. Therefore we can compute free differentials for $xyx-yxy$ instead of $r=xyx(yxy)^{-1}$.
Accordingly we compute

$$\begin{aligned}\frac{\partial}{\partial x}(xyx-yxy) &= \frac{\partial}{\partial x}(xyx)-\frac{\partial}{\partial x}(yxy)\\ &= 1+xy-y\\ &\overset{\alpha_*}{\mapsto} t^2-t+1\in\mathbb{Z}[t,t^{-1}].\end{aligned}$$

Similarly

$$\begin{aligned}\frac{\partial}{\partial y}(xyx-yxy) &= \frac{\partial}{\partial y}(xyx)-\frac{\partial}{\partial y}yxy\\ &= x-1-yx\\ &\overset{\alpha_*}{\mapsto} -(t^2-t+1)\in\mathbb{Z}[t,t^{-1}].\end{aligned}$$

Hence one has

$$A=\begin{pmatrix}(t^2-t+1) & -(t^2-t+1)\end{pmatrix},$$

and

$$\begin{aligned}\frac{\det A_2}{t-1} &= -\frac{\det A_1}{t-1}\\ &= \frac{t^2-t+1}{t-1}.\end{aligned}$$

By changing this presentation to $\langle x,y,z \mid xyx(yxy)^{-1}, xyz^{-1}\rangle$, the Alexander matrix is changed to

$$A=\begin{pmatrix}(t^2-t+1) & -(t^2-t+1) & 0\\ 1 & t & -1\end{pmatrix}.$$

In this case the abelianization α is given by

$$\alpha(x)=\alpha(y)=t, \alpha(z)=t^2.$$

From this Alexander matrix, we obtain

$$\begin{aligned}\frac{\det A_1}{t-1} &= \frac{t^2-t+1}{t-1},\\ \frac{\det A_2}{t-1} &= -\frac{t^2-t+1}{t-1},\\ \frac{\det A_3}{t^2-1} &= \frac{t(t^2-t+1)+(t^2-t+1)}{t^2-1}=\frac{t^2-t+1}{t-1}.\end{aligned}$$

Therefore the Alexander polynomial of the trefoil knot is given by

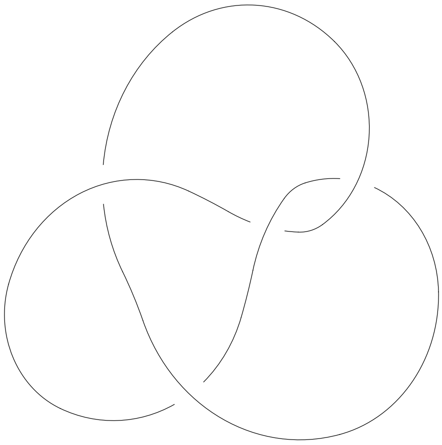
$$\Delta_{3_1}(t)=t^2-t+1$$

up to $\pm t^s$.

Example 3.11. Let us now consider the Figure-eight knot 4_1.
Take a presentation of $G(4_1)$ as

$$G(4_1)=\langle x,y \mid wxw^{-1}=y\rangle$$

where $w=x^{-1}yxy^{-1}$.
Using this presentation, the abelianization $\alpha: G(4_1)\to\langle t\rangle$ is given by $\alpha(x)=\alpha(y)=t$.



4_1

Then one has

$$\begin{aligned}\frac{\partial}{\partial x}(wxw^{-1}y^{-1}) &= \frac{\partial w}{\partial x} + w\frac{\partial x}{\partial x} - wxw^{-1}\frac{\partial w}{\partial x}\\ &= (1-y)\frac{\partial w}{\partial x} + w\\ &\overset{\alpha_*}{\mapsto} (1-t)\alpha_*\left(\frac{\partial w}{\partial x}\right) + 1\end{aligned}$$

and

$$\begin{aligned}\alpha_*\left(\frac{\partial w}{\partial x}\right) &= \alpha_*\left(\frac{\partial}{\partial x}(x^{-1}yxy^{-1})\right)\\ &= \alpha_*(-x^{-1} + x^{-1}y)\\ &= -t^{-1} + 1.\end{aligned}$$

Consequently it is seen that

$$\begin{aligned}\alpha_*\left(\frac{\partial}{\partial x}(wxw^{-1}y^{-1})\right) &= (1-t)(-t^{-1}+1)+1\\ &= -t^{-1}+1+1-t(-t^{-1}+1)\\ &= -t^{-1}+1+1+1-t\\ &= -t^{-1}+3-t.\end{aligned}$$

Similarly one has

$$\begin{aligned}\alpha_*\left(\frac{\partial}{\partial y}(wxw^{-1}y^{-1})\right) &= \alpha_*\left((1-y)\frac{\partial w}{\partial x} - 1\right)\\ &= (1-t)(t^{-1}-1)-1\\ &= t^{-1}-3+t.\end{aligned}$$

Hence we obtain

$$A = \begin{pmatrix} -t^{-1}+3-t & t^{-1}-3+t \end{pmatrix}$$

and

$$\begin{aligned}\frac{\det A_1}{\alpha_*(x_1)-1} &= \frac{t^{-1}-3+t}{t-1}\\ &= -\frac{1}{t}\frac{(-t^2+3t-1)}{t-1},\end{aligned}$$

$$
\begin{aligned}
\frac{\det A_2}{\alpha_*(x_2)-1} &= -\frac{t^{-1}-3+t}{t-1} \\
&= \frac{1}{t}\frac{(-t^2+3t-1)}{t-1}.
\end{aligned}
$$

Finally, the Alexander polynomial of the figure-eight knot is given by

$$\Delta_{4_1}(t) = -t^2+3t-1$$

up to $\pm t^s$.

4. Reidemeister torsion

In this section we explain the theory of the Reidemeister torsion, which is an invariant of a compact CW-complex with a linear representation of the fundamental group.

Let K be a knot in S^3 and $G(K)$ the knot group of K. We take an open tubular neighborhood $N(K) \subset S^3$ of K and the exterior $E(K) = S^3 \setminus N(K)$ of K. The knot exterior $E(K)$ is a compact 3-manifold with a torus boundary. Note that $\pi_1(E(K))$ is isomorphic to $G(K)$ by natural inclusion $E(K) \to S^3 \setminus K$.

Here we consider the abelianization $\alpha : G(K) \to T = \langle t \rangle \subset GL(1;\mathbb{Q}(t))$ as a 1-dimensional representation over $\mathbb{Q}(t)$. Here $\mathbb{Q}(t)$ denotes the one variable rational function field over $\mathbb{Q}$.

Now we can define Reidemeister torsion

$$\tau_\alpha(E(K)) \in \mathbb{Q}(t)$$

of $E(K)$ for α. We mention the following well-known theorem by Milnor [39] before giving the definition of Reidemeister torsion.

Theorem 4.1 (Milnor)**.**

$$\frac{\Delta_K(t)}{t-1} = \tau_\alpha(E(K)).$$

Remark 4.2. Both the left and right hand sides are well defined up to $\pm t^s$.

4.1. Algebraic definitions

Recall the definition of Reidemeister torsion.

Let C_* be a chain complex over a field $\mathbb{F}$ as

$$0 \longrightarrow C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} C_{m-2} \longrightarrow \ldots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0.$$

Because $0 \longrightarrow Z_q(= Ker\partial_q) \longrightarrow C_q \xrightarrow{\partial_q} B_{q-1}(= Im\partial_q) \longrightarrow 0$ is exact, then we have an isomorphism

$$C_q \cong Z_q \oplus B_{q-1},$$

which is not canonical. Note that a pair of bases of Z_q and B_{q-1} gives a basis of C_q.

Definition 4.3. *A chain complex C_* is called acyclic if $B_q = Z_q$ for $q = 0,1,\cdots,m$, that is, if all homology groups $H_*(C_*) = 0$.*

From here we assume that C_* is acyclic and further that a basis $\mathbf{c}_q$ of C_q is given for any q. That is, C_* is a based acyclic chain complex of finite dimensional vector spaces over $\mathbb{F}$.

Here take a basis $\mathbf{b}_q$ on B_q for any q.

On the above exact sequence

$$0 \longrightarrow Z_q \longrightarrow C_q \xrightarrow{\partial_q} B_{q-1} \longrightarrow 0,$$

take a lift $\tilde{\mathbf{b}}_{q-1}$ of $\mathbf{b}_{q-1}$. Now a pair $(\mathbf{b}_q, \tilde{\mathbf{b}}_{q-1})$ gives a basis on C_q. The two bases $\mathbf{c}_q$ and $(\mathbf{b}_q, \tilde{\mathbf{b}}_{q-1})$ givean isomorphism

$$C_q \cong B_q \oplus B_{q-1}.$$

For any two bases $\mathbf{b}=\{b_1,\cdots,b_n\}, \mathbf{c}=\{c_1,\cdots,c_n\}$ of a vector space V over $\mathbb{F}$, there exists a non-singular matrix $P=(p_{ij})\in GL(n;\mathbb{F})$ such that $b_j=\sum_{i=1}^{n} p_{ji}c_i$.

Definition 4.4. *P is called the transformation matrix from $\boldsymbol{c}$ to $\boldsymbol{b}$.*

Under this definition, we simply write $\left(\mathbf{b}_q, \tilde{\mathbf{b}}_{q-1}/\mathbf{c}_q\right)$ for the transformation matrix from $\mathbf{c}_q$ to $(\mathbf{b}_q, \tilde{\mathbf{b}}_{q-1})$ and $\left[\mathbf{b}_q, \tilde{\mathbf{b}}_{q-1}/\mathbf{c}_q\right]$ for the determinant $\det\left(\mathbf{b}_q, \tilde{\mathbf{b}}_{q-1}/\mathbf{c}_q\right)$.

Lemma 4.5. *The determinant $[\boldsymbol{b}_q, \tilde{\boldsymbol{b}}_{q-1}/\boldsymbol{c}_q]$ is independent on choices of a lift $\tilde{\boldsymbol{b}}_{q-1}$. Hence we can simply write $[\boldsymbol{b}_q, \boldsymbol{b}_{q-1}/\boldsymbol{c}_q]$ for it.*

Proof. Take another lift $\hat{\mathbf{b}}_{q-1}$ of $\mathbf{b}_{q-1}$ on C_q. For example, one vector v in $\tilde{\mathbf{b}}_{q-1}$ is replaced by another vector v' in $\hat{\mathbf{b}}_{q-1}$. But v, v' map to the same vector in B_{q-1}. Here

$$0 \longrightarrow Z_q \longrightarrow C_q \longrightarrow B_{q-1} \longrightarrow 0$$

is an exact sequence, so the difference $v-v'$ belongs to $Z_q=B_q$. Hence $v-v'$ can be expressed as a linear combination of the vectors of $\mathbf{b}_q$. Then by the definition of the determinant, it can be seen that

$$\left[\mathbf{b}_q, \tilde{\mathbf{b}}_{q-1}/\mathbf{c}_q\right]=\left[\mathbf{b}_q, \hat{\mathbf{b}}_{q-1}/\mathbf{c}_q\right].$$

Therefore the determinant is not changed. □

Definition 4.6. *The torsion $\tau(C_*)$ of a based chain complex $(C_*, \{\boldsymbol{c}_q\})$ is defined by*

$$\tau(C_*)=\frac{\prod_{q:odd}[\boldsymbol{b}_q, \boldsymbol{b}_{q-1}/\boldsymbol{c}_q]}{\prod_{q:even}[\boldsymbol{b}_q, \boldsymbol{b}_{q-1}/\boldsymbol{c}_q]}\in\mathbb{F}\setminus\{0\}.$$

Lemma 4.7. *The torsion $\tau(C_*)$ is independent of choices of $\boldsymbol{b}_0,\cdots,\boldsymbol{b}_m$.*

Proof. Assume $\mathbf{b}'_q$ is another basis of B_q.
In the definition of $\tau(C_*)$, the difference when using $\mathbf{b}'_q$ instead of $\mathbf{b}_q$ is only in the two terms $\left[\mathbf{b}'_q, \mathbf{b}_{q-1}/\mathbf{c}_q\right]$ and $\left[\mathbf{b}_{q+1}, \mathbf{b}'_q/\mathbf{c}_{q+1}\right]$. By standard arguments of linear algebra,

$$\left[\mathbf{b}'_q, \mathbf{b}_{q-1}/\mathbf{c}_q\right]=\left[\mathbf{b}_q, \mathbf{b}_{q-1}/\mathbf{c}_q\right]\left[\mathbf{b}'_q/\mathbf{b}_q\right],$$

$$\left[\mathbf{b}_{q+1}, \mathbf{b}'_q/\mathbf{c}_{q+1}\right]=\left[\mathbf{b}_{q+1}, \mathbf{b}_q/\mathbf{c}_{q+1}\right]\left[\mathbf{b}'_q/\mathbf{b}_q\right].$$

Since $\left[\mathbf{b}'_q/\mathbf{b}_q\right]$ appears in both the denominator and the numerator of the definition, they can be cancelled. □

Example 4.8. Put $m=4$. Now consider

$$C_*: 0\to C_4\to C_3\to C_2\to C_1\to C_0\to 0.$$

As $\mathbf{b}_4$ and $\mathbf{b}_{-1}$ are zero, then by the definition, one has

$$\begin{aligned}\tau(C_*)&=\frac{[\mathbf{b}_4, \mathbf{b}_3/\mathbf{c}_4][\mathbf{b}_2, \mathbf{b}_1/\mathbf{c}_2][\mathbf{b}_0, \mathbf{b}_{-1}/\mathbf{c}_0]}{[\mathbf{b}_3, \mathbf{b}_2/\mathbf{c}_3][\mathbf{b}_1, \mathbf{b}_0/\mathbf{c}_1]}\\&=\frac{[\mathbf{b}_3/\mathbf{c}_4][\mathbf{b}_2, \mathbf{b}_1/\mathbf{c}_2][\mathbf{b}_0/\mathbf{c}_0]}{[\mathbf{b}_3, \mathbf{b}_2/\mathbf{c}_3][\mathbf{b}_1, \mathbf{b}_0/\mathbf{c}_1]}.\end{aligned}$$

In this case, the number of factors in the denominator and the number of factors in the numerator are not same. However it can be seen that $\tau(C_*)$ is independent of choices of $\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$.

Example 4.9. Next we put $m=3$. Here

$$C_*: 0 \to C_3 \to C_2 \to C_1 \to C_0 \to 0.$$

As $\mathbf{b}_3$ and $\mathbf{b}_{-1}$ are zero, then one has

$$\begin{aligned}\tau(C_*) &= \frac{[\mathbf{b}_2, \mathbf{b}_1/\mathbf{c}_2][\mathbf{b}_0, \mathbf{b}_{-1}/\mathbf{c}_0]}{[\mathbf{b}_3, \mathbf{b}_2/\mathbf{c}_3][\mathbf{b}_1, \mathbf{b}_0/\mathbf{c}_1]} \\ &= \frac{[\mathbf{b}_2, \mathbf{b}_1/\mathbf{c}_2][\mathbf{b}_0/\mathbf{c}_0]}{[\mathbf{b}_2/\mathbf{c}_3][\mathbf{b}_1, \mathbf{b}_0/\mathbf{c}_1]}.\end{aligned}$$

In this case the numbers of factors are same. Similarly it can be seen that $\tau(C_*)$ is independent of choices of $\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$

The following lemma is well-known as Mayer-Vietoris argument for a torsion invariant. See [40] for the proof.

Lemma 4.10. *Let $0 \to C'_* \to C_* \to C''_* \to 0$ be a short exact sequence of based chain complexes. Assume that the bases of C_* are given as pairs of $(\boldsymbol{c}'_*, \mathbf{c}''_*)$ where $\{\boldsymbol{c}'_*\}, \{\boldsymbol{c}''_*\}$ are bases of C'_*, C''_*. If two of C'_*, C_*, C''_* are acyclic, then the third one is also acyclic and*

$$\tau(C_*) = \pm\tau(C'_*)\tau(C''_*).$$

Remark 4.11. The reason why the signs $\pm$ appear in the right hand side is the following. To define the torsions we use the following isomorphisms

- $C'_* \cong Z'_* \oplus B'_*$, $C_* \cong Z_* \oplus B_*$, $C''_* \cong Z''_* \oplus B''_*$.

On the other hand, to get this formula, we use

- $C_* \cong C'_* \oplus C''_* \cong Z'_* \oplus B'_* \oplus Z''_* \oplus B''_*$.

Here the signs appear as we need to change orders of vectors in general.

4.2. Geometric settings

Now we apply this torsion invariant of chain complexes to the following geometric situation. Let X be a finite CW-complex and $\tilde{X}$ the universal covering of X. We lift a CW-complex structure of X on $\tilde{X}$. The fundamental group $\pi_1 X$ acts on $\tilde{X}$ from the right-hand side as deck transformations. By applying the cellular approximation theorem, we may assume that this action is free and cellular under taking subdivisions if it is needed. Then the chain complex $C_*(\tilde{X};\mathbb{Z})$ has the structure of a chain complex of free $\mathbb{Z}[\pi_1 X]$-modules.
Let $\rho : \pi_1 X \to GL(V)$ be an n-dimensional linear representation over a field $\mathbb{F}$. Using the representation ρ, V admits a structure of a $\mathbb{Z}[\pi_1 X]$-module, which is denoted by V_ρ. Define the chain complex $C_*(X;V_\rho)$ by $C_*(\tilde{X};\mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1 X]} V_\rho$. Here we choose a preferred basis of $C_i(X;V_\rho)$ for any i as

$$(\tilde{u}_1 \otimes \mathbf{e}_1, \ldots, \tilde{u}_1 \otimes \mathbf{e}_n, \ldots, \tilde{u}_d \otimes \mathbf{e}_1, \ldots, \tilde{u}_d \otimes \mathbf{e}_n)$$

where $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a basis of V, $\{u_1, \ldots, u_d\}$ are the i-cells giving a basis of $C_i(X;\mathbb{Z})$ and $\{\tilde{u}_1, \ldots, \tilde{u}_d\}$ are lifts of them in $C_i(\tilde{X};\mathbb{Z})$.
Now we suppose that $C_*(X;V_\rho)$ is acyclic, namely all homology groups $H_*(X;V_\rho)$ are vanishing. In this case we call ρ an acyclic representation.

Definition 4.12. *Reidemeister torsion of X for a representation ρ is defined by*

$$\tau_\rho(X) = \tau(C_*(X;V_\rho)) \in \mathbb{F} \setminus \{0\}.$$

Remark 4.13. Reidemeister torsion $\tau_\rho(X)$ does not depend on the choices up to $\pm f$, where $f \in Im\{\det\circ\rho : \pi_1(X) \to \mathbb{F}\setminus\{0\}\}$. See [40] for the proof.

We apply the Reidemeister torsion for a knot K in S^3 as follows. Fix a CW-complex structure on $E(K)$. We take its universal cover

$$\tilde{E}(K) \to E(K)$$

and also a lift of the CW-complex structure of $E(K)$ to $\tilde{E}(K)$. By applying the cellular approximation theorem, we may assume that $G(K)$ acts freely and cellularly on $\tilde{E}(K)$ from the right as deck transformations.
Now we can consider the abelianization $\alpha : G(K) \to \langle t \rangle \subset GL(1; \mathbb{Q}(t))$ as a 1-dimensional representation of $G(K)$ over the rational function field $\mathbb{Q}(t)$.
Hence the chain complex of $E(K)$ with $\mathbb{Q}(t)_\alpha$-coefficients is defined by

$$C_*(E(K); \mathbb{Q}(t)_\alpha) = C_*(\tilde{E}(K); \mathbb{Z}) \otimes_{\mathbb{Z}G(K)} \mathbb{Q}(t)_\alpha.$$

Here we take bases $\mathbf{c}_i$ for $C_i(E(K); \mathbb{Q}(t)_\alpha)$ as

$$(\tilde{u}_1 \otimes \mathbf{1}, \ldots, \tilde{u}_d \otimes \mathbf{1})$$

by using lifts of i-cells $\{u_1, \ldots, u_d\}$ in $E(K)$ and a basis $\mathbf{1}$ for the 1-dimensional vector space $\mathbb{Q}(t)$ over itself as we explained.
Reidemeister torsion of $E(K)$ can be defined as

$$\tau_\alpha(E(K)) = \tau(C_*(E(K); \mathbb{Q}(t)_\alpha)) \in \mathbb{Q}(t) \setminus \{0\}$$

up to $\pm t^s$.
From Milnor's theorem, some properties of Reidemeister torsion induce properties of Alexander polynomial. For example, recall one of the well known properties, which was first proved by Seifert. This can proved by using properties of Reidemeister torsion.

Theorem 4.14 (Seifert [50], Milnor [39])**.** *For any knot K, it holds*

$$\Delta_K(t^{-1}) = \Delta_K(t)$$

up to $\pm t^s$.

We also have the following fact on the Alexander polynomial for a slice knot. A slice knot is defined as follows. Now we consider $S^3 = \partial B^4$.

Definition 4.15. *A knot $K \subset S^3$ is called a slice knot if there exists an embedded disk $D \subset B^4$ such that $\partial D = K \subset S^3 = \partial B^4$.*

The next theorem is a well-known and classical theorem. It can be proved by using Reidemeister torsion.

Theorem 4.16 (Fox-Milnor [15])**.** *If K is a slice knot, then the Alexander polynomial $\Delta_K(t)$ has a form of $\Delta_K(t) = \pm t^s f(t) f(t^{-1})$ where $f(t) \in \mathbb{Z}[t]$.*

5. Order and obstruction

Here we would like to mention two more things related with the Alexander polynomial;

- an order of $H_1(E(K); \mathbb{Q}[t, t^{-1}]_\alpha)$.
- an obstruction to deform an abelian representation.

It is seen that $H_1(E(K); \mathbb{Q}[t, t^{-1}]_\alpha) \cong H_1(E(K)_\infty; \mathbb{Q})$ as a $\mathbb{Q}[t, t^{-1}]$-module where $E(K)_\infty \to E(K)$ is the $\mathbb{Z}$-covering corresponding to the abelianization epimorphism $\alpha : G(K) = \pi_1(E(K)) \to \mathbb{Z} = \langle t \rangle$.
The order of a finitely generated module over a principal ideal domain is defined as follows. This is a generalization of the order of an abelian group.
Let M be a finitely generated $\mathbb{Q}[t, t^{-1}]$-module without free parts. From the structure theorem of a finitely generated module over a principal ideal domain, one has

$$M \cong \mathbb{Q}[t, t^{-1}]/(p_1) \oplus \cdots \oplus \mathbb{Q}[t, t^{-1}]/(p_k)$$

where $p_1, \cdots, p_k \in \mathbb{Q}[t, t^{-1}]$ such that

$$\mathbb{Q}[t, t^{-1}] \ (p_1) \supset (p_2) \supset \cdots \supset (p_k) \neq (0).$$

Definition 5.1. *The order ideal $ord(M)$ of M is defined by*

$$ord(M) = (p_1 \cdots p_k) \subset \mathbb{Q}[t, t^{-1}].$$

In the case of $H_*(E(K); \mathbb{Q}[t, t^{-1}]_\alpha)$, the following proposition holds.

Proposition 5.2.

- $ord(H_1(E(K); \mathbb{Q}[t, t^{-1}]_\alpha)) = (\Delta_K(t))$.
- $ord(H_0(E(K); \mathbb{Q}[t, t^{-1}]_\alpha)) = (t - 1)$.

See [41] as a reference.
Next we mention that the Alexander polynomial is an obstruction to deform a 1-dimensional abelian representation

$$\alpha_a : G(K) \to \mathbb{C}^* = \mathbb{C} \setminus \{0\} \subset \mathbb{C} \rtimes \mathbb{C}^* \subset GL(2; \mathbb{C}).$$

Let $G(K) = \langle x_1, \cdots, x_n \mid r_1, \cdots, r_{n-1} \rangle$ be a Wirtinger presentation of $G(K)$. By putting $t = a \neq 0$, one has a 1-dimensional abelian representation

$$\alpha_a = \alpha|_{t=a} : G(K) \ni x_i \mapsto a \in \mathbb{C}.$$

We put $\rho_a(x_i) = \begin{pmatrix} a & b_i \\ 0 & 1 \end{pmatrix} \in GL(2; \mathbb{C})$ for the image of x_i. Now a map

$$\rho_a : \{x_1, \ldots, x_n\} \to GL(2; \mathbb{C})$$

is given. If all $b_1, \cdots, b_n = 0$, then clearly ρ_a gives a representation

$$\rho_a : G(K) \ni x_i \mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in GL(2; \mathbb{C}).$$

However it is also an abelian representation. Assume $b_i \neq 0$ for some i. Here we consider the following problem.

Problem 5.3. *When can ρ_a be extended as a non abelian representation ?*

The answer is given by the next theorem.

Theorem 5.4 (de Rham [11])**.** *A map ρ_a gives a representation if and only if $\Delta_K(a) = 0$.*

Remark 5.5. One motivation for Wada to define twisted Alexander polynomial is to generalize such an obstruction for a higher dimensional representations.

6. Twisted Alexander polynomial

Historically, the first two studies to give a generalization of the Alexander polynomial are due to Lin [36] and Wada [55]. In this paper we follow the definition due to Wada, because it is most computable by using free differentials and it can be related to Reidemeister torsion of $E(K)$ directly.
Recall K is a knot in S^3 and $G(K)$ is the knot group. For simplicity we consider a representation of $G(K)$ in a 2-dimensional unimodular group over a field $\mathbb{F}$. From this assumption the twisted Alexander polynomial is well-defined up to t^{2s} $(s \in \mathbb{Z})$

Remark 6.1. Wada defined the twisted Alexander polynomial for any finite presentable group with an epimorphism onto a free abelian group and a $GL(l; R)$-representation over a Euclidean domain R.

Fix a presentation as

$$G(K) = \langle x_1, \dots, x_n \mid r_1, \dots, r_{n-1} \rangle$$

with deficiency one. Let $\rho : G(K) \to SL(2;\mathbb{F})$ be a representation. Let $M(2;\mathbb{F})$ be the matrix algebra of 2×2 matrices over $\mathbb{F}$. We write

$$\rho_* : \mathbb{Z}G(K) \to \mathbb{Z}SL(2;\mathbb{F}) \cong M(2;\mathbb{F})$$

for a ring homomorphism induced by ρ and

$$\alpha_* : \mathbb{Z}G(K) \to \mathbb{Z}\mathbb{Z} = \mathbb{Z}\langle t \rangle \cong \mathbb{Z}[t, t^{-1}]$$

for a ring homomorphism induced by α. By taking the tensor product of them, we obtain an induced ring homomorphism

$$\rho_* \otimes \alpha_* : \mathbb{Z}G(K) \to M(2;\mathbb{F}) \otimes \mathbb{Z}[t, t^{-1}] \cong M\big(2;\mathbb{F}[t, t^{-1}]\big)$$

and

$$\Phi : \mathbb{Z}F_n \to M\big(2;\mathbb{F}[t, t^{-1}]\big)$$

the composite of $\mathbb{Z}F_n \to \mathbb{Z}G(K)$ induced by the presentation and

$$\rho_* \otimes \alpha_* : \mathbb{Z}G(K) \to M\big(2;\mathbb{F}[t, t^{-1}]\big).$$

Definition 6.2. *The $(n-1) \times n$ matrix A_ρ whose (i,j) component is the 2×2 matrix*

$$\Phi\left(\frac{\partial r_i}{\partial x_j}\right) \in M\big(2;\mathbb{F}[t, t^{-1}]\big),$$

is called the twisted Alexander matrix of a knot group $G(K) = \langle x_1, \dots, x_n \mid r_1, \dots, r_{n-1} \rangle$ associated to ρ.

Remark 6.3. This matrix A_ρ can be considered as

$$\begin{aligned} A_\rho &\in M\big((n-1) \times n; M\big(2;\mathbb{F}[t, t^{-1}]\big)\big) \\ &= M\big(2(n-1) \times 2n; \mathbb{F}[t, t^{-1}]\big). \end{aligned}$$

Let $A_{\rho,k}$ be the $(n-1) \times (n-1)$ matrix obtained from A_ρ by removing the k-th column. Then one has

$$\begin{aligned} A_{\rho,k} &\in M\big((n-1) \times (n-1); M\big(2;\mathbb{F}[t, t^{-1}]\big)\big) \\ &= M\big(2(n-1) \times 2(n-1); \mathbb{F}[t, t^{-1}]\big). \end{aligned}$$

By similar arguments as for Alexander polynomials, the following two lemmas can be seen.

Lemma 6.4. *There exists k such that* $\det \Phi(x_k - 1) \neq 0$.

Lemma 6.5. $(\det A_{\rho,k})(\det \Phi(x_j - 1)) = (\det A_{\rho,j})(\det \Phi(x_k - 1))$ *for any j, k.*

Remark 6.6. The signs $\pm$ do not appear in the case of even dimensional unimodular representations.

From the above two lemmas, we can define the twisted Alexander polynomial of $G(K)$ associated to $\rho : G(K) \to SL(2;\mathbb{F})$ to be a rational expression as follows.

Definition 6.7. *The twisted Alexander polynomial of K for ρ is defined by*

$$\Delta_{K,\rho}(t) = \frac{\det A_{\rho,k}}{\det \Phi(x_k - 1)}$$

for any k such that $\det \Phi(x_k - 1) \neq 0$.

This gives an invariant of K with ρ. The following proposition can be proved by using similar arguments as in the case of the Alexander polynomial.

Proposition 6.8.

Up to ct^{2s} $(c \in \mathbb{F}, s \in \mathbb{Z})$, $\Delta_{K,\rho}(t)$ is an invariant of $(G(K), \rho)$. Namely, it does not depend on choices of a presentation.

Now we assume that we always take a Wirtinger presentation of $G(K)$. Hence we assume the deficiency is always one. In this case one has a more strict invariant as follows. However the deficiency is changed by the Tietze transformation (I).
Now we introduce the strong Tietze transformations for a presentation of a group.
(I_a): Replace a relator r_i by its inverse r_i^{-1}.
(I_b): Replace a relator r_i by its conjugate wr_iw^{-1}.
(I_c): Replace a relator r_i by $r_ir_k(i \neq k)$.

Remark 6.9. The deficiency is not changed by $(\mathrm{I}_\mathrm{a}), (\mathrm{I}_\mathrm{b}), (\mathrm{I}_\mathrm{c}), (\mathrm{II})$ or their inverses.

One can prove the following. See [55] for a proof.

Proposition 6.10. *Any Wirtinger presentation of G(K) can be transformed to any other Wirtinger presentation of $G(K)$ by an application of a finite sequence of the Tietze transformations* $(\mathrm{I}_\mathrm{a}), (\mathrm{I}_\mathrm{b}), (\mathrm{I}_\mathrm{c}), (\mathrm{II})$ *and their inverses.*

By applying the above proposition and the same arguments as in section 3, one has the following.

Proposition 6.11. *For any K, the polynomial $\Delta_{K,\rho}(t)$ defined by a Wirtinger presentation of $G(K)$ is an invariant of $(G(K), \rho)$ up to t^{2s} $(s \in \mathbb{Z})$.*

Remark 6.12.

- The above holds up to $\pm t^{ls}$ for an l-dimensional representation.
- On the other hand, by using only the theory of Reidemeister torsion, without the arguments of Tietze transformations, we can see $\Delta_{K,\rho}(t)$ is well-defined up to t^{2s} $(s \in \mathbb{Z})$.

In general the twisted Alexander polynomial $\Delta_{K,\rho}(t)$ depends on a representation ρ. However the following proposition can be proved easily.

Definition 6.13. *Two representations $\rho, \rho' : G(K) \to SL(2;\mathbb{F})$ are called conjugate if there exists $P \in SL(2;\mathbb{F})$ such that $\rho(x) = P\rho'(x)P^{-1}$ for any $x \in G(K)$.*

Proposition 6.14. *If two representations ρ and ρ' are conjugate, then $\Delta_{K,\rho}(t) = \Delta_{K,\rho'}(t)$ up to t^s.*

Example 6.15. If K is the trivial knot, we can take the presentation as $G(K) = \langle x \rangle$ and the abelianization $\alpha : \langle x \rangle \ni x \mapsto t \in \langle t \rangle$. In this case, any representation $\rho : G(K) \to SL(2;\mathbb{C})$ is given by just one matrix $X = \rho(x) \in SL(2;\mathbb{C})$. By definition, one has

$$\begin{aligned} \Delta_{K,\rho}(t) &= \frac{1}{\det(t\rho(x) - I)} \\ &= \frac{1}{(\lambda_1 t - 1)(\lambda_2 t - 1)} \end{aligned}$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity matrix, and λ_1, λ_2 are the eigenvalues of $\rho(x)$.

Example 6.16. Let $\rho = \mathbf{1} : G(K) \ni x \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in SL(2;\mathbb{C})$ be a 2 dimensional trivial representation. Then

$$\mathbf{1} \otimes \alpha = \alpha \oplus \alpha : G(K) \ni x \mapsto \begin{pmatrix} \alpha(x) & 0 \\ 0 & \alpha(x) \end{pmatrix} \in GL(2;\mathbb{C}(t)).$$

Hence it can be seen that

$$\begin{aligned} \Delta_{K,\mathbf{1}}(t) &= \frac{\Delta_K(t)}{t-1} \cdot \frac{\Delta_K(t)}{t-1} \\ &= \left(\frac{\Delta_K(t)}{t-1}\right)^2 \end{aligned}$$

Example 6.17. Let $\rho_a : G(K) \ni x \mapsto \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in SL(2;\mathbb{C})(a \in \mathbb{C} \setminus \{0\})$ be an abelian representation. By direct computation, one has

$$\begin{aligned}\Delta_{K,\rho_a}(t) &= \frac{\Delta_K(at)}{at-1} \cdot \frac{\Delta_K(a^{-1}t)}{a^{-1}t-1} \\ &= \left(\frac{\Delta_K(at)}{t-a}\right)\left(\frac{\Delta_K(a^{-1}t)}{t-a^{-1}}\right).\end{aligned}$$

Therefore we obtain

$$\begin{aligned}\lim_{a\to 1} \Delta_{K,\rho_a}(t) &= \Delta_{K,\mathbf{1}}(t) \\ &= \left(\frac{\Delta_K(t)}{t-1}\right)^2.\end{aligned}$$

From these above examples, the twisted Alexander polynomial is not a polynomial in general.
However, under a mild assumption on ρ, the twisted Alexander polynomial is a Laurent polynomial.

Proposition 6.18 (Kitano-Morifuji [29])**.** *If $\rho : G(K) \to SL(2;\mathbb{F})$ is not an abelian representation, then $\Delta_{K,\rho}(t)$ is a Laurent polynomial with coefficients in $\mathbb{F}$.*

6.1. Figure-eight knot

Let us see the figure-eight knot 4_1 again. The knot group $G(4_1)$ has a presentation as

$$G(4_1) = \langle x, y \mid wx = yw \rangle \ (w = x^{-1}yxy^{-1}).$$

Remark 6.19. Here the generators x and y are conjugate by w. This is the point to treat $SL(2;\mathbb{C})$-representations for a 2-bridge knot.

For simplicity, we write X to denote $\rho(x)$ for $x \in G(K)$. The next lemma can be seen by elementary arguments of linear algebra.

Lemma 6.20. *Let $X, Y \in SL(2,\mathbb{C})$. If X and Y are conjugate and $XY \neq YX$, then there exists $P \in SL(2;\mathbb{C})$ such that*

$$PXP^{-1} = \begin{pmatrix} s & 1 \\ 0 & 1/s \end{pmatrix},\ PYP^{-1} = \begin{pmatrix} s & 0 \\ u & 1/s \end{pmatrix}.$$

For any irreducible representation ρ, we may assume that its representative of the conjugacy class which contains ρ is given by

$$\rho_{s,u} : G(4_1) \to SL(2;\mathbb{C})$$

such that

$$\rho_{s,u}(x) = X = \begin{pmatrix} s & 1 \\ 0 & 1/s \end{pmatrix},$$

$$\rho_{s,u}(y) = Y = \begin{pmatrix} s & 0 \\ u & 1/s \end{pmatrix}$$

where $s, u \in \mathbb{C} \setminus \{0\}$.

Remark 6.21. Because

$$tr(X) = s + \frac{1}{s},\ tr(X^{-1}Y) = 2 - u,$$

it is seen that the space of conjugacy classes of irreducible representations can be parametrized by the traces of $X, X^{-1}Y$.

We compute the matrix

$$R = WX - YW = \rho(w)\rho(x) - \rho(y)\rho(w)$$

to get the defining equations of the space of conjugacy classes of irreducible representations.
One has each entry of $R = (R_{ij})$:

- $R_{11} = R_{22} = 0$,
- $R_{12} = 3 - \frac{1}{s^2} - s^2 - 3u + \frac{u}{s^2} + s^2u + u^2$,
- $R_{21} = -3u + \frac{u}{s^2} + s^2u + 3u^2 - \frac{u^2}{s^2} - s^2u^2 - u^3 = -uR_{12}$.

Hence $R_{12} = 0$ is the equation of the space of conjugacy classes of irreducible representations.
This equation

$$3 - \frac{1}{s^2} - s^2 - 3u + \frac{u}{s^2} + s^2u + u^2 = 0$$

can be solved in u:

$$u = \frac{-1 + 3s^2 - s^4 \pm \sqrt{1 - 2s^2 - s^4 - 2s^6 + s^8}}{2s^2}.$$

By applying $\dfrac{\partial}{\partial y}$ to $wx - yw$, one has

$$\begin{aligned}
\frac{\partial(wx - yw)}{\partial y} &= \frac{\partial w}{\partial y} - 1 - y\frac{\partial w}{\partial y} \\
&= (1-y)\frac{\partial w}{\partial y} - 1 \\
&= (1-y)\frac{\partial}{\partial y}(x^{-1}yxy^{-1}) - 1 \\
&= (1-y)(x^{-1} - wx) - 1.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
A_{\rho,1} &= \Phi\left(\frac{\partial(wx - yw)}{\partial y}\right) \\
&= (\Phi(1) - \Phi(y))(\Phi(x^{-1}) - \Phi(w)\Phi(x)) - \Phi(1) \\
&= (I - tY)(t^{-1}X^{-1} - tWX) - I.
\end{aligned}$$

Note that $\Phi(w) = W$ because $\alpha(w) = 1$.
Substituting

$$u = \frac{-1 + 3s^2 - s^4 \pm \sqrt{1 - 2s^2 - s^4 - 2s^6 + s^8}}{2s^2}$$

to each entry and doing direct computations, the numerator is given as

$$\det A_{\rho,1} = \frac{1}{t^2} - \frac{3}{st} - \frac{3s}{t} + 6 + \frac{2}{s^2} + 2s^2 - \frac{3t}{s} - 3st + t^2$$

Remark that it does not depend on two choices of u.
On the other hand, one has

$$\det(tX - I) = t^2 - \left(s + \frac{1}{s}\right)t + 1.$$

Finally we obtain

$$\begin{aligned}\Delta_{4_1,\rho_{s,u}}(t) &= \frac{\det A_{\rho,1}}{\det(tX-I)}\\ &= \frac{1}{t^2}-\frac{2\left(1+s^2\right)}{st}+1\\ &= \frac{1}{t^2}\left(t^2-2\left(s+\frac{1}{s}\right)t+1\right)\\ &= \frac{1}{t^2}(t^2-2(tr(X))t+1).\end{aligned}$$

Remark 6.22. We mention two things. The reason for the second one is explained in section 7.

- $\Delta_{4_1,\rho_{s,u}}(t)$ is a Laurent polynomial because $\rho_{s,u}$ is not abelian.
- $\Delta_{4_1,\rho_{s,u}}(t)$ is monic (explain later) because 4_1 is fibered.

6.2. Torus knots

We can consider that $\Delta_{K,\rho}(t)$ is a Laurent polynomial (up to some powers of t) valued function on the space of conjugacy classes of $SL(2;\mathbb{C})$-irreducible representations. In general a twisted Alexander polynomial is not constant on this space. For example, in the case of the figure-eight knot as we discussed above, it is depending on the trace of the image of the meridian.
On the other hand, the following holds for a (p,q)-torus knot $T(p,q)\subset S^3$.

Theorem 6.23 (Kitano-Morifuji [30]). *For any (p,q)-torus knot $T(p,q)$, $\Delta_{T(p,q),\rho}(t)$ is a locally constant function on each connected component of the space of conjugacy classes of $SL(2;\mathbb{C})$-irreducible representations.*

Let $G(p,q)=\langle x,y \mid x^p=y^q\rangle$ be the knot group of $T(p,q)$. Let $m\in G(p,q)$ be the meridian given by $x^{-r}y^s$ where $ps-qr=1$ and $z=x^p=y^q$ a center element of the infinite order. Now let $\rho: G(p,q)\to SL(2;\mathbb{C})$ be an irreducible representation.
Recall that the center of $SL(2;\mathbb{C})$ is $\{\pm I\}$. Hence one has $Z=\rho(z)=\pm I$ by the irreducibility of ρ. Then this implies

$$X^p=\pm I, Y^q=\pm I.$$

Here we may choice the eigenvalues of X and Y as

- $\lambda^{\pm 1}=e^{\pm\sqrt{-1}\pi a/p}$ such that $0<a<p$,
- $\mu^{\pm 1}=e^{\pm\sqrt{-1}\pi b/q}$ such that $0<b<q$.

Now we get

$$tr(X)=2\cos\frac{\pi a}{p}, tr(Y)=2\cos\frac{\pi b}{q},$$

and further

$$X^p=(-I)^a, Y^q=(-I)^b.$$

Remark 6.24. In any case one has $X^{2p}=Y^{2q}=I$.

Proposition 6.25 (Johnson [25]). *Any conjugacy class of irreducible representations is uniquely determined for a given triple of traces*

$$(tr(X), tr(Y), tr(M))$$

such that

- $tr(X)=2\cos\frac{\pi a}{p}$,

- $tr(Y) = 2\cos\frac{\pi b}{q}$,
- $Z = (-I)^a$,
- $tr(M) \neq 2\cos\pi(\frac{ra}{p} \pm \frac{sb}{q})$,
- $0 < a < p$, $0 < b < q$, $a \equiv b \bmod 2$,
- $r, s \in \mathbb{Z}$ *such that* $pq - rs = 1$.

Corollary 6.26.

- *A pair of* (a, b) *determines a connected component of conjugacy classes.*
- *Each connected component of the conjugacy classes can be parametrized by* $tr(M) \in \mathbb{C} \setminus \left\{2\cos\pi\left(\frac{ra}{p} \pm \frac{sb}{q}\right)\right\}$ *under fixing* (a, b).

Here we give a proof that twisted Alexander polynomial is constant on each connected component.

Proof. We use this parametrization to compute twisted Alexander polynomials. By applying Fox's differential to $r = x^p y^{-q}$, one has

$$\frac{\partial r}{\partial x} = 1 + x + \cdots + x^{p-1}.$$

Remark that $\alpha : G(K) \to \langle t \rangle$ is defined by $\alpha(x) = t^q, \alpha(y) = t^p$, and $\alpha(m) = t$.
By the definition, we obtain

$$\begin{aligned}
\Delta_{T(p,q),\rho}(t) &= \frac{\Phi(\frac{\partial r}{\partial x})}{\Phi(y-1)} \\
&= \frac{\det(I + t^q X \cdots + t^{(p-1)q} X^{p-1})}{\det(t^p Y - I)} \\
&= \frac{(1 + \lambda t^q + \cdots + \lambda^{p-1} t^{(p-1)q})(1 + \lambda^{-1} t^q + \cdots + \lambda^{-(p-1)} t^{-(p-1)q})}{1 - (\mu + \mu^{-1}) t^p + t^{2p}}.
\end{aligned}$$

Hence it can be seen that $\Delta_{T(p,q),\rho}(t)$ is determined by (p, q) and eigenvalues $(\lambda, \mu) = (e^{\sqrt{-1}\pi a/p}, e^{\sqrt{-1}\pi b/q})$ such that $0 < a < p, 0 < b < q$. This means that it cannot be varied locally. □

Now we consider the case of $(2, q)$-torus knot for simplicity. Here the connected components consists of $\frac{q-1}{2}$ components parametrized by odd integer b with $0 < b < q$.

Theorem 6.27 (Kitano-Morifuji [30])**.** *The Twisted Alexander polynomial of* $T(2, q)$ *is given by*

$$\Delta_{T(2,q),\rho_b}(t) = \left(t^2 + 1\right) \prod_{0<k<q,\ k:\text{odd},\ k\neq b} \left(t^2 - \xi_k\right)\left(t^2 - \bar{\xi}_k\right),$$

where $\xi_k = \exp\left(\sqrt{-1}\pi k/q\right)$.

Example 6.28. In particular, for the trefoil knot $3_1 = T(2, 3)$, there is just one connected component. For any irreducible representation ρ, we have

$$\begin{aligned}
\Delta_{K,\rho}(t) &= \frac{t^6 + 1}{t^4 - t^2 + 1} \\
&= t^2 + 1.
\end{aligned}$$

6.3. Reidemeister torsion, orders, and an obstruction

Here we mention the relation of the twisted Alexander polynomial with Reidemeister torsion, an order ideal and an obstruction of a representation.
For simplicity, we treat a representation over $\mathbb{C}$. By taking a tensor product of

$$\bar{\alpha} : G(K) \cong \pi_1(E(K)) \ni x \mapsto \alpha(x)^{-1} \in \langle t \rangle \subset GL(1; \mathbb{Z}[t, t^{-1}])$$

and

$$\rho : G(K) \cong \pi_1(E(K)) \to SL(2; \mathbb{C}),$$

we have

$$\rho \otimes \bar{\alpha} : G(K) \cong \pi_1(E(K)) \to GL(2; \mathbb{C}[t, t^{-1}]) \subset GL(2; \mathbb{C}(t)).$$

Further we can define a chain complex $C_*(E(K); \mathbb{C}(t)^2_{\rho\otimes\bar{\alpha}})$ by $\rho \otimes \bar{\alpha}$. We assume that this chain complex is acyclic, namely, all homology groups $H_*(E(K); \mathbb{C}(t)^2_{\rho\otimes\bar{\alpha}}) = 0$. Here we can define Reidemeister torsion

$$\tau_{\rho\otimes\bar{\alpha}}(E(K)) \in \mathbb{C}(t).$$

Under the acyclicity condition, we have the following.

Theorem 6.29 (Kitano [28])**.** *Up to t^{2s} ($s \in \mathbb{Z}$), it holds that*

$$\Delta_{K,\rho}(t) = \tau_{\rho\otimes\bar{\alpha}}(E(K)).$$

More generally by considering a twisted homology $H_*(E(K); \mathbb{C}[t, t^{-1}]^l_{\rho\otimes\alpha})$, we can consider the order of $H_*(E(K); \mathbb{C}[t, t^{-1}]^l_{\rho\otimes\alpha})$, which is a generalization of the Alexander polynomial as a generator of an order ideal. This is corresponding to the numerator of $\Delta_{K,\rho}(t)$ for a Wirtinger presentation.
Here we do not mention the details on the relation between twisted Alexander polynomials and order ideals. Please see [35].
In the last part of this section, we explain how the twisted Alexander polynomial is related to an obstruction to deform an representation.
Here assume $G(K) = \langle x_1, \cdots, x_n \mid r_1, \cdots, r_{n-1} \rangle$ is a Wirtinger presentation. Let $\rho : G(K) \to SL(2; \mathbb{C})$ be a representation with $X_i = \rho(x_i)$. Put another matrix $\tilde{X}_i = \begin{pmatrix} \alpha X_i & \mathbf{b}_i \\ \mathbf{0} & 1 \end{pmatrix} \in GL(3; \mathbb{C})$ where $\alpha \in \mathbb{C} \setminus \{0\}$ and $\mathbf{b}_i \in \mathbb{C}^2$.
Now we consider the next problem.

Problem 6.30. *When does the map $\tilde{\rho}_\alpha : \{x_1, \ldots, x_n\} \ni x_i \mapsto \tilde{X}_i$ give a representation $\tilde{\rho}_\alpha : G(K) \to GL(3; \mathbb{C})$?*

As a generalization of the theorem by de Rham (Theorem 5.4), one has the following.

Theorem 6.31 (Wada, unpublished)**.** *Assume α is not an eigenvalue of X_1. Then $\tilde{\rho}_\alpha : G(K) \to GL(3; \mathbb{C})$ is a representation if and only if the numerator of $\Delta_{K,\rho}(\alpha)$ is vanishing.*

Hence we can say that the twisted Alexander polynomial is an obstruction to deform a $GL(2; \mathbb{C})$-representation $G(K) \ni x_i \mapsto \alpha X_i \in GL(2; \mathbb{C})$ in $GL(2; \mathbb{C}) \ltimes \mathbb{C}^2 \subset GL(3; \mathbb{C})$.

7. Fibered knot

A twisted Alexander polynomial is an invariant for $G(K)$ with a representation. In general it is not easy to find a linear representation of $G(K)$.
There are two directions to do it by using a computer.

- a finite quotient (an epimorphism onto a finite group).
- a linear representation over a finite field.

7.1. A finite quotient

Suppose we have a finite quotient, which is an epimorphism onto a finite group G

$$\gamma : G(K) \to G.$$

Here G acts naturally on G and its group rings $\mathbb{Z}G$, $\mathbb{Q}G$. Then by using γ, $G(K)$ also acts on G, $\mathbb{Z}G$ and $\mathbb{Q}G$.
Note that $\dim_{\mathbb{Q}}(\mathbb{Q}G) = |G|$ where $|G|$ is the order of G. Then this gives a $|G|$-dimensional linear representation

$$\tilde{\gamma} : G(K) \to GL(|G|;\mathbb{Q}).$$

Further $\mathrm{Im}\tilde{\gamma} \subset GL(|G|;\mathbb{Z})$ and $\mathrm{Im}(\det \circ \tilde{\gamma}) = \{\pm 1\} \in \mathbb{Z}$ because $G(K)$ acts on $\mathbb{Z}G$. Hence the twisted Alexander polynomial $\Delta_{K,\tilde{\gamma}}(t)$ of K is well defined up to $\pm t^s$.
If K is the trivial knot, then a twisted Alexander polynomial has a form of

$$\Delta_{K,\rho} = \frac{1}{(\lambda_1 t - 1)\cdots(\lambda_l t - 1)}$$

for any l-dimensional representation ρ. Here $\lambda_1, \cdots, \lambda_l$ are eigenvalues of the image of a generator of $E(K) \cong \mathbb{Z}$.
Now the following holds.

Theorem 7.1 (Silver-Williams [53]). *If K is not trivial, then there exists a finite quotient $\gamma : G(K) \to G$ such that $\Delta_{K,\tilde{\gamma}}(t) \neq \dfrac{1}{(\lambda_1 t - 1)\cdots(\lambda_l t - 1)}$. That is, twisted Alexander polynomials distinguish the trivial knot.*

7.2. Fibered knot

Recall the definition of a fibered knot.

Definition 7.2. *A knot K is called a fibered knot of genus g if $E(K)$ admits a structure of a fiber bundle*

$$E(K) = S \times [0,1]/(x,1) \sim (\varphi(x), 0)$$

over S^1 where S is a compact connected oriented surface S of genus g and $\varphi : S \to S$ is an orientation preserving diffeomorphism.

The following classical result is well known.

Theorem 7.3 (Stallings [54], Neuwirth [46]). *A knot K is a fibered knot of genus g if and only if the commutator subgroup $[G(K), G(K)]$ is a free group of rank $2g$.*

In general it is not easy to check this condition on $[G(K), G(K)]$. The next proposition and its corollary are well known and useful to detect fiberedness. Now we fix a symplectic basis of $H_1(S;\mathbb{Z})$.

Proposition 7.4. *If K is a fibered knot with a fiber surface S of genus g, then Alexander polynomial $\Delta_K(t)$ is given by*

$$\Delta_K(t) = \det(t\varphi_* - I : H_1(S;\mathbb{Z}) \to H_1(S;\mathbb{Z}))$$

where φ_* is the induced isomorphism on $H_1(S;\mathbb{Z})$ by φ and I is the identity matrix of rank $2g$.

Corollary 7.5. *If K is a fibered knot of genus g, then $\Delta_K(t)$ is monic and its degree is $2g$.*

In general we define the monicness for a Laurent polynomial over a commutative ring R as follows.

Definition 7.6. *A Laurent polynomial $f(t)$ over R is monic if its coefficient of highest degree is a unit in R.*

Now we are considering twisted Alexander polynomials of K for $SL(l;\mathbb{F})$-representations over a field. Since any non zero element in a field is always a unit, then the above definition of the monicness does not make sense. However for any $SL(n;\mathbb{F})$-representation, twisted Alexander polynomial is well-defined as a rational expression up $\pm t^s$. Hence we can define the monicness of $\Delta_{K,\rho}(t)$ as follows.

Definition 7.7. *A twisted Alexander polynomial $\Delta_{K,\rho}$ is monic if the highest degree coefficients of the denominator and the numerator are ± 1.*

Generalization to the twisted case is given as follows.

Theorem 7.8 (Cha [6], Goda-Morifuji-Kitano [20].)**.** *If K is fibered, then $\Delta_{K,\rho}$ is monic for any $SL(l,\mathbb{F})$-representation ρ.*

If K is fibered, then $G(K)$ has the deficiency one presentation defined by its fiber bundle structure. By using this, it is clear that $\Delta_{K,\rho}(t)$ is monic. However it is not clear this presentation can be transformed by strong Tietze transformations. In [20] the above claim was proved for the Reidemeister torsion.
To make refinement of the above results, we need the notion of Thurston norm. Here the abelianization $\alpha: G(K) \to \mathbb{Z}$ can be considered as an integral 1-cocylce on $G(K)$. Hence it can be consider as $[\alpha] \in H^1(G(K);\mathbb{Z}) = H^1(E(K);\mathbb{Z})$. Now as one has

$$H^1(E(K);\mathbb{Z}) \cong H_2(E(K), \partial E(K);\mathbb{Z})$$

by Poincaré duality, there exists a properly embedded surface $S = S_1 \cup \cdots \cup S_k$ whose homology class $[S]$ is dual to $[\alpha]$. This surface S may be not connected in general.
Now Thurston norm $||\alpha||_T$ is defined by the following.

Definition 7.9.

$$||\alpha||_T = \min_{S \subset E(K)} \{ \chi_-(S) \mid [S] = \Sigma_i [S_i] \text{ is dual to } [\alpha] \}$$

where

$$\begin{aligned} \chi_-(S) &= \sum_{i=1}^{k} \max\{-\chi(S_i), 0\} \\ &= \sum_{i:\chi(S_i)<0} -\chi(S_i). \end{aligned}$$

Example 7.10. If K is a fibered knot of genus g, then the fiber surface S gives a homology class which is dual to $[\alpha]$. Here the euler characteristic $\chi(S) = 2 - 2g - 1 = 1 - 2g$.
Hence one has

- $||\alpha||_T = 2g - 1$,
- $\deg(\Delta_K(t)) = 2g$.

Therefore we can see that

$$\begin{aligned} ||\alpha||_T &= \deg(\Delta_K(t)) - 1 \\ &= \deg(\tau_\alpha(E(K)) \end{aligned}$$

where the degree of $\tau_\alpha(E(K))$ is defined by $\deg(\Delta_K(t)) - \deg(t-1)$.

This can be generalized for the twisted Alexander polynomial. The next result was a turning point to detect the fiberedness of a 3-manifold.

Theorem 7.11 (Friedl-Kim [16])**.** *Let K be a fibered knot. For any representation $\rho: G(K) \to SL(l;\mathbb{F})$, it holds that*

- *$\Delta_{K,\rho}(t)$ is monic,*
- *$l||\alpha||_T = \deg(\Delta_{K,\rho}(t))$.*

Furthermore the converse is true.

Theorem 7.12 (Friedl-Vidussi [17])**.** *If the following two conditions hold*

- $\Delta_{K,\tilde{\gamma}}(t)$ *is monic,*
- $|G| \cdot ||\alpha||_T = \deg(\Delta_{K,\tilde{\gamma}}(t))$*,*

for any representation $\tilde{\gamma} : G(K) \to GL(|G|;\mathbb{Q})$ *induced by a finite quotient* $\gamma : G(K) \to G$*, then* K *is a fibered knot and the genus of* K *is given by*

$$g = \frac{\deg(\Delta_{K,\tilde{\gamma}}(t)) + |G|}{2|G|}.$$

Proof. Here we explain only an outline of the proof of the theorem by Friedl-Vidussi.
Take a Seifert surface $S \subset E(K)$ such that $[S]$ is dual to $[\alpha]$, and its open neighborhood

$$N(S) = S \times (-1,1) \subset S \times [-1,1] \subset E(K).$$

Here we consider a submanifold

$$M = E(K) \setminus N(K),$$

which is called a sutured manifold.
Take a natural inclusion

$$\iota : S \to S \times \{1\} \subset M.$$

From the condition on twisted Alexander polynomials, we can see that $\iota_* : H_*(S) \cong H_*(M)$ for any twisted coefficient.
This implies that the natural inclusion induces an isomorphism

$$\iota_* : \pi_1 S \cong \pi_1 M.$$

Therefore we can prove that $S \times I \cong M$ and M admits a trivial fiber bundle structure over an interval. Finally $E(K)$ admits a structure of a fiber bundle over a circle. □

To detect fiberedness, it seems we need to compute Thurston norm $||\alpha||_T$. In general it is difficult. However we do not need to. For a non-fibered knot, we can see the vanishing of a twisted Alexander polynomial.

Theorem 7.13 (Friedl-Vidussi [19])**.** *If* K *is not fibered, then there exists a representation* ρ *such that* $\Delta_{K,\rho}(t) = 0$*.*

7.3. DFJ-conjecture

In this subsection we assume that K is a hyperbolic knot.

Definition 7.14. *A knot* K *is a hyperbolic knot if* $S^3 \setminus K$ *admits a complete Riemannian metric of constant sectional curvature -1. In other words,* $S^3 \setminus K$ *is the quotient of the three-dimensional hyperbolic space* $\mathbb{H}^3$ *by a subgroup of hyperbolic isometries* $Isom_+(\mathbb{H}^3)$ *acting freely and properly discontinuously.*

Remark 7.15*.* It is well known that $Isom_+(\mathbb{H}^3) \cong PSL(2;\mathbb{C})$.

Let K be a hyperbolic knot. Then there exists a holonomy representation

$$\bar{\rho}_0 : G(K) \to PSL(2;\mathbb{C})$$

and a lift

$$\rho_0 : G(K) \to SL(2;\mathbb{C})$$

with $tr(\rho_0(m)) = 2$. Here $m \in G(K)$ is a meridian.
If K is a fibered knot of genus g, then the twisted Alexander polynomial $\Delta_{K,\rho_0}(t)$ is monic polynomial of degree $4g-2$.
Dunfield, Friedl and Jackson claim that it is enough to consider the monicness of $\Delta_{K,\rho_0}(t)$ for only ρ_0 to detect the fiberedness of a hyperbolic knot.

Conjecture 7.16 (Dunfield-Friedl-Jackson [13]).

- $\Delta_{K,\rho_0}(t)$ *detects Thurston norm of* α*, that is, the genus of* K *can be described by the degree of* $\Delta_{K,\rho_0}(t)$.
- *A hyperbolic knot* K *is fibered if and only if* $\Delta_{K,\rho_0}(t)$ *is a monic polynomial.*

Theorem 7.17 (Dunfield-Friedl-Jackson [13]). *The DFJ-conjecture is true for all 313,209 hyperbolic knots with at most 15 crossings.*

Further it holds for any twist knot.

Theorem 7.18 (Morifuji [43]). *The DFJ-conjecture is true for any twist knot.*

Remark 7.19.

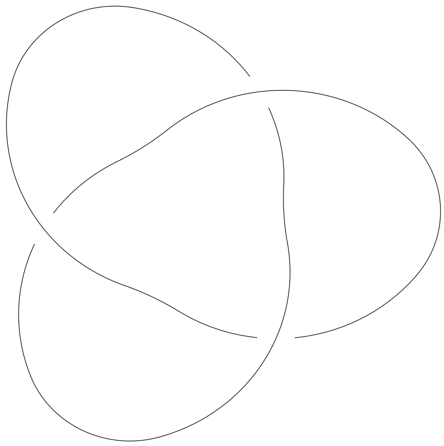
- Morifuji and Tran [45] treated twisted Alexander polynomials of a 2-bridge knot for parabolic representations in connection with the DFJ-conjecture. Here a representation ρ is called a parabolic representation if $tr(\rho(m)) = 2$.
- Recently Agol and Dunfield [1] showed that we can detect the Thurston norm of K by from $\Delta_{K,\rho_0}(t)$ in a large class of hyperbolic knots.

8. Epimorphism between knot groups

For the rest of this paper, as one application of the twisted Alexander polynomial, we treat some topics on epimorphisms between knot groups.

Definition 8.1. *For two knots* K_1, K_2*, we write* $K_1 \geq K_2$ *if there exists an epimorphism* $\varphi : G(K_1) \to G(K_2)$ *which maps a meridian of* K_1 *to a meridian of* K_2.

Let us start from a simple example: $8_5 \geq 3_1$.



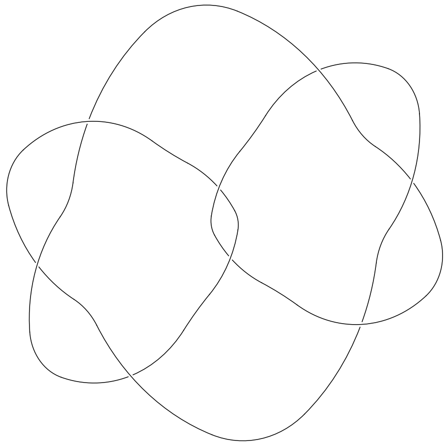
3_1

Example 8.2. The two knots 8_5 and 3_1 have the following presentations:

$$\begin{aligned} G(8_5) = \langle y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8 \mid\ & y_7 y_2 y_7^{-1} y_1^{-1}, y_8 y_3 y_8^{-1} y_2^{-1}, y_6 y_4 y_6^{-1} y_3^{-1}, \\ & y_1 y_5 y_1^{-1} y_4^{-1}, y_3 y_6 y_3^{-1} y_5^{-1}, y_4 y_7 y_4^{-1} y_6^{-1}, \\ & y_2 y_8 y_2^{-1} y_7^{-1} \rangle. \end{aligned}$$

and

$$G(3_1) = \langle x_1, x_2, x_3 \mid x_3 x_1 x_3^{-1} x_2^{-1}, x_1 x_2 x_1^{-1} x_3^{-1} \rangle.$$



8_5

If generators of $G(8_5)$ are mapped to the following generators of $G(3_1)$ as

$$y_1 \mapsto x_3,\ y_2 \mapsto x_2,\ y_3 \mapsto x_1,\ y_4 \mapsto x_3,$$
$$y_5 \mapsto x_3,\ y_6 \mapsto x_2,\ y_7 \mapsto x_1,\ y_8 \mapsto x_3,$$

any relator in $G(8_5)$ goes to the trivial element in $G(3_1)$. For example, it can be seen that

$$y_7 y_2 y_7^{-1} y_1^{-1} \mapsto x_1 x_2 x_1^{-1} x_3^{-1} = 1,$$
$$y_8 y_3 y_8^{-1} y_2^{-1} \mapsto x_3 x_1 x_3^{-1} x_2^{-1} = 1.$$

Hence this gives an epimorphism from $G(8_5)$ onto $G(3_1)$, which maps a meridian to a meridian. Therefore, we can write

$$8_5 \geq 3_1.$$

The geometric reason why there exists an epimorphism from $G(8_5)$ to $G(3_1)$ is

- 8_5 has a period 2, namely, it is invariant under some π-rotation of S^3,
- 3_1 is the quotient knot of 8_5 by this π-rotation.

Here we define a period of a knot as follows.

Definition 8.3. *A knot K in S^3 has a period $q > 1$ if there exists an orientation preserving periodic diffeomorphism $f : (S^3, K) \to (S^3, K)$ of order q such that the set of fixed points $Fix(f)$ is homeomorphic to S^1 in S^3 which is disjoint from K.*

Remark 8.4. By the positive answer for the Smith conjecture, we can see that the fixed point set is the unknot. See [42] for the Smith conjecture.

If K is a periodic knot of order q, this means that there exists an action of $\mathbb{Z}/q\mathbb{Z}$ on (S^3, K). Now the quotient space of S^3 by this action is topologically S^3 and the image of K by the quotient map is a knot in S^3 again.
The following problem is a fundamental problem.

Problem 8.5. *When and how does there exists an epimorphism between two given knot groups ?*

There are some geometric situations for the existence of a epimorphism as follows.

- To the trivial knot $\bigcirc$ from any knot K, there exists an epimorphism

  $$\alpha : G(K) \to G(\bigcirc) = \mathbb{Z}.$$

  This is just the abelianization

  $$G(K) \to G(K)/[G(K), G(K)] \cong \mathbb{Z}.$$

  This map can always be realized as a collapse map between knot exteriors with degree one.

- There exist two epimorphisms from any composite knot to each factor knot
$$G(K_1 \sharp K_2) \to G(K_1), G(K_2).$$
They are also just induced by collapse maps with degree one.
- In general a degree one map between knot exteriors induces an epimorphism, as we shall explain precisely later.
- Let K be a knot with a period q. Its quotient map $(S^3, K) \to (S^3, K') = (S^3, K)/\sim$ induces an epimorphism
$$G(K) \to G(K').$$
- For any knot K, we take the composite knot $K \sharp \bar{K}$ where $\bar{K}$ is the mirror image of K. The mirror image of K is defined as the image of K by a reflection of K along $\mathbb{R}^2$. Here we put a knot
$$\begin{aligned} K &\subset \mathbb{R}^2 \times (-\infty, 0) \\ &\subset \mathbb{R}^3 \subset S^3 = \mathbb{R}^3 \cup \{\infty\}. \end{aligned}$$
This reflection can be naturally extended to S^3. Then there exist epimorphisms
$$G(K \sharp \bar{K}) \to G(K)$$
between them. This epimorphism is induced from a quotient map
$$(S^3, K \sharp \bar{K}) \to (S^3, K)$$
of a reflection $(S^3, K \sharp \bar{K})$, whose degree is zero.
- There is the Ohtsuki-Riley-Sakuma construction for epimorphisms between 2-bridge links. Please see [47] for details.

First we recall the definition of the mapping degree.
Take any proper map
$$\varphi : (E(K_1), \partial E(K_1)) \to (E(K_2), \partial E(K_2))$$
between two knot exteriors. This map φ induces a homomorphism
$$\varphi_* : H_3(E(K_1), \partial E(K_1); \mathbb{Z}) \to H_3(E(K_2), \partial E(K_2); \mathbb{Z}).$$

Definition 8.6. *The degree of φ is defined to be the integer d satisfying*
$$\varphi_*[E(K_1), \partial E(K_1)] = d[E(K_2), \partial E(K_2)]$$
where $[E(K_i), \partial E(K_i)]$ is a generator of $H_3(E(K_i), \partial E(K_i); \mathbb{Z}) \cong \mathbb{Z}$ under the induced orientation from S^3 for $i = 1, 2$.

Proposition 8.7. *If $\varphi_* : G(K_1) \to G(K_2)$ is induced from a degree d map, then this degree d can be divisible by the index $n = [G(K_2) : \varphi_*(G(K_1))]$. Namely d/n is an integer.*

In particular if $d = 1$, then the index n should be 1 and hence $\varphi_*(G(K_1)) = G(K_2)$. Therefore we obtain the following.

Corollary 8.8. *If there exists a degree one map*
$$\varphi : (E(K_1), \partial E(K_1)) \to (E(K_2), \partial E(K_2)),$$
then φ induces an epimorphism
$$\varphi_* : G(K_1) \to G(K_2).$$

Remark 8.9. As explained later, there exist epimorphisms induced from

- a non zero degree map, but not degree one map,
- a degree zero map.

8.1. Determination on a partial order

For the set of isomorphism classes of knots, Definition 8.1 provides a partial order by using epimorphisms.

Proposition 8.10. *The relation $K \geq K'$ gives a partial order on the set of the prime knots. Namely this relation $\geq$ satisfies the followings;*

1. $K \geq K$.
2. $K \geq K',\ K' \geq K \Rightarrow K = K'$.
3. $K \geq K',\ K' \geq K'' \Rightarrow K \geq K''$.

Proof. The only non trivial claim is the second one,

$$K \geq K',\ K' \geq K \Rightarrow K = K'.$$

Here are two facts that we need to prove.

- Any knot group $G(K)$ is Hopfian, namely any epimorphism $G(K) \to G(K)$ is an isomorphism. See [22] as a reference for example.
- The knot group $G(K)$ determines the knot type for a prime knot K [21].

Now we assume $K \geq K',\ K' \geq K$. Then there exist two epimorphisms $\varphi_1 : G(K) \to G(K'),\ \varphi_2 : G(K') \to G(K)$. Here the composition of two epimorphisms $\varphi_2 \circ \varphi_1 : G(K) \to G(K)$ is an isomorphism because $G(K)$ is Hopfian.
Similarly the other $\varphi_1 \circ \varphi_2 : G(K') \to G(K')$ is an isomorphism, too. Hence $G(K)$ is isomorphic to $G(K')$. Because K and K' are prime knots, then $K = K'$. □

Remark 8.11.

- To say facts, here we do not use the assumption that an epimorphim preserves a meridian. However we need this assumption to determine the partial order.
- Cha and Suzuki [8] proved that there exist pairs of knots only with an epimorphism which does not preserve a meridian. Namely they admit an epimorphism, but never do an meridian preserving epimorphism.

To determine partial orders, fundamental tools are

- Alexander polynomial,
- Twisted Alexander polynomial.

The following fact on the Alexander polynomial is well known. As a reference, see [10] for example.

Proposition 8.12. *If $K_1 \geq K_2$, then $\Delta_{K_1}(t)$ can be divisible by $\Delta_{K_2}(t)$.*

This can be generalized to the twisted Alexander polynomial as follows.

Theorem 8.13 (Kitano-Suzuki-Wada [34])**.** *If $K_1 \geq K_2$ realized by an epimorpshim $\varphi : G(K_1) \to G(K_2)$, then $\Delta_{K_1,\rho_2\circ\varphi}(t)$ can be divisible by $\Delta_{K_2,\rho_2}(t)$ for any representation $\rho_2 : G(K_2) \to SL(l;\mathbb{F})$.*

By using these criterion for $SL(2;\mathbb{Z}/p\mathbb{Z})$-representations over a finite prime field $\mathbb{Z}/p\mathbb{Z}$, we can check the non-existence. For the rest, we find epimorphisms between knot groups by using a computer and obtain the following list.

Theorem 8.14 (Kitano-Suzuki [31], Horie-Kitano-Matsumoto-Suzuki [23])**.**

$$\left.\begin{array}{l} 8_5, 8_{10}, 8_{15}, 8_{18}, 8_{19}, 8_{20}, 8_{21}, 9_1, 9_6, 9_{16}, 9_{23}, 9_{24}, 9_{28}, 9_{40}, \\ 10_5, 10_9, 10_{32}, 10_{40}, 10_{61}, 10_{62}, 10_{63}, 10_{64}, 10_{65}, 10_{66}, 10_{76}, 10_{77}, \\ 10_{78}, 10_{82}, 10_{84}, 10_{85}, 10_{87}, 10_{98}, 10_{99}, 10_{103}, 10_{106}, 10_{112}, 10_{114}, \\ 10_{139}, 10_{140}, 10_{141}, 10_{142}, 10_{143}, 10_{144}, 10_{159}, 10_{164} \end{array}\right\} \geq 3_1$$

$$\left.\begin{array}{l} 11a_{43}, 11a_{44}, 11a_{46}, 11a_{47}, 11a_{57}, 11a_{58}, 11a_{71}, 11a_{72}, 11a_{73}, \\ 11a_{100}, 11a_{106}, 11a_{107}, 11a_{108}, 11a_{109}, 11a_{117}, 11a_{134}, 11a_{139}, \\ 11a_{157}, 11a_{165}, 11a_{171}, 11a_{175}, 11a_{176}, 11a_{194}, 11a_{196}, \\ 11a_{203}, 11a_{212}, 11a_{216}, 11a_{223}, 11a_{231}, 11a_{232}, 11a_{236}, \\ 11a_{244}, 11a_{245}, 11a_{261}, 11a_{263}, 11a_{264}, 11a_{286}, 11a_{305}, 11a_{306}, \\ 11a_{318}, 11a_{332}, 11a_{338}, 11a_{340}, 11a_{351}, 11a_{352}, 11a_{355}, \\ 11n_{71}, 11n_{72}, 11n_{73}, 11n_{74}, 11n_{75}, 11n_{76}, 11n_{77}, 11n_{78}, 11n_{81}, \\ 11n_{85}, 11n_{86}, 11n_{87}, 11n_{94}, 11n_{104}, 11n_{105}, 11n_{106}, 11n_{107}, 11n_{136}, \\ 11n_{164}, 11n_{183}, 11n_{184}, 11n_{185}, \end{array}\right\} \geq 3_1$$

$$\left.\begin{array}{l} 9_{18}, 9_{37}, 9_{40}, 9_{58}, 9_{59}, 9_{60}, 10_{122}, 10_{136}, 10_{137}, 10_{138}, \\ 11a_5, 11a_6, 11a_{51}, 11a_{132}, 11a_{239}, 11a_{297}, 11a_{348}, 11a_{349}, \\ 11n_{100}, 11n_{148}, 11n_{157}, 11n_{165} \end{array}\right\} \geq 4_1$$

$$11n_{78}, 11n_{148} \geq 5_1$$

$$10_{74}, 10_{120}, 10_{122}, 11n_{71}, 11n_{185} \geq 5_2$$

$$11a_{352} \geq 6_1$$

$$11a_{351} \geq 6_2$$

$$11a_{47}, 11a_{239} \geq 6_3$$

8.2. Hasse diagram

Now let us consider a Hasse diagram. It is an oriented graph for a partial ordering as follows.

- a vertex : a prime knot
- an oriented ege : if $K_1 \geq K_2$, then we draw it from the vertex of K_1 to the one of K_2.

Naturally the following problem arises.

Problem 8.15. *How can we understand the structure of this Hasse diagram of the prime knots under this partial order ?*

By using Kawauchi's imitation theory [26], the next theorem can be proved.

Theorem 8.16 (Kawauchi)**.** *For any knot K, there exists a hyperbolic knot $\tilde{K}$ such that there exists an epimorphism from $G(\tilde{K})$ onto $G(K)$ induced by a degree one map.*

As a similar application of Kawauchi's theory, we can see the following.

Proposition 8.17. *For any knot K, there exists a hyperbolic knot K' such that there exist two epimorphisms from $G(K')$ onto $G(K)$ as follows. The one is induced by degree one map and another one is induced by degree zero map.*

From the above proposition, there exists an epimorphism from a hyperbolic knot to any knot. On the other hand, the following fact is known. See [51, 32].

Fact 8.18. *For any torus knot K, if there exists an epimorphism $\varphi : G(K) \to G(K')$, then K' is a torus knot, too.*

Further we can see this Hasse diagram is not so simple as follows. The following proposition can be also proved by using Kawauchi's imitation theory.

Proposition 8.19. *For any two prime knots K_1 and K_2, there exists a prime knot K such that $K \geq K_1$ and $K \geq K_2$.*

In our list of partial ordering, knots

$$3_1, 4_1, 5_1, 5_2, 6_1, 6_2, 6_3$$

are minimal elements in the set of prime knots with up to 11-crossings.
Here in fact, we can prove that they are minimal in the set of all prime knots.

Theorem 8.20 (Kitano-Suzuki[33])**.** *The knots $3_1, 4_1, 5_1, 5_2, 6_1, 6_2, 6_3$ are minimal elements in the set of all prime knots.*

By the above results, the following problem appears naturally.

Problem 8.21. *If $K_1 \geq K_2$, then is the crossing number of K_1 greater than the one of K_2?*

It is clear in the list. If it is true in general, it gives another proof of the theorem by Agol and Liu.

Theorem 8.22 (Agol-Liu [2])**.** *Any knot group $G(K)$ surjects onto only finitely many knot groups.*

Remark 8.23*.* This statement was called the Simon's conjecture. See [27].

8.3. Epimorphisms induced by degree zero maps

Boileau, Boyer, Reid and Wang proved the following.

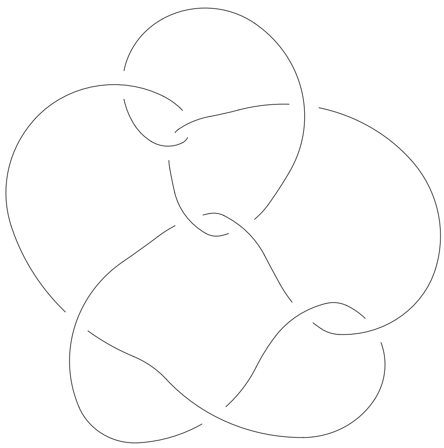
Proposition 8.24 (Boileau-Boyer-Reid-Wang [4])**.** *Any epimorphism between 2-bridge hyperbolic knots is always induced from a non zero degree map.*

On the other hand, there are some interesting examples in our list as follows.

Example 8.25. Here $10_{59},\ 10_{137}$ are 3-bridge hyperbolic knots. From the list one has $10_{59},\ 10_{137} \geq 4_1$, that is, there exist epimorphisms

$$G(10_{59}), G(10_{137}) \to G(4_1).$$

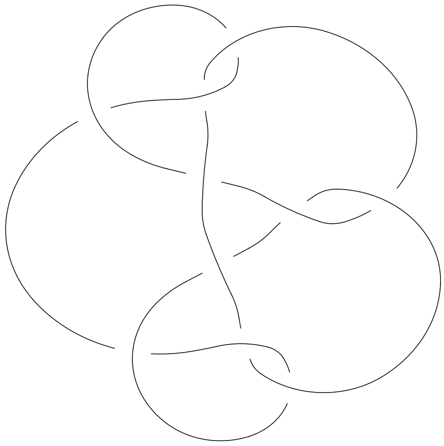
However there is no non-zero degree map between them. Namely any epimorphism induced by a proper map between these knot exteriors is induced from a degree zero map.



10_{59}

Here recall the Alexander module of a knot. We take a $\mathbb{Z}$-covering

$$E_\infty(K) \to E(K)$$



10_{137}

associated to $\alpha : G(K) \to \mathbb{Z} \cong < t >$. Here a group ring $\mathbb{Z}[\mathbb{Z}] \cong \mathbb{Z}[t, t^{-1}]$ acts on $H_1(E_\infty(K); \mathbb{Z})$ and it gives a structure of a module over a Laurent polynomial ring $\mathbb{Z}[t, t^{-1}]$ on $H_1(E_\infty(K); \mathbb{Z})$. This module is called the Alexander module of K over $\mathbb{Z}$.

Remark 8.26. If we consider the Alexander module over $\mathbb{Q}$, a generator of its order ideal is just the Alexander polynomial of K.

To see that there are no non-zero degree maps, we have to study the structure of Alexander modules. The following facts are well known in the theory of surgeries on compact manifolds. For example, see in the book by Wall [56].

Fact 8.27. *If there exits an epimorphism*

$$\varphi_* : G(K) \to G(K')$$

induced from a non zero degree map (resp. a degree one map)

$$E(K) \to E(K'),$$

then its induced epimorphism

$$H_1(E_\infty(K); \mathbb{Q}) \to H_1(E_\infty(K'); \mathbb{Q})$$

between their Alexander modules over $\mathbb{Q}$ (resp. over $\mathbb{Z}$) is split over $\mathbb{Q}$ (resp. $\mathbb{Z}$).

Remark 8.28. The twisted Alexander module version of the above fact may be a refinement of the divisibility of twisted Alexander polynomials.

Example 8.29. By similar observation for Alexander modules, we can see the followings.

- $9_{24} \geq 3_1$ and $11a_5 \geq 4_1$.
- Any epimorphism induced by a proper map between these knot exteriors is induced only from an degree zero map.

Remark 8.30. Here 10_{59}, $10_{137}, 9_{24}$ are Montesinos knots given as follows.

- $10_{59} = M(-1; (5, 2), (5, -2), (2, 1))$,
- $10_{137} = M(0; (5, 2), (5, -2), (2, 1))$,
- $9_{24} = M(-1; (3, 1), (3, 2), (2, 1))$.

Why does there exist an epimorphism between them ? We can explain the reason from the geometric observation by Ohtsuki-Riley-Sakuma in [47].
Here let

$$\varphi : G(K) \to G(K')$$

be an epimorphism. We take a simple closed curve $\gamma \subset S^3 \setminus K$ whose homotopy class belongs to $\mathrm{Ker}\varphi \subset G(K)$. Then if γ is an unknot in S^3, by taking the surgery along γ, we get a new knot $\tilde{K}$ in S^3 such that there exists an epimorphism $G(\tilde{K}) \to G(K')$.
Now we can apply this construction to a pair of $(4_1\sharp\bar{4}_1 = 4_1\sharp 4_1, 4_1)$. First we recall that there exists an epimorphism

$$G(4_1\sharp\bar{4_1}) \to G(4_1)$$

which is a quotient map of a reflection. Then it is induced from a degree zero map. By surgery along some simple closed curve, one has both of

$$G(10_{59}) \to G(4_1),$$

and

$$G(10_{137}) \to G(4_1).$$

More generally we can see the following by applying this construction to any 2-bridge knot. It was not written explicitly, but essentially in [47] by Ohtsuki, Riley and Sakuma.

Proposition 8.31. *For any 2-bridge knot K, there exists a Montesinos knot $\tilde{K}$ such that there exists an epimorphism*

$$G(\tilde{K}) \to G(K)$$

induced from a degree zero map $E(\tilde{K}) \to E(K)$.

Return to the list of knots with up to 10-crossings. We can find epimorphisms explicitly, but have not found all epimorphisms if they exist.
For the epimorpshism we could find, the following partial order relations can be realized by epimorphisms induced from degree zero maps.

$$\left.\begin{array}{l} 8_{10},\ 8_{20},\ 9_{24},\ 10_{62},\ 10_{65},\ 10_{77}, \\ 10_{82},\ 10_{87},\ 10_{99},\ 10_{140},\ 10_{143} \end{array}\right\} \geq 3_1$$

$$10_{59},\ 10_{157} \geq 4_1$$

In this list, Montesinos knots appear as above.

Remark 8.32. The other knots are given by Conway's notation [9] as follows:

- $10_{82} = 6 * *4.2$,
- $10_{87} = 6 * *22.20$,
- $10_{99} = 6 * *2.2.20.20$

About the above degree zero maps, it might be understood from this classification.

8.4. Problems

Finally we put a list of problems.

- Characterize a minimal knot in the set of prime knots under the partial order.
- Characterize an epimorphism induced from a degree zero map.
- If K_1, K_2 are hyperbolic knots and $K_1 \geq K_2$, then is the hyperbolic volume of $S^3 \setminus K_1$ greater than or equal to the one of $S^3 \setminus K_2$?
- How strong is twisted Alexander polynomial for a representation over a finite field ?
  - To determine the non-existence of an epimorphism.
  - To detect the fiberedness.

For example, is it true that K is fibered if any twisted Alexander polynomial is monic for any 2-dimensional unimodular representation over a finite prime field ?

- By using twisted Alexander module, give a generalization of the method to determine existence of epimorphism by using Alexander module.

- Find a skein relation for twisted Alexander polynomial.

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