WBLN

Winter Braids Lecture Notes

Patrick Popescu-Pampu

Complex singularities and contact topology

PATRICK POPESCU-PAMPU

Abstract

This text is a greatly expanded version of the mini-course I gave during the school *Winter Braids VI* organized in Lille between 22–25 February 2016. It is an introduction to the study of interactions between singularity theory of complex analytic varieties and contact topology. I concentrate on the relation between the smoothings of singularities and the Stein fillings of their contact boundaries. I tried to explain basic intuitions and facts in both fields, for the sake of the readers who are not accustomed with one of them.

Contents

1. Introduction

Let (X, x) be an isolated singularity of equidimensional complex analytic set. One may look at privileged representatives of it by choosing a representative in $(\mathbb{C}^n, 0)$ and by taking its intersections with sufficiently small Euclidean balls centered at the origin. One *almost* gets in this way a compact manifold with boundary: the only non-smooth point is the origin. By a fundamental theorem of Whitney, such a representative is homeomorphic to a cone over the boundary. As this boundary is independent of the choices of embedding and small ball, we will call it simply *the boundary of the singularity* (X, x), and we will denote it by $\partial(X, x)$.

As boundary of a complex manifold, which is always naturally oriented, $\partial(X, x)$ gets also a natural orientation. In the sequel we will always consider it endowed with this orientation. This is a non-trivial supplementary structure, as in dimension at least 3 the orientable manifolds do not necessarily have orientation-reversing self-diffeomorphisms.

Not all closed oriented manifolds are boundaries of compact oriented manifolds: Thom's cobordism theory provides precise measuring tools for this phenomenon. What about the singularity boundaries? Well, Hironaka's celebrated theorem implies that there exists a *resolution of singularities* of (X, x). Namely, informally speaking, one may replace the point x of X by a compact analytic space, such that the resulting space $\tilde{X}$ is *smooth*. In this process, one does not touch the boundary of X, therefore $\partial(X, x)$ is also the boundary of an oriented *manifold*. In fact, as soon as X is of dimension at least 2, one gets an infinite number of homeorphism types of resolutions, therefore an infinite number of homeomorphism types of *fillings* of $\partial(X, x)$ (here and in the sequel, we simply say that a compact oriented manifold F whose boundary is identified with a given closed oriented manifold M is a *filling* of M).

If *resolutions* of (X, x) lead to an infinite number of fillings, by contrast *deformations* of (X, x) lead necessarily to a finite number of them. Informally speaking, a deformation of (X, x) is a germ of family of complex analytic spaces having a special member identified with (X, x). By repeating the construction with the choice of small ball for the total space of the deformation, one may associate a *generic fiber* over each irreducible component of the parameter space of a given deformation, well-defined up to homeomorphisms. Its boundary may be canonically identified up to isotopy with $\partial(X, x)$. Therefore, when the generic fiber is smooth, one gets again a filling of $\partial(X, x)$. One says in this case that the corresponding irreducible component is a *smoothing component*, and that the generic fiber is the associated *Milnor fiber*.

A fundamental theorem of Grauert states that there exists a so called *versal* deformation from which all other deformations may be obtained by base-change. In particular, the Milnor fibers of its smoothing components give all the Milnor fibers obtainable from deformations. As the parameter space of a versal deformation has a finite number of irreducible components, one gets in this way a finite number of fillings of $\partial(X, x)$.

The aim of this text is to explain that *contact topology* may be useful in order to understand the topological structure of the Milnor fibers of a given isolated singularity. The reason is that:

- the boundary $\partial(X, x)$ is canonically a *contact manifold* and that the Milnor fibers of (X, x) are *Stein fillings* of this manifold;

- given a contact manifold, there are serious constraints on the topological types of its Stein fillings; for instance, a theorem of Eliashberg states that any Stein filling of the contact boundary of $(\mathbb{C}^2, 0)$ (which is the standard contact 3-sphere) is diffeomorphic to a ball.

The general question we want to examine is:

How to characterize Milnor fibers among the Stein fillings of the contact boundary of an isolated singularity?

As a generalization of Eliashberg's theorem stated above, Némethi and the author proved Lisca's conjecture that for *cyclic quotient singularities* (that is, the singularities of normal toric surfaces),

the Milnor fibers give all possible diffeomorphism types of Stein fillings of their contact bound-
aries. It is completely unknown how to characterize the surface singularities for which one has an
analogous theorem. The present notes describe the state of the art about these questions.

I would like to mention that in recent years Mark McLean has proved very interesting results
relating analytical invariants of isolated complex singularities of dimension at least 3 and contact
topological invariants of their links. I won't say anything about his works here, sending the inter-
ested reader to the original papers [122] and [123].

This text is intended to be an introduction to the topology and contact topology of isolated com-
plex singularities and of their smoothings, for students who know the basic languages of complex
algebraic or analytic geometry as well as of differential topology. I tried to describe important intu-
itions and examples. The majority of results are given without full proofs, but they are accompanied
with heuristical explanations whenever possible. Each section concludes with a list of references for
a deeper study of its material.

As a starting point for writing the present notes, I used my habilitation [162]. The recent texts
which are closest in spirit to it are the surveys [135] of Némethi, [17] of Bhupal and Stipsicz and
[149] of Ozbagci. If the first survey concentrates on the Milnor fibers of normal surface singularities,
the two other ones focus on the basic techniques for studying the topological structure of Stein
fillings of given contact 3-manifolds.

Let me briefly describe the contents of the various sections of the paper:

- Subsection 2.1 contains an explanation of basic notions about complex analytic singulari-
  ties (*dimension, irreducible components, local ring, multiplicity*). Subsection 2.2 introduces
  the basic classes of singularities of arbitrary dimension discussed in the sequel (*hypersur-
  faces, complete intersections, Cohen-Macaulay, normal, Gorenstein* and *quotient singulari-
  ties*). Subsection 2.3 describes the general technique of study of singularities through their
  modifications (in particular the *blow-ups*) and *resolutions*.

- Subsection 3.1 explains basic facts about intersection theory of divisors on smooth surfaces
  (*intersection numbers*, the *arithmetic genus* and the *adjunction formula*). Subsection 3.2
  presents the *exceptional divisor* of a resolution of normal surface singularity and basic ob-
  jects associated with it (its *weighted dual graph*, its *intersection form*, the *Lipman semigroup*,
  the *fundamental cycle* and the *anti-canonical cycle*). Subsection 3.3 explains the basic facts
  about the topological structure of normal surface singularities (the notion of *boundary* of
  such a singularity, the *plumbing* of special 3-manifolds and 4-manifolds, the notion of *graph
  manifold*, the fact that the boundaries of normal surface singularities are graph manifolds
  and the way to deduce the homology of the boundary from the weighted dual graph of a
  resolution). Subsection 3.4 introduces two notions of *genus* for normal surface singularities
  and two classes of such singularities defined in terms of those genera, the *rational* and the
  minimally elliptic singularities. Several subclasses are emphasized (the *Kleinian*, the *cyclic
  quotients*, the *simple elliptic* and the *cusp singularities*). Section 3 concludes with Figure 3.6,
  which represents (by an Euler-Venn diagram) the various inclusion relations between the
  classes of surface singularities discussed so far.

- Subsection 4.1 presents the *method initiated by Milnor* for studying the topology of isolated
  complex hypersurface singularities. Subsection 4.2 presents basic facts about *deformations*
  of singularities (the notion of *deformation*, *miniversal deformations*, *smoothings and their
  Milnor fibers*, *Grauert's theorem* about the existence of miniversal deformations of isolated
  singularities). Subsection 4.3 presents basic facts about the Milnor fibers of the smoothings
  of normal surface singularities (various formulae for the inertia index of their intersection
  form in terms of invariants of the singularity and of the dimension of the associated smooth-
  ing component). Subsection 4.4 presents in a detailed way Pinkham's first example of normal
  surface singularity with two smoothing components, and describes the structure of the cor-
  responding Milnor fibers.

- Subsection 5.1 presents basic *analogies between real affine geometry and complex geometry*, which are useful whenever one begings to think seriously about symplectic or contact aspects of complex manifolds. Subsection 5.2 presents basic facts about *strict plusrisubharmonic functions* and *Stein manifolds* (differential-geometric objects associated to such a function and the theorem about the homotopy type of such a manifold). Subsection 5.3 presents basic facts about *contact structures, contact manifolds* and their various types of *fillings* (Stein, holomorphic, strong and weak symplectic).

- Subsection 6.1 introduces the notion of *contact boundary* of an isolated complex singularity and basic theorems about it (the fact that a holomorphic function with isolated critical point defines a Milnor open book which supports the contact structure on the boundary in the sense of Giroux and, as a consequence, the fact that in complex dimension 2 this contact structure is determined by the topological type). Subsection 6.2 contains a description of the *types of normal surface singularities for which it was proved that their Milnor fibers exhaust their Stein fillings*, with details about the combinatorial objects appearing in the special case of cyclic quotient singularities.

- Section 7 contains a list of open questions about contact topological aspects of singularities.

2. Generalities about complex analytic singularities and their resolutions

2.1. What is a singularity?

By definition, a **complex analytic set** is a Hausdorff topological space which may be covered by an atlas whose charts are identifed with the following models: simultaneous zero-loci inside open subsets of $\mathbb{C}^n$ of sets of holomorphic functions. Moreover, one asks the changes of charts to be also holomorphic. One may then define easily the notions of **holomorphic function** and **holomorphic map** from one complex analytic set to another one. In particular, one may speak about **holomorphic isomorphisms**, also called **holomorphic equivalences**.

If $f_1, ..., f_p$ are holomorphic functions on a complex analytic set X, we will denote by $Z(f_1, ..., f_p) \hookrightarrow X$ their common zero-locus, which is a closed complex analytic subset of X.

Complex analytic sets may have *singular points*:

Definition 2.1. Let X be a complex analytic set. A point $x \in X$ is **a singular point of** X if there does not exist a neighborhood of x in X, which is holomorphically equivalent to an open set in some $\mathbb{C}^n$. A point of X which is not singular is called **regular**. If all the points of X are regular, then we say that X is a **complex manifold**.

One may show that the subset Sing X of singular points of X is a closed complex analytic subset of X, strictly included in X (see [90, Corollary 6.3.4 and Remark 4.3.7]).

According to common usage among singularity theorists, we use also the following vocabulary, *in which we allow by abuse of language the point x to be regular on X*:

Definition 2.2. A **singularity** is a germ (X, x) of complex analytic set. In this case, one says that X is a **representative** of the singularity and that x is its **base point**.
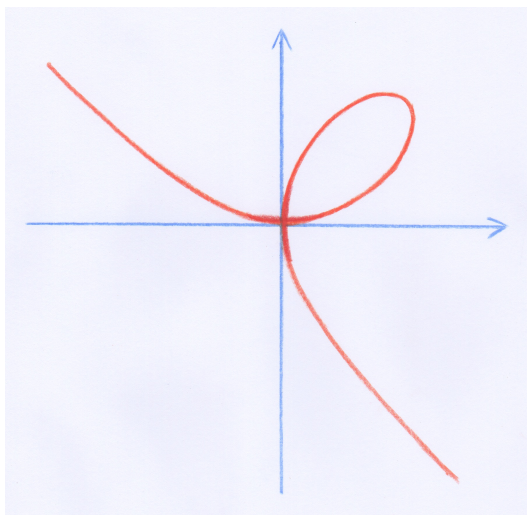
Figure 2.1. The folium of Descartes

A singularity is **irreducible** if it cannot be written as a union of two singularities different from it. An **irreducible component** of a singularity is an irreducible subsingularity which is maximal for inclusion. A singularity is **isolated** if there exists a representative of it which is smooth outside its base point.

One may show that each singularity has a finite number of irreducible components.

The notion of irreducibility may be also defined for global analytic sets, not only for germs. It may be shown that any irreducible germ has an irreducible representative. The converse is not true, in the sense that irreducible complex analytic sets may have reducible germs at some points:

Example 2.3. Consider the *folium of Descartes*, defined by the equation $x^3 + y^3 - 3xy = 0$. It may be shown that it is an irreducible complex analytic set, but that its singularity at the origin has two irreducible components, which may be guessed by looking at the locus of real points (see Figure 2.1).

If X is an irreducible complex analytic set, then it may be shown that its subset of regular points is connected and locally isomorphic to an open set of $\mathbb{C}^n$, for some $n \in \mathbb{N}$. In this case, one says that n is the **complex dimension** of X. When X is reducible, its complex dimension is by definition the maximal dimension of its irreducible components. If all of them have the same dimension, then X is called **equidimensional**. We will use the same vocabulary when speaking about singularities and we will denote by $\dim(X, x)$ the complex dimension of the singularity (X, x).

Example 2.4. If X is the union of a plane and a transversal line inside $\mathbb{C}^3$, then its irreducible components have dimension 2 and 1 respectively. Therefore the complex dimension of X is 2.

Another essential invariant of singularities is their *multiplicity*, which is a local version of the degree of a projective variety:

Definition 2.5. Let (X, x) be a singularity. Fix an embedding $(X, x) \hookrightarrow (\mathbb{C}^n, 0)$. Then the **multiplicity** $m(X, x)$ of (X, x) is the number of intersection points of a sufficiently small representative of X with a generic affine subspace of $\mathbb{C}^n$ of complementary dimension to that of (X, x), not passing through 0 but very close to 0.

The previous definition is rather intuitive, but needs some work to be made precise. What means to be "generic" and "very close" and why does one get a number which is independent of the choice of embedding? One may find an answer to those questions in de Jong and Pfister [90, Theorem 4.2.24].

In order to avoid those technicalities, one turns usually to a more algebraic view of singularities, through their *local rings*. Recall first the definition of such rings:

Definition 2.6. A commutative ring is called **local** if it has only one maximal ideal.

The ring $\mathbb{C}\{z_1, \ldots, z_n\}$ of power series in the variables $z_1, \ldots, z_n$, which are convergent in a neighborhood of $0 \in \mathbb{C}^n$ (we will simply speak in the sequel of **convergent power series**), is an example of local ring. Its maximal ideal consists of the series whose constant term vanishes. This example allows to explain the qualificative "*local*". Indeed, $\mathbb{C}\{z_1, \ldots, z_n\}$ consists of the holomorphic functions defined *locally* in a neighborhood of the point 0 in the complex analytic manifold $\mathbb{C}^n$. Its unique maximal ideal corresponds then to the functions vanishing at 0.

A power series $f \in \mathbb{C}\{z_1, \ldots, z_n\}$ is called **reduced** if its prime factorization inside $\mathbb{C}\{z_1, \ldots, z_n\}$ has only factors of multiplicity 1. One may show that this is equivalent to the fact that the quotient ring $\mathbb{C}\{z_1, \ldots, z_n\}/(f)$ of convergent series modulo multiples of f has no non-trivial *nilpotent* elements. This motivates the following definition:

Definition 2.7. A local ring is called **reduced** if its only nilpotent element is 0.

According to Definition 2.2, singularities may be defined by the vanishing of a set of convergent power series. The simplest case is that of **hypersurface singularities**, when one takes only one power series f. Note that the product of the distinct prime factors of f is reduced and defines the same hypersurface singularity as f. Therefore, one may assume that f is reduced. In this case, the ring of restrictions to $X = Z(f)$ of the convergent power series on $\mathbb{C}^n$ is again local and it may be canonically identified with the quotient ring $\mathbb{C}\{z_1, \ldots, z_n\}/(f)$ considered above. We call it the **local ring of the hypersurface singularity** $(X, 0)$**.** More generally:

Definition 2.8. The **local ring $\mathcal{O}_{X,x}$ of the complex analytic set X at its point** x is the ring of germs of holomorphic functions defined on X in a neighborhood of x.

One may show that such a ring is indeed always local, its unique maximal ideal $\mathfrak{m}_{X,x}$ consisting of the holomorphic functions which vanish at x. Moreover, such a local ring is always reduced, because a function which admits a power identically vanishing coincides with the zero function.

If $(X, 0) \subset (\mathbb{C}^n, 0)$ is defined as the germ at 0 of $Z(f_1, \ldots, f_p)$, where $f_1, \ldots, f_p \in \mathbb{C}\{z_1, \ldots, z_n\}$ generate the ideal of convergent power series vanishing on $(X, 0)$, then one gets a canonical identification:

$$\mathcal{O}_{X,0} \simeq \mathbb{C}\{z_1, \ldots, z_n\}/(f_1, \ldots, f_p).$$

More generally, one associates to each open set U of a complex analytic set X, the ring of holomorphic functions defined (at least) on U. One gets in this way a sheaf of rings, called the **structure sheaf** $\mathcal{O}_X$ of X. The local ring $\mathcal{O}_{X,x}$ may be canonically identified to the ring of germs at x of sections of this structure sheaf.

In fact, in order to get continuity properties for various numerical invariants in reasonable families of complex sets, as well as more functorial constructions, one needs to allow also complex analytic sets with structure sheaves *admitting nilpotent elements*. For instance, one allows as local ring of a hypersurface singularity $(Z(f), 0) \subset (\mathbb{C}^n, 0)$ the quotient $\mathbb{C}\{z_1, \ldots, z_n\}/(f)$, even when f is not reduced. In such a generality, one speaks about **complex analytic spaces** instead of *sets*.

It is the following theorem of Samuel [173] concerning the local ring of a singularity, which is turned usually into a definition of its *multiplicity*:

Theorem 2.9. *Let $\mathcal{O}$ be the local ring of the singularity (X, x) and $\mathfrak{m}$ be its maximal ideal. Then the function:*

$$
\begin{array}{rccc}
f: & \mathbb{Z}_+ & \longrightarrow & \mathbb{Z}_+ \\
& k & \longrightarrow & \dim_{\mathbb{C}}\left(\dfrac{\mathcal{O}}{\mathfrak{m}^k}\right)
\end{array}
$$

becomes a polynomial function of degree $\dim(X, x)$ *for k large enough. The leading coefficient of this polynomial is equal to* $\dfrac{m(X, x)}{\dim(X, x)!}$.

In order to get a general view of the main concepts of complex analytic geometry, one may read Fischer's book [55]**. A carefully written introduction to singularities of complex analytic spaces is de Jong and Pfister's book** [90]**.**

2.2. Some classes of singularities in arbitrary dimension

Let us come back to a hypersurface singularity $(X, 0)$ defined by a reduced convergent power series f. One may describe in the following way its singular locus:

Proposition 2.10. *Let $(X, 0)$ be the hypersurface singularity in $\mathbb{C}^n$ defined by the vanishing of the reduced convergent power series $f \in \mathbb{C}\{z_1, ..., z_n\}$. Then $\operatorname{Sing} X$ is the subset $Z\left(\dfrac{\partial f}{\partial z_1}, \dots, \dfrac{\partial f}{\partial z_n}\right)$ of X defined by the vanishing of all the partial derivatives of f.*

The fact that a point of a representative of X is regular whenever there exists a partial derivative $\dfrac{\partial f}{\partial z_i}$ which does not vanish at it, is a consequence of *the holomorphic version of the implicit function theorem* (see [90, Theorem 3.3.1]). It is a subtler point to understand why the remaining points of the representative belong necessarily to the singular locus, as introduced in Definition 2.1 (see [90, Section 4.3]).

Example 2.11. Assume that $f(z_1, ..., z_n) := z_1^{a_1} + \cdots + z_n^{a_n}$, where $a_1, ..., a_n \in \mathbb{N}^*$. Proposition 2.10 allows to see immediately that the hypersurface $Z(f)$ of $\mathbb{C}^n$ is smooth outside the origin, and that it is also smooth at the origin if and only if at least one of the exponents a_k is equal to 1. When all the a_k are ≥ 2, one gets therefore an isolated hypersurface singularity, called a **Pham-Brieskorn singularity**. This name honors the works [157] of Pham and [20] of Brieskorn. One may learn details about the relation between those works and their influence on the development on the topological study of singularities in Brieskorn's paper [23].

The singularities which are not hypersurfaces need at least two functions in order to be defined as subvarieties of a smooth germ $(\mathbb{C}^n, 0)$. It may be shown that each function drops the ambient dimension by at most one. In fact one has the following result, which is a particular case of **Krull's principal ideal theorem** (see [90, Section 4.1]):

Proposition 2.12. *Let (X, x) be a singularity and $f \in \mathcal{O}_{X,x}$. Then $\dim Z(f) \geq \dim X - 1$, with equality whenever f is not a zero-divisor in $\mathcal{O}_{X,x}$.*

Geometrically, the fact that an element of the local ring of a singularity is not a zero-divisor means that its zero-locus does not contain any irreducible component of the singularity.

Example 2.13. (continuation of Example 2.4, see Figure 2.2). Let X be the union of the plane $Z(z_1)$ of the coordinates (z_2, z_3) and of the z_1-axis inside $\mathbb{C}^3$. The local ring of the singularity $(X, 0)$ is isomorphic to $\mathbb{C}\{z_1, z_2, z_3\}/(z_1 z_2, z_1 z_3)$. The vanishing locus $Z(z_1) \hookrightarrow X$ is simply the plane of coordinates (z_2, z_3), therefore it has the same dimension as $(X, 0)$. This is due to the fact that z_1 is a zero-divisor in the local ring $\mathbb{C}\{z_1, z_2, z_3\}/(z_1 z_2, z_1 z_3)$. Note that this example shows also that the implication stated in Proposition 2.12 cannot be extended into an equivalence. Indeed, $Z(z_2) \hookrightarrow X$ is the union of the z_1-axis and of the z_3-axis, therefore $\dim Z(z_2) = \dim X - 1$, but z_2 is a zero-divisor in the local ring $\mathcal{O}_{X,0}$.

Unlike the previous example, the following generalizations of hypersurface singularities are always equidimensional:

Definition 2.14. A singularity is called a **complete intersection** if it is analytically isomorphic to $Z(f_1, ..., f_p) \subset (\mathbb{C}^n, 0)$, where $f_1, \dots, f_p \in \mathbb{C}\{z_1, ..., z_n\}$ and for any $k \in \{1, \dots, p\}$, f_k is not a zero-divisor in the local ring $\mathbb{C}\{z_1, ..., z_n\}/(f_1, \dots, f_{k-1})$.
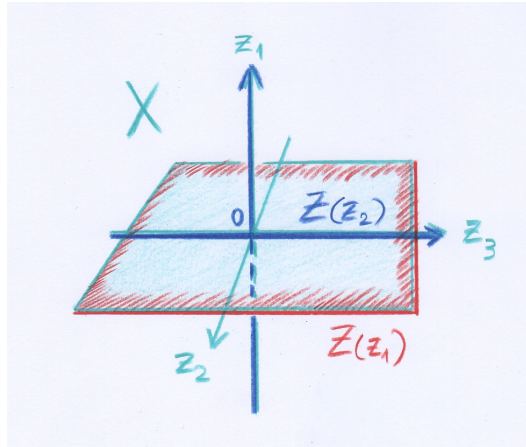
Figure 2.2. An illustration of Krull's principal ideal theorem

Let us introduce the following standard vocabulary used in this situation:

Definition 2.15. If (X, x) is a singularity, then a finite sequence $(f_1, ..., f_p) \in \mathcal{O}_{X,x}$ is called a **regular sequence** on (X, x) if for any $k \in \{1, ..., p\}$, f_k is not a zero-divisor in the local ring $\mathcal{O}_{X,x}/(f_1, \ldots, f_{k-1})$.

Therefore, complete intersection singularities are those defined by a regular sequence on a smooth germ. One may show that they are a particular case of *Cohen-Macaulay singularities*, which are maximal from the view-point of existence of regular sequences:

Definition 2.16. A singularity (X, x) is called **Cohen-Macaulay** if it has a regular sequence with $\dim(X, x)$ elements.

Even if they are more general than complete intersections, Cohen-Macaulay singularities are also necessarily equidimensional (see [90, Corollary 6.5.8]). This gives a way to see topologically that the singularity of Example 2.13 is not Cohen-Macaulay.

Even if a singularity is equidimensional, one may conclude sometimes that it is not Cohen-Macaulay simply by looking at its singular locus. Indeed, one has the following theorem of Hartshorne (see [44, Theorem 18.12]):

Theorem 2.17. *A Cohen-Macaulay singularity is connected in codimension* 1, *that is, one cannot disconnect it by removing a subsingularity of codimension at least* 2.

Example 2.18. As a variation of Example 2.13, consider the union of the planes of coordinates (z_1, z_2) and (z_3, z_4) in $\mathbb{C}^4$. One gets an equidimensional singularity of dimension 2 by taking the germ at the origin 0. As one may disconnect it by removing 0, which is a subsingularity of codimension 2, Theorem 2.17 shows that this singularity is not Cohen-Macaulay. Therefore, it is nor a complete intersection.

Let us introduce another general class of singularities, which will be very important in the sequel:

Definition 2.19. A singularity is called **normal** if it is irreducible and if one of the following equivalent properties holds:

1. its local ring is integrally closed in its field of fractions;

2. any bounded function defined on a representative of the singularity and holomorphic outside a strict subsingularity, extends to a holomorphic function over the full singularity;

3. the codimension of the singular set is at least 2 and there exists a regular sequence of length at least 2.

A complex analytic set is called **normal** if all its germs are normal.

It is a theorem that the previous properties are equivalent (see [90, Theorems 4.4.11, 4.4.15]). Reexpressed usually in the language of commutative algebra the third property is called *Serre's criterion* (see [90, Theorem 4.4.11] or [44, Section 11.2]).

One may show that there exists a regular sequence of length at least 2 on a singularity (X, x) if and only if any germ $f \in \mathcal{O}_{X,x}$ which is not a zero divisor may be extended to a regular sequence of length at least 2. Therefore, in order to show that a singularity is not normal, it is enough to find a germ f which does not have this property. This argument is used in Example 2.21 below.

By *Riemann's extension theorem*, all complex manifolds are normal. More generally, one has the following theorem, which explains why we introduced the notion of normality as a companion to that of being Cohen-Macaulay:

Theorem 2.20.

1. *A surface singularity (that is, an equidimensional germ of dimension* 2*) is normal if and only if it is an isolated Cohen-Macaulay singularity.*

2. *If the singular locus of a Cohen-Macaulay singularity is of codimension at least* 2*, then the singularity is normal.*

This theorem may be proved using Serre's criterion (3) stated in Definition 2.19. Note that the conditions of irreducibility and isolatedness of the singularity alone do not imply that a surface singularity is normal:

Example 2.21. This example is taken from [90, Example 6.5.6 (6)]. Consider the germ at the origin of the surface X of $\mathbb{C}^4$ described parametrically by the map $(s, t) \to (x, y, z, u) = (s, st, t^2, t^3)$. The Jacobian matrix being of rank 2 for $(s, t) \neq (0, 0)$, this map is an immersion outside the origin. It is easy to check that it is moreover injective. Therefore, $(X, 0)$ is an irreducible isolated surface singularity. One may check (see again [90, Example 6.5.6 (6)]) that it is not Cohen-Macaulay, as the restriction of $x - z$ to its local ring cannot be extended to a regular sequence of length 2. By point (1) of Theorem 2.20, one sees that $(X, 0)$ is not normal.

Cohen-Macaulay singularities are not necessarily irreducible, in contrast with normal ones. Therefore, one may be surprised by the implication of Theorem 2.20 stating that *isolated* Cohen-Macaulay singularities are normal, therefore irreducible. Note that this fact may be seen as a consequence of Theorem 2.17. Indeed, an *isolated* singularity (X, x) with at least two irreducible components may be disconnected just by removing the point x, which shows that it is not connected in codimension 1.

It may be shown that each complex analytic set X may be *normalized*, in the following sense:

Definition 2.22. A **normalization** of a complex analytic set X is a normal analytic set $\overline{X}$ endowed with a finite (that is, proper with finite fibers) surjective morphism $\nu : \overline{X} \to X$ which is an isomorphism over a dense open subset of X.

It is a theorem that a normalization morphism is unique *up to a unique isomorphism over X.* That is, if $\nu_1 : \overline{X}_1 \to X$ and $\nu_2 : \overline{X}_2 \to X$ are both normalizations of X, then there exists a unique isomorphism $\phi : \overline{X}_1 \to \overline{X}_2$ such that $\nu_2 \circ \phi = \nu_1$. This allows to speak about *the* normalization morphism.

The change of topology produced by the normalization morphism is partially described by the following proposition:

Proposition 2.23. *Let X be a complex analytic set. The normalization morphism $\nu : \overline{X} \to X$ is a homeomorphism if and only if all the germs of X at its various points are irreducible singularities. More generally, for each $x \in X$, the cardinal of $\nu^{-1}(x)$ is equal to the number of irreducible components of the singularity (X, x).*

Informally speaking, the combination of this proposition with the fact that normal complex analytic sets have a singular locus of codimension at least 2 may be expressed in the following way:
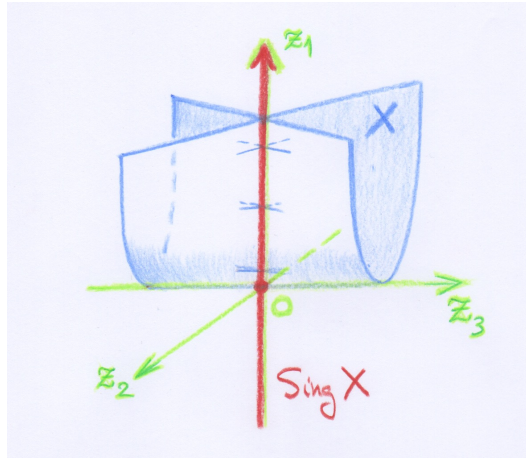
Figure 2.3. Whitney's umbrella
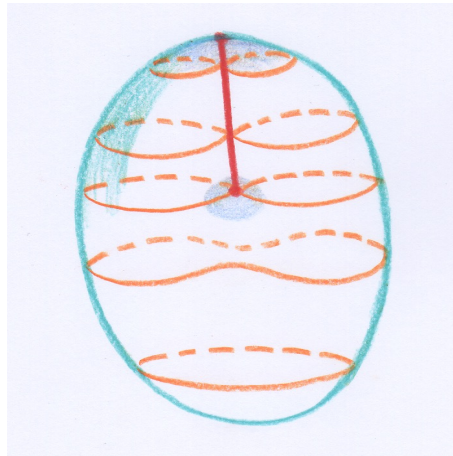
Figure 2.4. A cross cap

normalization separates the local irreducible components and removes the singular locus of codimension 1.

Example 2.24. This example is treated with more details in [90, Example 4.4.7 (5)]. Consider the surface $X = Z(f)$ in $\mathbb{C}^3$, where $f(z_1, z_2, z_3) = z_2^2 - z_1 z_3^2$. Its set of points with real coordinates is called **Whitney's umbrella** (see Figure 2.3). This denomination refers to Whitney's papers [203] and [204], in whose sections 4 and 3 respectively its portion of pure dimension 2 was presented as a model for the two singularities of a "*cross cap*" (one of the standard models of the real projective plane in $\mathbb{R}^3$, see Figure 2.4) at the ends of its segment of self-intersection. Whitney proved in those papers that such singularities appear generically on images of smooth maps from surfaces to 3-manifolds. He also proved an extension of this property for generic maps from n-dimensional manifolds to $(2n-1)$-dimensional ones, for arbitrary $n \geq 2$. Let us come back to $X = Z(f) \subset \mathbb{C}^3$. It may be seen as the image of the map $\nu : \mathbb{C}^2 \to \mathbb{C}^3$ defined by $(s, t) \to (z_1 = s^2, z_2 = st, z_3 = t)$. One may verify easily using this formula that ν is a normalization of X, according to Definition 2.22. It is also immediate to check that the fibre above a point of the z_1-axis which is distinct from the origin consists of two points. This corresponds to the fact that at such a point two sheets of X intersect transversally. By contrast, the fibre above the origin consists of a single point, the origin of $\mathbb{C}^2$, which corresponds to the fact that $(X, 0)$ is an irreducible singularity.

Complete intersections are not only Cohen-Macaulay singularities, they are moreover *Gorenstein*, in the following sense:

Definition 2.25. A Cohen-Macaulay singularity (X, x) is **Gorenstein** if its dualising module $\omega_{X,x}$ is free as an $\mathcal{O}_{X,x}$-module.

The **dualising module** $\omega_{X,x}$ is the germ at x of the **dualising sheaf** ω_X, which is well-defined on any Cohen-Macaulay set (that is, a complex analytic set all of whose germs are Cohen-Macaulay). In restriction to the smooth locus $X \setminus \mathrm{Sing}X$, the dualising sheaf ω_X is simply the sheaf of holomorphic differential forms of maximal degree. It is more complicated to understand what it means along the singular locus, and we won't enter into details in this text, as we don't need this generality. We will be only interested by the case in which X is not only Cohen-Macaulay but moreover *normal*. Then the situation is simpler:

Proposition 2.26. *Suppose that X is a* Cohen-Macaulay *and a* normal *complex analytic set. Then*

$$\omega_X \simeq i_* \omega_{X \setminus \mathrm{Sing}X}$$

where $X \setminus \mathrm{Sing}X \overset{i}{\hookrightarrow} X$ denotes the inclusion morphism. In particular, (X, x) is Gorenstein if and only if there exists a nowhere-vanishing holomorphic form of maximal degree defined on the smooth locus of some neighborhood of x.

Example 2.27. Let us explain why hypersurface singularities are Gorenstein, in the particular case of isolated surface singularities. Consider such a singularity $(X, 0) = (Z(f), 0) \subset (\mathbb{C}^3, 0)$, where $f \in \mathbb{C}\{z_1, z_2, z_3\}$. In restriction to X, one has the equality:

$$df = 0 \iff \frac{\partial f}{\partial z_1} dz_1 + \frac{\partial f}{\partial z_2} dz_2 + \frac{\partial f}{\partial z_3} dz_3 = 0$$

Taking successively its exterior product with dz_1, dz_2, dz_3, one gets the following equality in restriction to X:

$$(2.1) \qquad \frac{dz_1 \wedge dz_2}{\dfrac{\partial f}{\partial z_3}} = \frac{dz_2 \wedge dz_3}{\dfrac{\partial f}{\partial z_1}} = \frac{dz_3 \wedge dz_1}{\dfrac{\partial f}{\partial z_2}}.$$

By the *holomorphic version of the implicit function theorem* (see [90, Theorem 3.3.1]), the first expression is holomorphic and non-vanishing over the open subset $X \setminus Z\left(\dfrac{\partial f}{\partial z_3}\right)$ of X. One has the analogous properties for the second and third expressions. Therefore, the rational 2-form defined by any one of the three expressions in equality (2.1) is non-zero and holomorphic outside the origin, by our assumption that $(X, 0)$ has an isolated singularity and by Proposition 2.10. Being a complete intersection, $(X, 0)$ is automatically Cohen-Macaulay. Being an isolated singularity, we know by Theorem 2.20 (1) that $(X, 0)$ is normal. We conclude then that $(X, 0)$ is Gorenstein using Proposition 2.26.

Both Cohen-Macaulay and Gorenstein singularities behave well under hyperplane sections using non-zero divisors (see Bruns and Herzog [24, Proposition 3.1.19] or Ishii [86, Proposition 5.3.12]):

Proposition 2.28. *Let (X, x) be a singularity and $f \in \mathcal{O}_{X,x}$ a non-zero divisor. Assume that the complex analytic subgerm $(Z(f), x) \hookrightarrow (X, x)$ defined by f is reduced. Then:*

- *(X, x) is Cohen-Macaulay if and only if $(Z(f), x)$ is Cohen-Macaulay;*

- *(X, x) is Gorenstein if and only if $(Z(f), x)$ is Gorenstein.*

Before ending this section, let us introduce another class of singularities of arbitrary dimension:

Definition 2.29. A **quotient singularity** is a singularity analytically isomorphic to a germ obtained as a quotient of a smooth germ by a finite group of holomorphic automorphisms.

Quotient singularities are normal, as proved by Cartan [27]. By a local linearization theorem, one may show that in all dimensions quotient singularities are isomorphic to germs of the form $\mathbb{C}^n/G$, where G is a finite subgroup of $GL(n, \mathbb{C})$.

Say that an element of the general linear group $GL(n, \mathbb{C})$ is a **complex reflection** if it fixes point-wise a hyperplane. By a theorem of Chevalley [31], the quotient of $\mathbb{C}^n$ by a finite group generated by complex reflections is again isomorphic to $\mathbb{C}^n$. Now, if $G \subset GL(n, \mathbb{C})$ is an arbitrary finite group, its subgroup G_c generated by complex reflections is a normal subgroup, therefore one may construct the quotient $\mathbb{C}^n/G$ as a two-step quotient $(\mathbb{C}^n/G_c)/(G/G_c)$. One can show that the induced linear action of G/G_c on $\mathbb{C}^n/G_c \simeq \mathbb{C}^n$ contains no non-trivial complex reflections, that is, it is a so-called *small* linear group:

Definition 2.30. A finite subgroup $G \subset GL(n, \mathbb{C})$ is called **small** if it contains no complex reflection.

We see that any quotient singularity is obtainable as the germ at 0 of the quotient of $\mathbb{C}^n$ by a *small* finite linear group. Moreover, Prill [167] proved that the corresponding linear representation is encoded in the analytical structure of the corresponding quotient singularity.

Example 2.31. The simplest quotient singularity is obtained as the quotient of $\mathbb{C}^2$ by the antipodal involution $\sigma : (t_1, t_2) \to (-t_1, -t_2)$ (more precisely, by the linear group of order 2 generated by this involution). Note that σ is not a complex reflection, therefore this linear group is small. One may compute the quotient by finding first its algebra of regular (polynomial) functions: it is the subalgebra of $\mathbb{C}[t_1, t_2]$ left invariant by the involution σ. An easy computation shows that this subalgebra is generated by $z_1 = t_1^2, z_2 = t_2^2, z_3 = t_1 t_2$. Therefore, it is isomorphic to $\mathbb{C}[z_1, z_2, z_3]/(z_1 z_2 - z_3^2)$. This shows that the quotient of $\mathbb{C}^2$ by the antipodal involution σ is isomorphic to the hypersurface singularity of $\mathbb{C}^3$ defined by the polynomial $z_1 z_2 - z_3^2$. Note that Proposition 2.10 allows to show immediately that this singularity is isolated.

For details about the algebraic aspects of Cohen-Macaulay singularities, one may consult Bruns and Herzog's treatise [24]**. Information about the topology of *isolated* such singularities may be found in Kollár** [95]**.**

For details about the notions of dualising module and Gorenstein singularities, one may consult Bruns and Herzog [24]**, Peternell and Remmert** [156]**, Eisenbud** [44]**, Reid** [168]**.**

2.3. Modifications, blow-ups and resolutions of singularities

One of the main ways to study singular complex analytic sets is to see them as images of smooth ones, that is, as images of complex manifolds. A priori one could look for such manifolds of arbitrary higher dimensions, but one usually restricts them in the following way:

Definition 2.32. Let X be a complex analytic set. A **resolution of singularities** of X is a morphism $\pi : \tilde{X} \to X$ such that:

1. $\tilde{X}$ is smooth;

2. π is proper (that is, the preimage of a compact subset of X is compact);

3. the restriction $\pi : \tilde{X} \setminus \pi^{-1}(\text{Sing } X) \to X \setminus \text{Sing } X$ is an isomorphism;

4. π realizes a bijection between the irreducible components of $\tilde{X}$ and X.

The **exceptional locus** $\text{Exc}(\pi)$ of π is the subspace $\pi^{-1}(\text{Sing } X)$ of $\tilde{X}$.

Note that some writers call *exceptional* only the irreducible components E of $\pi^{-1}(\text{Sing } X)$ whose image by π have a strictly smaller dimension than E.

Informally speaking, in order to resolve the singularities of X, one replaces its singular locus by another complex analytic set, such that the resulting complex set becomes *smooth*. The properness condition (2) is included in order to guarantee that one does not simply remove the singular locus, or that one does not replace it by something too small.

In fact, the process is subtler, not being describable only in topological terms. Indeed, a resolution of singularities may be a homeomorphism such that the restriction $\pi : \pi^{-1}(\text{Sing } X) \rightarrow \text{Sing } X$ is moreover an isomorphism (see Example 2.33 below), in which case one replaces Sing X by itself. How to understand then what changes if one passes from X to $\tilde{X}$? Well, one *adds* new holomorphic functions on X, changing its "structure sheaf".

Example 2.33. Let X be a complex analytic *curve*, that is, a complex analytic set of pure dimension 1. In this case there exists a unique resolution of singularities of X, up to a unique isomorphism over X. This resolution $\pi : \tilde{X} \rightarrow X$ is simply the *normalization* of the curve X, introduced in Definition 2.22. It may be obtained by gluing holomorphic parametrisations of the irreducible components of the germs of X at all its points. If one asks moreover that these parametrisations are homeomorphisms of representatives, one gets simply the *Riemann surface* associated to the curve X. The resolution π is a homeomorphism if and only if the germ of X is irreducible at each one of its points. More generally, for any $x \in X$, the fibre $\pi^{-1}(x)$ is a finite set which is in bijection with the set of irreducible components of the singularity (X, x) (see Proposition 2.23).

Example 2.34. Let $X \hookrightarrow \mathbb{C}^n$ be an algebraic cone with vertex at the origin 0, that is, the zero locus of a set of homogeneous polynomials in n variables. Denote by $\mathbb{P}(X) \hookrightarrow \mathbb{P}(\mathbb{C}^n)$ its projectivisation, that is, the set of lines contained in X and passing through 0. It is simply the projective subvariety of $\mathbb{P}(\mathbb{C}^n)$ defined by the same set of homogeneous polynomials. Assume that X is smooth outside 0, which is equivalent to the smoothness of $\mathbb{P}(X)$. One has a natural complex line bundle on $\mathbb{P}(X)$, whose fibre above a point representing a line is the line itself (its sheaf of holomorphic section is denoted usually $\mathcal{O}_{\mathbb{P}(X)}(-1)$ in algebraic geometry). Denote by $\tilde{X}$ the total space of this line bundle, and by $\pi : \tilde{X} \rightarrow X$ the natural morphism which associates to each point of a fiber of $\tilde{X}$ the same point seen on the corresponding line inside X. This morphism is a resolution of singularities of X, with exceptional set $\mathbb{P}(X)$, identified with the zero-section of $\tilde{X}$.

One may construct the morphism π alternatively in the following way. Consider the rational map:

$$
\begin{array}{ccc}
X & \cdots\rightarrow & \mathbb{P}(X) \\
x & \cdots\rightarrow & [x]
\end{array}
$$

where $[x]$ denotes the point of $\mathbb{P}(X)$ corresponding to the generating line of the cone passing through x. This map is well-defined outside 0. Then $\tilde{X}$ is the closure of the graph of this map in $X \times \mathbb{P}(X)$ and $\pi : \tilde{X} \rightarrow X$ is the natural projection on the first factor.

We have represented schematically the morphism π in Figure 2.5. In our drawing the total space $\tilde{X}$ looks like a trivial line bundle over $\pi^{-1}(0)$. In fact, over $\mathbb{C}$ this is never the case. Indeed, this line bundle is the dual of an *ample* line bundle, whose holomorphic sections vanish precisely along the hyperplane sections of $\mathbb{P}(X) \hookrightarrow \mathbb{P}(\mathbb{C}^n)$. By contrast, as the complex manifold $\mathbb{P}(X)$ is compact, the maximum modulus principle implies that the trivial line bundle $\mathbb{P}(X) \times \mathbb{C}$ has only constant sections.

Definition 2.35. Let $X \hookrightarrow \mathbb{C}^n$ be an algebraic cone with vertex at the origin 0. Assume that X is smooth outside 0. The morphism $\pi : \tilde{X} \rightarrow X$ constructed in Example 2.34 is called the **blow-up** of the point 0 in X.

The previous example shows that by blowing up the vertex of a cone with isolated singularity, one gets a resolution of singularities in the sense of Definition 2.32.

Let us comment a little more Definition 2.32. Condition (3) insures that one replaces only the singular locus, not a bigger complex subset of X. Finally, condition (4) insures that one does not include in $\tilde{X}$ some connected component which is a manifold projecting properly inside Sing X.
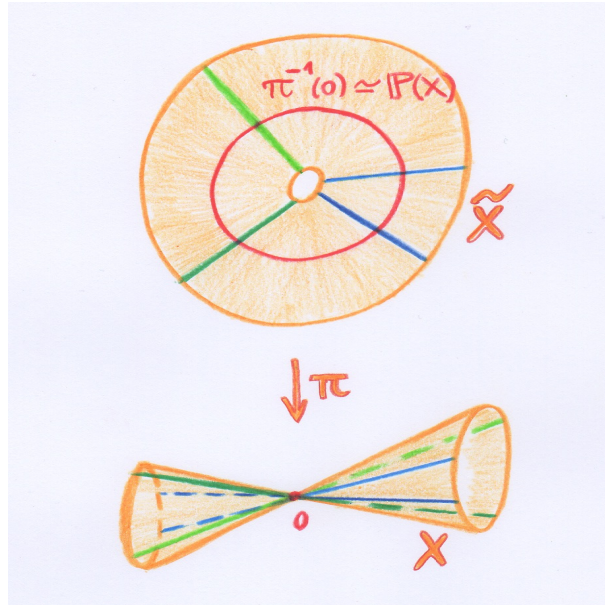
Figure 2.5. The blow-up of the vertex of a cone

Some writers do not impose condition (3) in the definition of a resolution of singularities. Its presence has the advantage that the boundary of a tubular neighborhood of the singular set may be canonically identified up to an isotopy to the boundary of a tubular neighborhood of the exceptional set. This is crucial if one is interested in the topological study of X in the neighborhood of Sing X, for instance in the case when X has an isolated singularity, which is the most important one in this article.

In general, one may hope to reach a resolution of singularities by composing special types of *modifications*:

Definition 2.36. Let X be a complex analytic set. A **modification** of it is a morphism $\pi : \tilde{X} \to X$ of complex sets such that:

1. π is proper;

2. π realizes a bijection between the irreducible components of $\tilde{X}$ and X;

3. there exists a closed complex analytic subset $I \hookrightarrow X$ which does not contain any irreducible component of X, such that the restriction $\pi : \tilde{X} \setminus \pi^{-1}(I) \to X \setminus I$ is an isomorphism.

The **indeterminacy locus** $\mathrm{Ind}(\pi)$ of the modification π is the minimal subspace I of X which has the property (3). The **exceptional locus** $\mathrm{Exc}(\pi)$ of π is the preimage $\pi^{-1}(\mathrm{Ind}(\pi))$ of the indeterminacy locus.

When π is a resolution of X in the sense of Definition 2.32, then its indeterminacy locus is the singular locus of X, which implies that its exceptional loci according to both definitions coincide. The following example presents a modification which is not a resolution of singularities:

Example 2.37. Consider again the situation of a cone, as in Example 2.34, but this time without the hypothesis that X is smooth outside 0. One may apply the same construction as in that example, getting a modification $\pi : \tilde{X} \to X$ which is still called the *blow-up of* 0 *in* X. Its indeterminacy locus is still the point 0. Therefore, if X has singularities outside 0, then π is not a resolution of singularities of X.

Given a modification of a complex analytic set X, it is important to look at the induced modifications on the complex subsets of X whose irreducible components are *not* contained in the indeterminacy locus:

Definition 2.38. Let X be a complex analytic set and let $\pi : \tilde{X} \to X$ be a modification of it. Consider a closed analytic subset $Y \hookrightarrow X$ without irreducible components contained in the indeterminacy locus $\mathrm{Ind}(\pi)$. Its **total transform** by the modification π is the full preimage $\pi^{-1}(Y)$. Its **strict transform** $\pi_s^{-1}(Y)$ by the modification π is the closure inside $\tilde{X}$ of the preimage of the part $\pi^{-1}(Y \setminus \mathrm{Ind}(\pi))$ of Y which is not contained in the indeterminacy locus.

The reason of this terminology is that the restriction $\pi : \pi_s^{-1}(Y) \to Y$ of π to the strict transform of Y is again a modification, which is not in general the case of the restriction $\pi : \pi^{-1}(Y) \to Y$ to the total transform of Y.

Example 2.39. Consider the blow-up $\pi : \tilde{X} \to X$ of a cone X with isolated singularity at 0, as in Example 2.34, illustrated on Figure 2.5. We assume that X is of dimension at least 2, which implies that $\mathbb{P}(X) \simeq \pi^{-1}(0)$ is of dimension at least 1. Consider one of the generating lines $Y \hookrightarrow X$ of the cone X. Then the strict transform of Y is the corresponding line seen as a fiber of the line bundle $\tilde{X} \to \mathbb{P}(X)$ and the restriction $\pi : \pi_s^{-1}(Y) \to Y$ is an isomorphism. But the total transform $\pi^{-1}(Y)$ has two irreducible components $\pi_s^{-1}(Y)$ and $\pi^{-1}(0)$. The restriction $\pi : \pi^{-1}(Y) \to Y$ is therefore not a modification, as condition (2) of Definition 2.36 is not satisfied.

In his 1964 paper [78], Hironaka proved the following fundamental theorem, which had been proved before only for varieties of complex dimension at most 3:

Theorem 2.40. *All complex algebraic varieties admit resolutions of singularities, obtainable moreover by sequences of blow-ups of* smooth *subvarieties.*

Let us explain the notion of *blow up* of a closed submanifold S of another complex manifold M. In the case when $M = \mathbb{C}^n$ seen as a cone with vertex at the origin and $S = 0$, the definition we are going to give specializes to Definition 2.35.

Informally speaking, the blow up of S in M replaces S by its *projectivised normal bundle* $\mathbb{P}(N_M S)$. More precisely, one has the following definition, which is to be contrasted with Definition 2.32:

Definition 2.41. Let S be a closed complex submanifold of the complex manifold M. The **blow up** of S in M is a morphism $\beta_S : \mathcal{B}_S M \to M$ of complex manifolds such that:

1. $\mathcal{B}_S M$ is smooth;

2. β_S is proper;

3. $\beta_S^{-1}(S)$ is a codimension 1 submanifold of $\mathcal{B}_S M$;

4. the restriction $\beta_S : \mathcal{B}_S M \setminus \beta_S^{-1}(S) \to M \setminus S$ is an isomorphism.

One may show that the blow-up of S in M exists and is unique up to a unique isomorphism above M. Moreover, one may show that the map which associates to each smooth germ of curve $(C, s) \hookrightarrow (M, s)$ transversal to S at a point $s \in S$, the intersection of its strict transform with $\beta_S^{-1}(S)$, identifies $\beta_S^{-1}(S)$ with the projectivized normal bundle $\mathbb{P}(N_M S)$.

More generally, one may blow up any complex analytic subspace of a complex analytic set (see Fischer [55, Chapter 4] or Peternell [155, Section 2]). We won't explain this generalization, as we don't need it in the sequel.

Hironaka's proof of Theorem 2.40 extends readily to complex analytic *germs*. Much more efforts were needed to extend it to complex analytic *sets*, but he achieved this with the help of Aroca and Vicente in the volumes [79], [3], [4].

When (X, x) is a curve singularity, the normalization morphism resolves it. This is no longer true in higher dimensions, but as explained after Proposition 2.23, normalization destroys nevertheless the
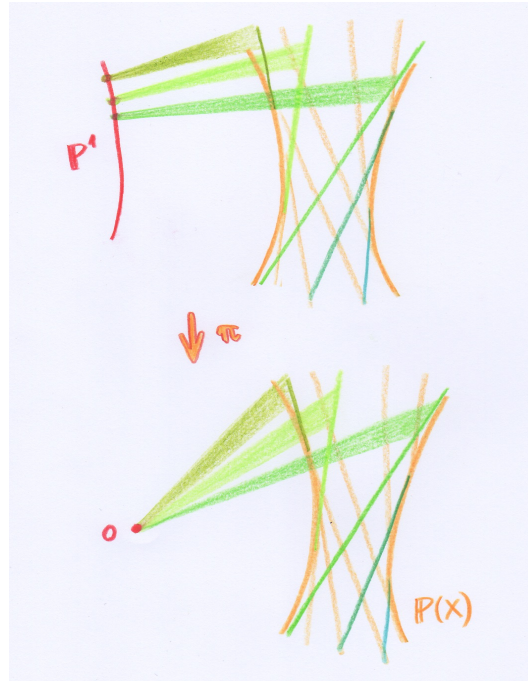
Figure 2.6. A small resolution of the cone over a smooth quadric surface

singular locus in codimension 1. It may be shown that when the analytic set X is normal, then the exceptional locus of any resolution of singularities of it has everywhere dimension ≥ 1. Therefore, when moreover $\dim X = 2$, this exceptional locus is a curve, that is, a divisor, in the total space $\tilde{X}$ of the resolution.

In general, for germs of arbitrary dimension, we say that a resolution is **divisorial** if its exceptional locus is a *divisor* in the total space, that is, if it has pure codimension 1 in it. Starting from dimension 3, there exist singularities admitting non-divisorial resolutions, and even resolutions with exceptional sets having everywhere codimension ≥ 2 (called **small resolutions**).

Example 2.42. The simplest example of a normal singularity which admits a small resolution is given by the hypersurface singularity at the origin 0 of the cone X over a smooth quadric in $\mathbb{P}^3$. For instance, one may take the cone in $\mathbb{C}^4$ defined by the equation $z_1 z_2 - z_3 z_4 = 0$. A smooth quadric is doubly ruled, that is, it is covered by two families of lines, each one being parametrized by $\mathbb{P}^1$. This shows that it is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Select one of the rulings, say, by the fibers of the first projection $p_1 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. One may consider the rational map:

$$
\begin{aligned}
X &\dashrightarrow \mathbb{P}^1 \\
x &\dashrightarrow p_1[x]
\end{aligned}
$$

where $[x]$ denotes the point of the quadric $\mathbb{P}(X)$ corresponding to the generating line of the cone passing through x. This map is well-defined outside 0. Denote by $\tilde{X}$ the closure of its graph in $X \times \mathbb{P}^1$ and by $\pi : \tilde{X} \to X$ the natural projection onto the first factor. One may show that this morphism is a resolution of singularities of X, with exceptional locus isomorphic with $\mathbb{P}^1$ (see Figure 2.6).

This construction may be seen as an analog of the blow-up construction of Example 2.34. Indeed, there one looked at the collection of generating lines of the cone X and $\tilde{X}$ was the total space of the associated line bundle. Here one looks at one collection of planes contained in the cone X and one looks at the total space $\tilde{X}$ of the associated plane bundle over $\mathbb{P}^1$.

Note that one may permute the roles of the two factors in the product decomposition $\mathbb{P}(X) \simeq \mathbb{P}^1 \times \mathbb{P}^1$, getting a second small resolution. It may be shown that any other resolution of X factors through at least one of them, but that none of these two small resolutions factors through the

other one. Therefore, the singularity $(X, 0)$ has no *minimal* resolution (in the sense that any other resolution factors through it).

For more details about modifications, one may consult Peternell [155]**. For more details about blow-ups of submanifolds of complex manifolds, one may consult Griffiths and Harris** [69, Page 603]**. For a gentle introduction to the use of blow-ups in resolution of singularities, one may consult Hauser** [75]**. The reader interested in proofs of Theorem 2.40 may consult Cutkosky's book** [35] **or Kollár's book** [94]**. For introductions to various techniques of resolution of singularities of surfaces, one may consult Lipman** [110]**, Faber and Hauser** [52] **and Popescu-Pampu** [164]**.**

3. Surface singularities

3.1. Divisors on smooth complex surfaces and their intersection numbers

We will study normal surface singularities (X, x) through their resolutions (see Definition 2.32). If the base point x is a singular point of X, then it gets replaced by a *divisor* on the total space of the resolution. For this reason, we start by explaining in this subsection the needed facts about divisors on smooth complex surfaces.

Definition 3.1. A **divisor** on a smooth complex surface S is an element $D := \sum_{i \in I} a_i D_i$ of the free abelian group generated by the closed irreducible curves of S. If the previous sum is irredundant (that is, if the irreducible curves D_i are pairwise distinct), then the coefficient $a_i \in \mathbb{Z}$ of the irreducible curve D_i in D is called the **multiplicity of** D_i **in** D. The **support** $|D|$ of D is the union of the irreducible curves of non-zero multiplicity in D. The divisor D is called **reduced** if all the closed irreducible curves appear in it with multiplicity 0 or 1. It is called **effective** if all the multiplicities a_i are non-negative.

Each meromorphic section of a line bundle on S defines a divisor, provided that it does not vanish or has poles on an infinite number of irreducible curves of S:

Definition 3.2. Let L be a holomorphic line bundle on the smooth complex surface S and s a meromorphic section of it. Consider a closed irreducible curve C on S. Denote by f_C a holomorphic function which defines C on S in a neighborhood of p (in particular, it vanishes at order 1 along it). The **order of vanishing $\mathrm{ord}_C(s)$ of** s **along** C is the unique integer a such that for any $p \in C$, the meromorphic section $f_C^{-a} s$ of L is in fact holomorphic and non-zero along C in a pointed neighborhood $U \setminus \{p\}$ of p in S. If $a > 0$ then C **is a zero of order** a **of** s and if $a < 0$ then C **is a pole of order** $-a$ **of** s.

The **divisor (s) of** s is the sum

$$(s) := \sum_C \mathrm{ord}_C(s) C,$$

taken over all irreducible closed curves on S. It is well defined only when there is a finite number of such curves with $\mathrm{ord}_C(s) \neq 0$.

In particular, one has the following special types of divisors:

- **principal divisors**, which are the divisors of the form (f), when f is a meromorphic function on S, that is, a meromorphic section of the trivial holomorphic line bundle $S \times \mathbb{C}$;

- **canonical divisors**, which are the divisors of the form (ω), when ω is a meromorphic 2-form, that is, a meromorphic section of the second exterior power $\Lambda^2 T^* S$ of the cotangent bundle $T^* S$ of S.

The function $\sin(x)$ defined on the complex plane with coordinates (x, y) is an example of non-zero holomorphic function which vanishes over an infinite number of irreducible curves, and which has therefore no divisor according to the previous definition. One could enlarge the definition by allowing also infinite sums, but this level of generality will not be needed in the sequel.

If $p \in S$ is a point belonging to the support $|D|$ of a divisor D, a **defining function** f_D **of** D **in the neighborhood of** p is a meromorphic function defined in a neighborhood of p such that $D = (f_D)$ in this neighborhood. Such a function necessarily exists whenever the neighborhood is sufficiently small, but it is not unique. Nevertheless, two such functions differ multiplicatively by a holomorphic function which vanishes nowhere in a neighborhood of p.

By contrast, a divisor D may not allow a *global* defining function, that is, there are smooth complex surfaces which admit non-principal divisors. This is the case for instance whenever D is a non-trivial *effective* divisor on a *connected and compact* complex surface. Indeed, in such a case any global holomorphic function is constant, by the *maximum modulus principle*, which shows that no such function defines globally the divisor D.

The fact that a divisor is not principal indicates only that it is not the divisor of a section of the *trivial* line bundle. In fact, each divisor D may be realized as the divisor of a meromorphic section of a suitable line bundle $L_S(D)$, which is moreover canonically attached to it (see Hartshorne [73, Section II.6] or Ishii [86, Section 5.2]).

If A and B are two divisors on a smooth surface S, one may define their **intersection number** $A \cdot B \in \mathbb{Z}$ in the following cases:

- when the intersection of the supports $|A|$, $|B|$ is finite;

- when at least one of the supports $|A|$, $|B|$ is compact.

Let us consider successively the two cases.

Definition 3.3. Assume that A and B are two *effective divisors* on the smooth complex surface S, whose supports intersect in a finite set. If $p \in |A| \cap |B|$ and f_A, f_B are defining holomorphic functions of A and B in a neighborhood of p, then the **intersection number** $(A \cdot B)_p$ **of** A **and** B **at** p is the dimension

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_{S,p}}{(f_A, f_B)},$$

where (f_A, f_B) denotes the ideal generated by f_A and f_B in the local ring $\mathcal{O}_{S,p}$ of holomorphic functions on S at p. The **(global) intersection number** $A \cdot B$ **of the divisors** A **and** B is the sum of local intersection numbers at all points of $|A| \cap |B|$.

When A is not necessarily irreducible but still compact and A, B are not necessarily effective but $|A| \cap |B|$ is still finite, then the intersection number $A \cdot B$ is defined by bilinearity, writing both A and B as differences of effective divisors.

The same strategy, of defining first the intersection number for *effective* divisors and extending it afterwards to arbitrary divisors, works also when the supports of the divisors share irreducible components, *provided that these common components are compact*:

Definition 3.4. Assume that A is a compact irreducible curve on the smooth complex surface S and that B is an arbitrary divisor on S. Then the **intersection number** $A \cdot B$ is equal to the degree of the pull-back $\nu^* L_S(B)$ of the line bundle associated to B by the normalization morphism $\nu : \tilde{A} \to A$.

When A and B are not necessarily effective but A is still assumed compact, their intersection number $A \cdot B$ is defined by bilinearity, writing both A and B as differences of effective divisors.

One may show that definitions 3.3 and 3.4 are compatible. Namely, when A is compact, then both of them lead to the same notion of intersection number.

The intersection product is by construction bilinear and satisfies the following important property:

Proposition 3.5. *If D is a divisor with compact support and f is a meromorphic function on the complex surface S, then $D \cdot (f) = 0$.*

If D is a *compact* effective divisor on the smooth complex surface S, then it may be interpreted as a (non-necessarily reduced) compact curve, with associated structure sheaf $\mathcal{O}_D$ defined as the quotient of the structure sheaf $\mathcal{O}_S$ of S by the sheaf $\mathcal{O}_S(-D)$ of ideals of holomorphic functions vanishing along D. The local rings of the structure sheaf $\mathcal{O}_D$ are not reduced along the irreducible components of D with multiplicity different from 0 and 1, which explains the denomination "reduced divisor" introduced in Definition 3.1. More generally, for any compact but not necessarily reduced algebraic curve, one has an associated notion of genus, which generalizes the classical Riemannian genus of a compact Riemann surface:

Definition 3.6. The **arithmetic genus** $p_a(D)$ of the compact and not necessarily reduced curve D is by definition equal to $1 - \chi(\mathcal{O}_D)$.

When the curve D is situated, as in our case, in a smooth complex surface, it is possible to compute its arithmetic genus only by computing intersection numbers inside the ambient surface:

Theorem 3.7. *(**The adjunction formula**) Assume that D is a compact effective divisor contained in the smooth complex surface S. Then*

$$p_a(D) := 1 + \frac{1}{2}(D^2 + K_S \cdot D)$$

where K_S is any canonical divisor on S (see Definition 3.2).

Assume now that C is an irreducible compact complex curve, possibly with singularities. We do not assume any more that C is contained in a smooth surface. Denote by $g(C)$ the *arithmetic genus of its normalization*. By a theorem of Riemann, $g(C)$ is equal to the topological genus of the underlying Riemann surface (that is, to one half of its first Betti number). In order to understand the relation between $p_a(C)$ and $g(C)$, let us introduce a measure of the complexity of a curve singularity, different from its multiplicity:

Definition 3.8. Let (C, p) be a curve singularity. Denote by $\nu : (\overline{C}, \overline{p}) \to (C, p)$ the normalization of C (therefore, $(\overline{C}, \overline{p})$ may be a multi-germ). The **delta-invariant** $\delta(C, p)$ of (C, p) is defined by:

$$\delta(C, p) := \dim_{\mathbb{C}}(\mathcal{O}_{\overline{C}, \overline{p}}/\mathcal{O}_{C, p}).$$

Let us come back to an irreducible compact complex curve C. The two genera $p_a(C)$ and $g(C)$ associated to it are related in the following way:

Proposition 3.9. *Let C be an irreducible compact complex curve. Both genera $p_a(C)$ and $g(C)$ are related by the following formula:*

$$p_a(C) = g(C) + \sum_{p \in C} \delta(C, p).$$

In particular, $p_a(C) \geq g(C)$, with equality if and only if the curve C is smooth.

One may find more details on intersection numbers of divisors on smooth complex surfaces in Barth, Hulek, Peters and Van de Ven [8, Sections II.9-10] **and Ishii** [86, Section 5.4]**.**

For more details on *arithmetic genera*, the *adjunction formula* and the *anti-canonical cycle*, we refer to Reid [168, Section 3.6]**, Barth, Hulek, Peters & Van de Ven** [8, Section II.11] **and Ishii** [86, Proposition 5.3.11]**.**

3.2. Objects associated to a resolution of surface singularity

Assume that (X, x) is a *normal* surface singularity. Usually one studies it using its resolutions. As in any dimension which is at least equal to 2, those resolutions are not unique. But one has instead (see Laufer [99, Theorem 5.9]):

Proposition 3.10. *The normal surface singularity (X, x) has a unique minimal resolution, in the sense that any other resolution factors through it. It may be characterized by the fact that its exceptional divisor does not contain any irreducible smooth rational curve of self-intersection -1.*

In higher dimensions, one has no analogous theorem, as may be understood already by looking at the cone over a smooth quadric surface, as in Example 2.42.

Smooth rational curves of self-intersection -1 appear in Proposition 3.10 for the following reason:

Proposition 3.11. *Consider the blow up $\beta : S \to \mathbb{C}^2$ of the origin in the complex affine plane $\mathbb{C}^2$. Denote by $E := \beta^{-1}(0)$ its exceptional locus. Then E is a smooth rational curve of self-intersection -1.*

Let us prove this proposition. For the reason explained in Example 2.34, E may be canonically identified with the projectivisation of $\mathbb{C}^2$ seen as a cone with vertex at the origin. Therefore, E is a smooth rational curve. Consider the line $L := Z(x) \subset \mathbb{C}^2$ and its strict transform L' by the blow up β (see Definition 2.38). Consider also the lift $\beta^* x$ to S of the defining function x of L. Its divisor may be decomposed in the following way:

$$(\beta^* x) = aE + L',$$

where $a \in \mathbb{Z}_+^*$ (as the function $\beta^* x$ vanishes only along E and L' and it has no poles). Let us show that $a = 1$. Consider for this a second line $H \neq L$ passing through the origin of $\mathbb{C}^2$ and its strict transform H' on S. By the construction of the blow up explained in Example 2.34, the morphism β restricts to an isomorphism from H' to H. As x vanishes with multiplicity 1 at the origin of H, we deduce that $\beta^* x$ vanishes also with multiplicity 1 at the origin of H'. But this order of vanishing is equal to a, as H' is transversal to E, being a fiber of S seen as a line bundle over E (see again Example 2.34). This shows that one has indeed $a = 1$. Apply now Proposition 3.5 to the compact divisor E and to the principal divisor $(\beta^* x)$ on S. One gets:

$$E \cdot (E + L') = 0 \implies E^2 = -E \cdot L' = -1.$$

The proposition is proved.

One may use the model of the blow up of the origin of $\mathbb{C}^2$ in a local chart, whenever one blows up a smooth point of a complex surface. For this reason, any such blow up creates a smooth rational curve of self-intersection -1. Therefore, if one starts from a smooth surface and one composes a finite sequence of blow ups, one gets a surface which contains necessarily at least one such curve. In fact, one has the following converse (see Hartshorne [73, Proposition 5.3] and Cutkosky [34, Corollary 6.3] for proofs in the algebraic category, which may be easily adapted to the analytic category):

Proposition 3.12. *Assume that $\pi : S_1 \to S$ is a proper holomorphic modification between smooth complex surfaces. Then:*

1. *the exceptional locus of π contains at least one smooth rational curve D of self-intersection -1;*

2. *π may be factored as $\pi = \psi \circ \beta$, where $\psi : S_2 \to S$ is a proper holomorphic modification between smooth complex surfaces and $\beta : S_1 \to S_2$ is the blow up of a point $p \in S_2$, whose exceptional locus is D;*

3. *π is a composition of blow ups of points.*

Note that this proposition is specific to dimension 2: as explained in [73, Remark 5.4.4] and [34, Example 6.4], in higher dimensions there are proper modifications between manifolds which are not compositions of blow ups of smooth centers.

Note also that the exceptional locus of a modification obtained starting from a smooth complex surface and composing blow ups has *normal crossings*, in the following sense:

Definition 3.13. A divisor D is said to have **normal crossings** if its reduced germ at any point of it is either smooth or the union of two smooth germs of curves intersecting transversally.

Let us return to our normal surface singularity (X, x). Consider any resolution $\pi : (\tilde{X}, E) \to (X, x)$ of it, where E denotes the reduced fibre over x. Therefore E can be seen as a connected reduced effective divisor in $\tilde{X}$, called **the exceptional divisor** of π. The divisor E has not necessarily normal crossings. But by blowing-up recursively the points at which E has not a normal crossing, one obtains canonically starting from π a **strict normal crossings resolution**, that is, one whose exceptional divisor has normal crossings and smooth irreducible components. If one starts this process from the minimal resolution, one obtains **the canonical strict normal crossings resolution**. It may be shown that any other strict normal crossings resolution factors through it.

One associates to the resolution of (X, x) its *weighted dual graph* (see several examples in Figure 3.4 below):

Definition 3.14. Let $\pi : (\tilde{X}, E) \to (X, x)$ be a resolution of the normal surface singularity (X, x). Its **weighted dual graph** $\Gamma(\pi)$ is obtained as follows:

- its vertices i correspond bijectively to the irreducible components E_i of E;

- there is an edge with multiplicity $e_{ij} := E_i \cdot E_j \geq 0$ between the distinct vertices i and j (if $e_{ij} = 0$, then one considers that there is no edge between i and j);

- each vertex i is weighted by the self-intersection number $-e_i := E_i^2$ of the associated component E_i, inside the smooth surface $\tilde{X}$;

- each vertex i is also weighted by the arithmetic genus p_i (see Definition 3.6) of the compact irreducible curve E_i.

A basic property of the graph $\Gamma(\pi)$, coming from the fact that (X, x) is normal, is that it is *connected*. Note that Proposition 3.9 implies that $p_i = 0$ if and only if E_i is a smooth rational curve.

Example 3.15. Let us assume that $X \hookrightarrow \mathbb{C}^n$ is a cone over a smooth algebraic curve $\mathbb{P}(X)$ of degree $d \geq 1$ in the projective space $\mathbb{P}^{n-1}$. Consider the blow up $\tilde{X}$ of 0 in X, as in Example 2.34. Then $\tilde{X}$ is the total space of a line bundle over $\mathbb{P}(X)$, which shows that the exceptional divisor of π is isomorphic to $\mathbb{P}(X)$. One may show by the same method as that used in the proof of Proposition 3.11 that the self-intersection number of this exceptional divisor in $\tilde{X}$ is $-d$. The associated dual graph has therefore only one vertex i, no edge, and $e_i = d$.

For simplicity, once a resolution π is fixed, we will denote the weighted dual graph by Γ.

Denote by $V(\Gamma)$ the set of vertices of Γ and by $\underline{e} \in \mathbb{Z}^{V(\Gamma)}$ the function which associates to each vertex i the integer e_i. To the weighted graph Γ is associated a canonical quadratic form on the real vector space $\mathbb{R}^{V(\Gamma)}$, called **the intersection form** of the resolution π:

$$Q(\underline{x}) := - \sum_{i \in V(\Gamma)} e_i x_i^2 + \sum_{\substack{(i,j) \in V(\Gamma)^2 \\ i \neq j}} e_{ij} x_i x_j.$$

The geometrical meaning of the intersection form is the following: if one associates to $\underline{x} \in \mathbb{R}^{V(\Gamma)}$ the divisor $\sum_{i \in V(\Gamma)} x_i E_i$ on the smooth surface $\tilde{X}$, then $Q(\underline{x})$ is its self-intersection number on $\tilde{X}$.

One has the following characterization of exceptional divisors of resolutions of normal surface singularities (see Laufer [99, Chapter 4]), whose first statement was proved by Du Val [195] and Mumford [130] and whose second statement was proved by Grauert [62]:

Theorem 3.16.

1. *The intersection form Q is negative definite. In particular, $e_i > 0$ for all $i \in V(\Gamma)$.*

2. *If the intersection form associated to a reduced and connected compact divisor E on a smooth surface is negative definite, then E can be contracted to a normal singular point of an analytic surface (that is, the germ of the surface along it is the total space of a resolution of singularities of a normal surface singularity).*

Consider now a germ $f \in m_{X,x}$, where $m_{X,x}$ denotes the maximal ideal of the local ring $\mathcal{O}_{X,x}$ of germs of holomorphic functions on the surface singularity (X, x). Lift f to the resolved surface $\tilde{X}$ and look at the associated principal divisor $(\pi^* f)$ of this lift. It may be uniquely decomposed as a sum:

$$(\pi^* f) = (\pi^* f)_e + (\pi^* f)_s$$

without common irreducible components, where $(\pi^* f)_e$ denotes its **exceptional part** (whose support is contained in E) and $(\pi^* f)_s$ denotes the **strict transform** on $\tilde{X}$ of the divisor (f). As the intersection number between $(\pi^* f)$ and each irreducible component E_i of E vanishes (see Proposition 3.5), one gets:

(3.1) $$(\pi^* f)_e \cdot E_i = -(\pi^* f)_s \cdot E_i \leq 0, \text{ for all } i \in I.$$

Therefore one is led to introduce the **Lipman semigroup** $\mathcal{L}(\pi)$ of π (the name makes reference to Lipman's work [109]), defined as:

$$\mathcal{L}(\pi) := \{D \in \sum_{i \in I} \mathbb{Z}E_i \mid D \cdot E_i \leq 0, \text{ for all } i \in I\}.$$

This set is a semigroup for the addition of divisors. On it we consider the partial order relation:

$$D_1 \geq D_2 \iff D_1 - D_2 \text{ is effective.}$$

A basic property of this semigroup is (see Zariski [206, Lemma 7.1]):

Proposition 3.17. *All the elements of the Lipman semigroup are effective divisors.*

As we have seen in the explanations leading to formula (3.1), the exceptional part of the divisor of the lift of any holomorphic function $f \in m_{X,x}$ belongs to the Lipman semigroup. The converse is not true in general, excepted for *rational* singularities (see Definition 3.32).

M. Artin showed in [6] that the set of non-zero divisors of the Lipman semigroup has a unique minimal element, which is essential in the study of the singularity. This motivates:

Definition 3.18. The minimal element of $\mathcal{L}(\pi) \setminus \{0\}$ is called the **fundamental cycle** Z_{num} of π.

Laufer [100, Proposition 4.1] showed that the fundamental cycle may be computed algorithmically:

Proposition 3.19. *Start from $Z_0 := E_{i_0}$, where i_0 is an arbitrary element of I. If Z_j is defined and there exists $i \in I$ such that $Z_j \cdot E_i > 0$, define $Z_{j+1} := Z_j + E_i$. Then this process stops after a finite number of steps and the last element in the sequence $Z_0, Z_1, \ldots$ is the fundamental cycle of π.*

We will also need to manipulate another cycle supported by the exceptional divisor E of the resolution π and defined, as the fundamental cycle, by intersection-theoretical properties: the *anticanonical cycle*.

Before explaining its definition, let us introduce supplementary notations. For each irreducible component E_i of E, denote by g_i the arithmetic genus of its normalization (recall from Definition 3.14 that we denote by p_i the arithmetic genus of E_i, and from Proposition 3.9 that $g_i = p_i$ if and only if E_i is smooth).

As the intersection form Q is negative definite (see Theorem 3.16), there exists a unique divisor *with rational coefficients* Z_K supported on E such that:

(3.2) $$Z_K \cdot E_i = -K_{\tilde{X}} \cdot E_i, \text{ for all } i \in V(\Gamma).$$

Indeed, by the adjunction formula (see Theorem 3.7), this translates into the following system of equations:

(3.3) $$Z_K \cdot E_i = 2 - 2p_i + E_i^2, \ \text{for all } i \in I,$$

which is a square system of affine equations with unknowns the coefficients of Z_K. The matrix of the associated homogeneous system (relative to any total order of the vertices of Γ) is a matrix of the intersection form Q. As Q is negative definite, this matrix is invertible.

The negative sign in the definition (3.2) of the cycle Z_K is motivated by the following consequence of propositions 3.9, 3.10, 3.17 and formula (3.3):

Proposition 3.20. *If the resolution π is minimal, then Z_K is an effective divisor.*

Here comes the announced definition:

Definition 3.21. The rational cycle Z_K defined by the equivalent systems (3.2) and (3.3) is called the **anti-canonical cycle** of E (or of the resolution π).

This name is motivated by the fact that whenever (X, x) is Gorenstein (see Definition 2.25), $-Z_K$ is a canonical divisor on $\tilde{X}$. Indeed, if the singularity (X, x) is Gorenstein, consider a non-vanishing holomorphic form defined in a pointed neighborhood of x. Therefore its lift to $\tilde{X}$ is meromorphic and its locus of zeros and poles is contained in E. This locus, considered with multiplicities, is by construction a *canonical divisor* on $\tilde{X}$ (see Definition 3.13). Therefore, it is exactly $-Z_K$, which shows that for Gorenstein singularities, Z_K *has integral coefficients*. This property being numeric (that is, depending only on intersection-theoretical properties) and common to all normal Gorenstein singularities, it motivates the introduction of the following notion, which plays a role in the statement of Theorem 3.39 below:

Definition 3.22. The normal surface singularity (X, x) is called **numerically Gorenstein** if Z_K is an integral divisor.

Not every numerically Gorenstein singularity is Gorenstein (see Laufer's theorem 3.44 below). But the author proved in [163] that any numerically Gorenstein normal surface singularity has the same topological type as a Gorenstein one. By contrast, one does not know how to characterize the topological types of hypersurface or complete intersection normal surface singularities. By the way, how is it possible to describe such a topological type? Next subsection is dedicated to this question.

A survey of the properties of normal surface singularities related to their intersection forms was written by Wall [201]**. More details on the various cycles attached to such a singularity and on their importance for classification questions may be found in Némethi's notes** [132] **and** [134]**.**

3.3. The topology of normal surface singularities

Two isolated singularities are called **topologically equivalent** if they have representatives which may be identified through a homeomorphism which sends one base point onto the other one and is orientation-preserving outside the base point (the orientations being iduced by the complex structures).

How to encode the topological equivalence class of an isolated complex singularity? The usual method is to start from some representative of it and to define suitable *tubular neighborhoods* of the base point, which are *cones* over a real smooth manifold, the *boundary* or *link* of the singularity. The topological type of the singularity is therefore captured by the topological structure of its boundary.

Let us be more precise. Consider an isolated singularity (X, x) of arbitrary dimension. By choosing an embedding of a representative of it in some complex affine space $(\mathbb{C}^n, 0)$, one may restrict to X
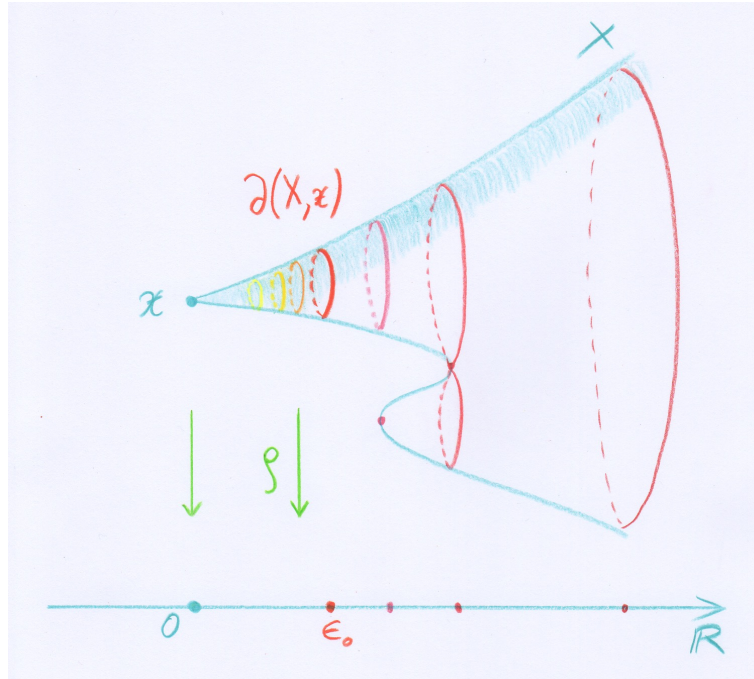
Figure 3.1. Passing from a rug function to a choice of boundary

the squared distance function to the origin

(3.4) $$\rho_0(z_1, ..., z_n) := |z_1|^2 + \cdots + |z_n|^2,$$

getting in this way a function:

$$\rho : (X, x) \to (\mathbb{R}, 0)$$

with the following properties:

- it is real-analytic;

- it is non-negative;

- $\rho^{-1}(0) = x$ for some choice of the representative X.

Consider more generally *any* function ρ with the previous properties, not necessarily coming from the squared-distance to the origin relative to some embedding. Following a denomination introduced by Thom [189] in a related context, we will call it a **rug functions** of (X, x) ("*fonction tapissante*" in French). One may show that a rug function is moreover proper and submersive in a pointed neighborhood of x in X. This implies that the sufficiently small positive levels $M_\epsilon := \rho^{-1}(\epsilon)$ of ρ are all smooth and pairwise diffeomorphic, which allows to define:

Definition 3.23. Let (X, x) be an isolated singularity and ρ be a rug function of (X, x). The **boundary** or **link** $\partial(X, x)$ of the singularity (X, x) is any positive level manifold M_{ϵ_0} of ρ such that all the levels M_ϵ are pairwise diffeomorphic manifolds for $\epsilon \in (0, \epsilon_0]$. One orients M_{ϵ_0} as the boundary of the complex manifold $\rho^{-1}(0, \epsilon_0)$.

The process leading to the construction of a representative level of the boundary of (X, x) is illustrated in Figure 3.1.

The boundary $\partial(X, x)$ is a closed oriented manifold, whose connected components are the boundaries of the irreducible components of (X, x). If (X, x) is of pure complex dimension n, then its boundary is of real dimension $2n - 1$. One may show that $\partial(X, x)$ is independent of the choice of

rug function ρ, up to orientation-preserving diffeomorphisms which are well-defined up to isotopy. Moreover, the previous construction allows to get the announced special conic representatives of the singularity:

Proposition 3.24. *Assume that the rug function ρ is proper on $\rho^{-1}[0, \epsilon_0]$ and that it is submersive on $\rho^{-1}(0, \epsilon_0]$. Then the pair $(\rho^{-1}[0, \epsilon_0], x)$ is homeomorphic to the cone over $\partial(X, x)$.*

Which oriented odd-dimensional manifolds appear as boundaries of isolated complex singularities? If such a singularity has several irreducible components, then its boundary is the disjoint union of the boundaries of its components. Therefore, let us restrict the previous question to the case of *irreducible* isolated singularities.

In complex dimension 1, the answer is simple: one gets only circles. In dimension 2 the answer is much more complicated, but it is also known (see Theorem 3.27 below). In higher dimensions the question is open. Nevertheless, one knows several constraints on the algebraic topological invariants of such boundaries (see Kollár [95]).

Let us restrict now to the study of boundaries of *isolated and irreducible surface singularities*. If (X, x) is such a singularity and $\nu : (\overline{X}, \overline{x}) \to (X, x)$ is its normalization, then the lift $\rho \circ \nu$ of any rug function of (X, x) is a rug function of $(\overline{X}, \overline{x})$. This allows to identify the boundaries of (X, x) and $(\overline{X}, \overline{x})$.

For this reason, *in the rest of this section we will assume that (X, x) is a normal surface singularity.*

The systematic study of the topological structure of boundaries of normal surface singularities started with Mumford's article [130], in which he proved that one could recognize whether a point on a normal surface was smooth only by looking at the topology of the boundary of the germ of surface at this point (see also Hirzebruch [81]):

Theorem 3.25. *If the boundary of a normal surface singularity (X, x) is simply connected, then x is a smooth point of X.*

In particular, as the boundary of a germ of surface at a smooth point is diffeomorphic to $\mathbb{S}^3$, this showed that one could not get a counterexample to Poincaré's conjecture by taking the boundary of a surface singularity.

In order to prove Theorem 3.25, Mumford described the boundary as the result of performing an operation which he called **plumbing** on suitable elementary 3-manifolds. The list of those elementary 3-manifolds and the instructions for plumbing them were determined by the weighted dual graph Γ (see Definition 3.14) of any *strict normal crossings* resolution of (X, x).

Example 3.26. Let us consider again the case of a cone $X \hookrightarrow \mathbb{C}^n$ over an irreducible smooth algebraic curve $\mathbb{P}(X)$ of degree $d \geq 1$, as in Example 3.15. Consider the blow-up $\pi : \check{X} \to X$ of 0 in X, whose exceptional divisor E is canonically identified with $\mathbb{P}(X)$. It is a strict normal crossing resolution of X. Then the tubular neighborhoods of 0 in X defined by a rug function lift to tubular neighborhoods of E in $\check{X}$. This allows to identify the boundary $\partial(X, 0)$ with the boundary of such a tubular neighborhood of E in $\check{X}$. As $\check{X}$ is the total space of a line bundle of degree $-d$ over $\mathbb{P}(X)$, its zero-section being E, we deduce that $\partial(X, 0)$ *is diffeomorphic to the total space of the circle bundle associated to this line bundle*. Such a circle bundle is determined up to fiber and orientation-preserving diffeomorphisms by its *Euler number*. When the circle bundle is associated, as is the case here, to a complex line bundle, then its Euler number is equal to the degree of this line bundle. Summarizing, we see that *the boundary of the singularity at the vertex of the cone over a smooth irreducible curve of degree d is a circle bundle with Euler number $-d$ over the underlying Riemann surface of the curve.* Note that, if one restricts to the case where $X = \mathbb{C}^2$, then the previous considerations lead to another proof of Proposition 3.11. The circle bundle structure which one gets on the boundary of a Euclidean ball centered at the origin of $\mathbb{C}^2$ is nothing else than the classical Hopf fibration on $\mathbb{S}^3 \simeq \partial(\mathbb{C}^2, 0)$.

As seen in Example 3.15, the dual graph of the previous resolution of the cone X has only one vertex i, with $e_i = d$ and g_i equal to the genus of the smooth projective curve $\mathbb{P}(X)$. In this case one
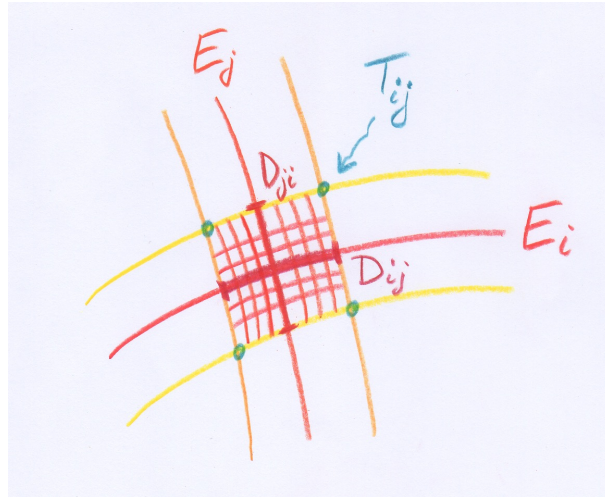
Figure 3.2. The local situation leading to a plumbing operation

has only one elementary manifold, which is the circle bundle of Euler number $-e_i$ over an oriented closed connected surface of genus g_i. Here one does not perform any plumbing operation.

For an arbitrary normal surface singularity (X, x), one needs to perform plumbing operations in order to reconstruct its boundary $\partial(X, x)$ only when one works with a strict normal crossing resolution of (X, x) whose exceptional divisor E has at least two irreducible components. But the elementary manifolds to be plumbed are still circle bundles over oriented surfaces, as in the previous example.

In order to understand this, let us think at another way of building a tubular neighborhood of E in $\tilde{X}$, without the help of a rug function:

- By choosing a Riemannian metric on $\tilde{X}$ in a neighborhood of E, one may use the associated exponential map in order to push the disc bundle structure on the normal bundle of each E_i into a disc bundle structure of a tubular neighborhood W_i of E_i. One has to choose discs of sufficiently small radii in the normal bundle.

- By requiring moreover that the metric be chosen such that, at each singular point p of E, the two components E_i and E_j meeting at p are orthogonal at p and totally geodesic in a neighborhood of p, one may ensure that E_i contains a fiber of the disc bundle of W_j in a neighborhood of p, and conversely after permuting i and j.

- Choose more carefully the tubular neighborhoods W_i and W_j such that in the neighborhood of p their intersection gets identified to a product $D_{ij} \times D_{ji}$ of discs by the two bundle projections (see Figure 3.2).

- Ensure also that the global intersection of W_i and W_j is the disjoint union of those local intersections with product structures.

- Take as tubular neighborhood W of E the union of the individual tubular neighborhoods W_i. If one imagines that those disc bundles W_i are abstract manifolds, then the operation of gluing which has to be performed in order to get W is called **plumbing**.

- The same name applies to the operation which allows to reconstruct ∂W from the circle bundles induced on the boundaries ∂W_i by the disc bundles on W_i. Locally near an intersection point p of E_i and E_j, one removes the solid torus $D_{ij} \times \partial D_{ji}$ from ∂W_i, symmetrically $D_{ji} \times \partial D_{ij}$ from ∂W_j and one identifies the resulting 2-dimensional tori, getting a torus T_{ij} inside ∂W (see again Figure 3.2).
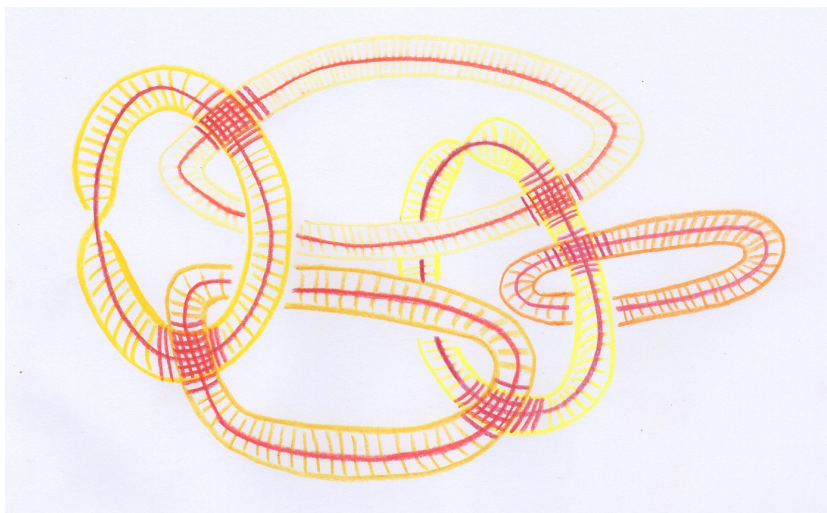
Figure 3.3. Plumbing annuli and Möbius bands

- In this way, the oriented 3-manifold ∂W appears decomposed into circle-bundles over compact surfaces with boundary using a finite set of pairwise disjoint tori. Along each such torus, the fibers of the two fibrations meeting along it on both sides have as intersection number ± 1.

The resulting manifold with corners W is illustrated in Figure 3.3. In order to suggest the complex 2-dimensional situation, we have drawn a real 2-dimensional picture, in which the disc-bundles over closed surfaces are replaced by segment-bundles over circles. In order to indicate that the disc-bundles are non-trivial (which is a consequence of the negative definiteness of the intersection form of the resolution, see Theorem 3.16), we have drawn both annuli and Möbius bands as such segment bundles.

It may be shown, starting from the study done by Mumford in [130], that this second way of constructing a tubular neighborhood of E leads to a 4-dimensional manifold with boundary and corners which is piecewise-diffeomorphic to the tubular neighborhoods constructed using lifts of rug functions. Using also Theorem 3.16, one arrives at:

Theorem 3.27. *The oriented boundaries of normal surface singularities are precisely the oriented* 3-*dimensional manifolds which may be obtained by plumbing circle bundles according to a negative-definite connected weighted graph.*

Therefore, the boundaries of normal surface singularities are particular 3-manifolds, obtained by plumbing circle bundles over surfaces following a weighted graph. For this reason, such 3-manifolds were named **graph-manifolds**. Their theory was started by Waldhausen [200]. He looked at the collection of tori which one gets in a graph manifold as images of the tori which were identified after taking out solid tori from the total spaces of the circle bundles. As explained above, on such a torus the intersection number of the fibers arriving from both sides has absolute value 1. Waldhausen considered then more general families of tori, by asking only that their complement be fiberable by circles, but forgetting the condition about intersection numbers. The class of 3-manifolds which admit such a *graph structure* is the same as before, but one has more possibilities of simplification: each time one finds two *parallel* tori, that is, disjoint 2-tori which cobound a thick torus $[0, 1] \times \mathbb{T}^2$, one can eliminate one of them, and obtain again a graph structure on the same 3-manifold. Waldhausen proved in [200] that, when the initial 3-manifold is **irreducible**, that is, indecomposable as a connected sum of two other 3-manifolds non-diffeomorphic to the 3-sphere, a minimal such collection of tori is in general a topological invariant of the 3-manifold:

Theorem 3.28. *With the exception of a finite explicit list of 3-manifolds, a minimal collection of tori which correspond to a graph structure on an irreducible closed 3-manifold is unique up to isotopy.*

Waldhausen described also a notation for graph structures and characterized using it the graph structures corresponding to the minimal collections of tori. His work was the starting point of a "calculus" elaborated by Neumann [142] for plumbing structures. Neumann applied his calculus to give an algorithm which allowed to determine if a given plumbing graph describes or not a singularity boundary. Using this algorithm, he showed:

Theorem 3.29. *The boundary of a normal surface singularity is irreducible. Its oriented topological type determines the weighted dual graph of the minimal strict normal crossings resolution up to isomorphism.*

Therefore, one may encode the oriented topological type of the singularity boundary by this graph. Moreover, one has an algorithmic way, given an oriented graph manifold, to determine if it is diffeomorphic to a singularity boundary or not.

Before Neumann's theorem 3.29, Sullivan [186] had given the first example of an irreducible graph manifold which was not a singularity boundary: the 3-dimensional torus. In fact even in higher dimensions, odd-dimensional tori cannot be boundaries of isolated singularities, as was proved first by Durfee and Hain [42] and rediscovered by myself (see [161, Corollary 5.3]).

Waldhausen's structure theorem for graph manifolds was extended later by Jaco & Shalen [87] and Johannson [88] into a structure theorem for any *irreducible* 3-manifolds. Namely, any such manifold contains a finite family of pairwise disjoint and non-parallel incompressible tori, minimal for the property that the components of their complement are either *Seifert-fiberable* or do not contain new incompressible tori (which are not boundary-parallel). Moreover, such a family is unique up to an isotopy. It is now called *a JSJ-family of tori*. This uniqueness theorem was the starting point of *Thurston's geometrization conjecture* about the structure of 3-manifolds.

In this section we have examined till now the way to understand the topological structure of the boundary $\partial(X, x)$ starting from a strict normal crossings resolution. Let us see now how such a resolution allows to compute the homology of the boundary.

The universal coefficients theorem shows that it is enough to compute the homology $H_*(\partial(X, x), \mathbb{Z})$ with integral coefficients. As the boundary is an oriented closed 3-manifold, the Poincaré duality theorem combined with the universal coefficients theorem expressing cohomology in terms of homology implies that:

$$H_2(\partial(X, x), \mathbb{Z}) \simeq H^1(\partial(X, x), \mathbb{Z}) \simeq (H_1(\partial(X, x), \mathbb{Z}))^*.$$

Here and in the sequel we use the notation M^* for the **dual** $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ of an abelian group M. Denote by $\mathrm{Tors}(M)$ the torsion subgroup of M.

The previous isomorphisms show that the whole homology of $\partial(X, x)$ is determined by the first homology group $H_1(\partial(X, x), \mathbb{Z})$. The following proposition, in which all homology groups are taken with integral coefficients, explains that this group is determined by the intersection form of a strict normal crossings resolution:

Proposition 3.30. *Let $\pi : (\tilde{X}, E) \to (X, x)$ be a strict normal crossings resolution of the normal surface singularity (X, x). Denote as before by $(E_i)_{i \in I}$ the irreducible components of E (identified with its support $|E|$) and by Γ the dual graph of E (seen simply as a topological space). Let Λ be the second homology group $H_2(E) \simeq \mathbb{Z}^I$ and $\Lambda^* \simeq \mathbb{Z}^I$ be its dual. If $\lambda : \Lambda \to \Lambda^*$ denotes the morphism of free abelian groups determined by the intersection form on the oriented 4-manifold $\tilde{X}$, then:*

1. *There exists a short exact sequence of abelian groups:*

$$0 \longrightarrow \bigoplus_{i \in I} H_1(E_i) \longrightarrow H_1(E) \longrightarrow H_1(\Gamma) \longrightarrow 0.$$

2. *There exists an exact sequence of abelian groups:*

$$\Lambda \overset{\lambda}{\longrightarrow} \Lambda^* \longrightarrow H_1(\partial(X, x)) \longrightarrow H_1(E) \longrightarrow 0.$$

3. $|\Lambda^*/\lambda(\Lambda)| = |\mathrm{Tors}(H_1(\partial(X,x)))|$.

4. $H_1(\partial(X,x))/\mathrm{Tors}(H_1(\partial(X,x))) \simeq H_1(E)$. *Therefore, the boundary $\partial(X,x)$ and the exceptional divisor E have the same first Betti number.*

Let us sketch a proof of this proposition, as it allows to understand better the relations between the boundary $\partial(X,x)$, the exceptional divisor E and its dual graph Γ.

We consider as before a conic representative X of the singularity and we identify the boundary $\partial(X,x)$ with the boundary $\partial\tilde{X}$ of the oriented smooth 4-dimensional manifold $\tilde{X}$.

One has the following two continuous maps between topological spaces:

$$\bigsqcup_{i\in I} E_i \xrightarrow{\ \nu\ } E \xrightarrow{\ \psi\ } \Gamma,$$

in which ν is the normalization morphism of E. The map ψ is not canonically defined, but only up to homotopy. In order to construct it, one chooses discs D_{ij} as in Figure 3.2 above and one foliates them by concentric circles centered at the chosen intersection point p of E_i and E_j. After doing this in the neighborhood of each singular point of E, one contracts to points the connected components of the complement inside E of the interiors of the various discs D_{ij}, as well as each circle of the various foliated discs. One gets in this way a graph isomorphic to the dual graph Γ. It is this quotient map $E \to \Gamma$ which we denote by ψ.

The morphisms of the exact sequence of point (1) of Proposition 3.30 are those induced by the maps ν and ψ.

Let us consider now the following neighborhood of $H_1(\partial\tilde{X})$ in the long exact homology sequence with integral coefficients of the pair $(\tilde{X}, \partial\tilde{X})$:

$$H_2(\tilde{X}) \longrightarrow H_2(\tilde{X}, \partial\tilde{X}) \longrightarrow H_1(\partial\tilde{X}) \longrightarrow H_1(\tilde{X}) \longrightarrow H_1(\tilde{X}, \partial\tilde{X}).$$

Using Poincaré-Lefschetz duality for the oriented compact manifold with boundary $\tilde{X}$, this exact sequence becomes:

(3.5) $$H_2(\tilde{X}) \longrightarrow H^2(\tilde{X}) \longrightarrow H_1(\partial\tilde{X}) \longrightarrow H_1(\tilde{X}) \longrightarrow H^3(\tilde{X}).$$

As $\tilde{X}$ retracts by deformation onto E, which has a structure of *CW*-complex of dimension 2, the previous exact sequence becomes:

(3.6) $$H_2(E) \longrightarrow H^2(E) \longrightarrow H_1(\partial\tilde{X}) \longrightarrow H_1(E) \longrightarrow 0.$$

As both abelian groups $\bigoplus_{u\in\mathcal{V}} H_1(E_u)$ and $H_1(\Gamma)$ are free, point (1) of the proposition implies that this is also the case for $H_1(E)$. As $\mathrm{Tors}(H^2(E)) \simeq \mathrm{Tors}(H_1(E))$ by the universal coefficents theorem, we get canonical isomorphisms:

$$H^2(E) \simeq H_2(E)^* \simeq \Lambda^*.$$

Therefore the exact sequence (3.6) becomes the exact sequence of point (2) of the proposition. Points (3) and (4) are then direct consequences of this exact sequence.

For more details on the construction of the boundary of a singularity, see Looijenga [114, Section 2.A]**, as well as Durfee** [40]**, which builds a general theory of tubular neighborhoods of compact real subvarieties in real algebraic geometry and Dutertre** [43]**, which extends it to possibly non-compact subvarieties.**

For more details about graph-manifolds and the JSJ-decompostion, one may consult Neumann [142]**, Neumann and Swarup** [143] **and Popescu-Pampu** [160]**.**

3.4. Rational and minimally elliptic surface singularities

Since Clebsch called *genus* the measure of complexity associated by Abel and Riemann to an algebraic curve (see [166, Chapter 20]), the term *genus* flourished as a measure of various complexities in algebraic and analytic geometry. This happened also in singularity theory. The following definition was introduced by Wagreich [197]:

Definition 3.31. Let (X, x) be a normal surface singularity. Its **geometric genus** is defined as:

$$p_g(X, x) := \dim_{\mathbb{C}} R^1 \pi_* \mathcal{O}_{\tilde{X}}$$

where $\pi : \tilde{X} \to X$ is any resolution of singularities. Its **arithmetic genus** is defined as:

$$p_a(X, x) := \sup_{Z \geq 0} p_a(Z)$$

where Z varies among the effective divisors supported by the exceptional divisor $\mathrm{Exc}(\pi)$ and $p_a(Z)$ denotes the arithmetic genus of Z, introduced in Definition 3.6.

In this definition, $R^1 \pi_*$ denotes the first right derived functor of the direct image functor π_*, sending sheaves of $\mathcal{O}_{\tilde{X}}$-modules into sheaves of $\mathcal{O}_X$-modules. Let us mention another viewpoint, for the sake of the reader who is not accustomed with this kind of yoga. Namely, one may show that if U is a conic representative of (X, x) chosen using a Euclidean squared distance function as explained at the beginning of Subsection 3.3 and if $\tilde{U}$ is its preimage by the chosen resolution, then:

$$p_g(X, x) = \dim_{\mathbb{C}} H^1(\tilde{U}, \mathcal{O}_{\tilde{U}}).$$

It is a theorem that both the geometric and the arithmetic genera introduced in Definition 3.31 are independent of the chosen resolution (see Wagreich [197, Section 1] or Behnke and Riemenschneider [11, Section 2.3]). One has always the inequalities (which were stated by Wagreich [197, Page 425]; for a proof, see Ishii [86, Theorem 7.2.14]):

$$p_g(X, x) \geq p_a(X, x) \geq p_a(Z_{num}) \geq 0.$$

Recall that the *fundamental cycle Z_{num}* was introduced in Definition 3.18.

By analogy with the fact that among smooth connected compact analytic curves, those of smallest genus are called *rational*, Michael Artin [6] introduced the same terminology for surface singularities:

Definition 3.32. A normal surface singularity (X, x) is called **rational** if its geometric genus vanishes.

M. Artin proved that rational singularities may be characterized topologically (see Artin [6, Proposition 1, Theorem 3], Bădescu [10, Theorem 3.21] or Ishii [86, Theorem 7.3.1]):

Theorem 3.33. *A normal surface singularity is rational if and only if one of the following equivalent facts happen:*

  1. *One has $p_a(X, x) = 0$.*

  2. *There exists a strict normal crossings resolution for which $p_a(Z_{num}) = 0$.*

  3. *For all the strict normal crossings resolutions, one has $p_a(Z_{num}) = 0$.*

Using Laufer's algorithm (see Proposition 3.19), we see that point (2) of Theorem 3.33 allows to determine readily from the knowledge of the weighted graph of a resolution whether a singularity is rational or not. Using Theorem 3.29, we deduce that:

Theorem 3.34. *Let (X, x) be a normal surface singularity. The topological type of its boundary $\partial(X, x)$ determines whether it is rational or not.*

In general, the minimal resolution of a normal surface singularity is not a composition of blow-ups of points, and its exceptional divisor has not necessarily normal crossings. The situation is different for rational surface singularities (see Lipman [109, Theorem 4.1], Bădescu [10, Theorem 3.23.3]):

Theorem 3.35. *Let (X, x) be a rational surface singularity. Then:*

  1. *The surface obtained by blowing up x in X is again normal and has only rational singularities.*

  2. *Starting from (X, x) and blowing-up recursively all non-smooth points, one gets in a finite number of steps the minimal resolution of (X, x).*

3. *The exceptional divisor of the minimal resolution of (X, x) has strict normal crossings, its irreducible components are rational and its dual graph is a tree.*

4. *All the resolutions of (X, x) have simple normal crossings divisors whose irreducible components are rational and whose dual graphs are trees.*

5. *Any element of the Lipman semigroup associated to any resolution of singularities of (X, x) may be realised as the exceptional part of a principal divisor.*

In fact, the last property may be strengthened into the following characterization of rational singularities (see Lipman [109, Theorem 12.1]):

Proposition 3.36. *Let $\pi : (\tilde{X}, E) \to (X, x)$ be a resolution of a normal surface singularity X. Then (X, x) is rational if and only if any germ D of effective divisor in the neighborhood of E such that $D \cdot E_i = 0$ for all $i \in I$ is principal.*

Theorem 3.33 allows to decide by a finite algorithm whether a normal surface singularity is rational or not whenever one knows the weighted dual graph of some strict normal crossing resolution of it. Let us present a different criterion of rationality, which does not use explicitly resolutions in its statement, even if they are needed in its proof (see Behnke and Riemenschneider [11, Theorem 2.7]):

Theorem 3.37. *Assume that $\phi : (Y, y) \to (X, x)$ is a finite (that is, proper with finite fibers) holomorphic morphism between normal surface singularities. If (Y, y) is rational, then (X, x) is also rational.*

As $(\mathbb{C}^2, 0)$ is rational, an immediate consequence of the theorem is (recall that quotient singularities were introduced in Definition 2.29):

Corollary 3.38. *Quotient surface singularities are all rational.*

Another proof of this fact, valid also in higher dimensions, may be found in Ishii [86, Corollary 7.4.10]. As we won't speak here about higher dimensional rational singularities, we decided not to include their definition in our text.

Among quotient surface singularities, the ones which may be realized as *hypersurface singularities* in $\mathbb{C}^3$ are exactly those described in the following theorem:

Theorem 3.39. *Let (X, x) be a normal surface singularity. Then the following properties are equivalent:*

1. *(X, x) is analytically isomorphic to the germ $(\mathbb{C}^2/G, 0)$, where G is a finite subgroup of $SL(2, \mathbb{C})$ (one says that it is **a Kleinian singularity**).*

2. *The anti-canonical cycle (see Definition 3.21) of the minimal resolution of (X, x) is trivial (one says that it is **a Du Val singularity**).*

3. *(X, x) is rational of multiplicity 2 (one says that it is **a rational double point**).*

4. *(X, x) is rational and numerically Gorenstein (see Definition 3.22).*

Kleinian singularities are not only numerically Gorenstein, but even Gorenstein. Indeed, G being a subgroup of $SL(2, \mathbb{C})$, it leaves invariant the non-vanishing holomorphic 2-form $dx \wedge dy$ on the affine plane $\mathbb{C}^2$ with coordinates (x, y). Therefore, this form descends to a holomorphic 2-form on the quotient $(\mathbb{C}^2/G, 0)$. This second form is non-vanishing outside the singular point of the quotient. As this quotient is a normal surface, it is Cohen-Macaulay, and Proposition 2.26 implies that it is moreover Gorenstein.

In fact, there are many more characterizations of Kleinian singularities than those stated in Theorem 3.39. They appeared historically in different contexts under different aspects, some of those contexts having led to the names emphasized in the previous theorem. More precisely:

Rational double points. $(o^2 \text{—} -2)$
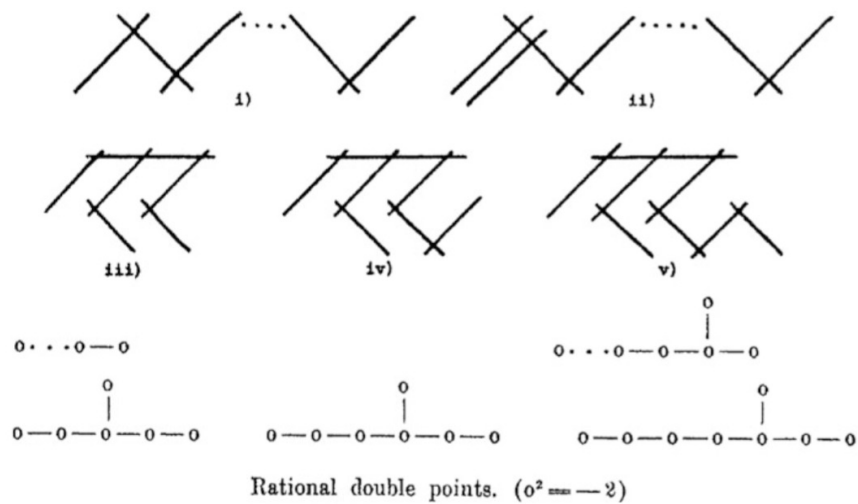
Figure 3.4. Exceptional divisors of Kleinian singularities and their dual graphs

- Klein [92] was studying the theory of invariants of finite subgroups of $SL(2, \mathbb{C})$;

- Du Val [194] was studying the isolated singularities of surfaces in $\mathbb{CP}^3$ which *do not affect the conditions of adjunction*, that is, such that the holomorphic 2-forms defined outside the singular point extended to *holomorphic* 2-forms on any resolution of the singular point (see [166, Chapter 33]);

- Artin [6] showed that any singularity having the same dual graph as those of Du Val's list (even without assuming that they were of embedding dimension 3), were rational of multiplicity 2 and embedding dimension 3.

One has the following "*ADE*" classification of Kleinian singularities:

(3.7)
$$
\begin{aligned}
\mathbf{A_n} &: \quad x^{n+1} + y^2 + z^2 = 0 \ (n \geq 1); \\
\mathbf{D_n} &: \quad x^{n-1} + xy^2 + z^2 = 0 \ (n \geq 4); \\
\mathbf{E_6} &: \quad x^4 + y^3 + z^2 = 0; \\
\mathbf{E_7} &: \quad x^3 y + y^3 + z^2 = 0; \\
\mathbf{E_8} &: \quad x^5 + y^3 + z^2 = 0.
\end{aligned}
$$

On the left is written the standard name of each singularity, and on the right is given a defining equation.

The dual graph of the minimal resolution is each time a tree of smooth rational curves with self-intersections -2 (that is, $\underline{p} = \underline{0}$ and $\underline{e} = \underline{2}$), the number of vertices of the graph associated to X_n being n, and the shape of the graph being the same as the one of the Coxeter diagram of the root lattice of the simple complex Lie algebra with the same name. In the sequel we will say that such weighted graphs are **Kleinian graphs**. Figure 3.4 representing them combines figures from Artin's papers [5] (top) and [6] (bottom). Case i) corresponds to the A_n singularities, case ii) to D_n's, case iii) to E_6, case iv) to E_7 and, finally, case v) to the E_8 singularity.

Let us pass now to the quotient surface singularities which are not Kleinian. They were classified by Brieskorn [21] (see also Matsuki [120, Chapter 4.6] or Ishii [86, Theorem 7.4.19]). The simplest ones are the quotients of $\mathbb{C}^2$ by *cyclic* subgroups of $GL(2, \mathbb{C})$. The quotient of a cyclic group by a subgroup being again cyclic, one may assume that this subgroup is small (see Definition 2.30). This
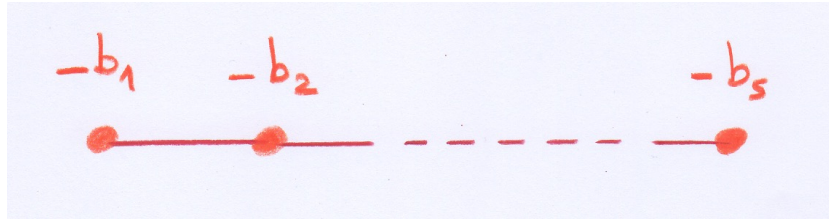
Figure 3.5. The dual graphs of the minimal resolutions of cyclic quotients

implies that a generator of it may be diagonalised in the form:

$$\begin{pmatrix} \xi & 0 \\ 0 & \zeta \end{pmatrix},$$

where ξ and ζ are roots of 1 *of the same order*. Denote by $p > 0$ this common order. Then one may express $\zeta = \xi^q$ with $0 \leq q < p$. The order of ζ being also equal to p, we see that q is coprime with p. This shows that the following definition exhausts all surface singularities obtainable as quotients of $\mathbb{C}^2$ by cyclic groups:

Definition 3.40. Let p, q be coprime integers such that $p > q > 0$. The **cyclic quotient** (or **Hirzebruch-Jung**) **singularity** $(\mathcal{X}_{p,q}, 0)$ is the germ at the image of the origin in $\mathbb{C}^2$ of the quotient $\mathcal{X}_{p,q}$ of $\mathbb{C}^2$ by the action $(\xi, (x, y)) \to (\xi x, \xi^q y)$ of the cyclic group $\{\xi \in \mathbb{C}, \ \xi^p = 1\} \simeq \mathbb{Z}/p\mathbb{Z}$. Its oriented boundary is the (oriented) **lens space** $L(p, q)$.

The alternative name "Hirzebruch-Jung" for cyclic quotient singularities comes from the fact that those singularities appear naturally in the Hirzebruch-Jung method (originating in Jung [91] and Hirzebruch [80]) of resolution of surface singularities by preliminary embedded resolution of the discriminant curve of a finite projection to a smooth surface. Namely, they are the singularities of the normalization of a surface having such a projection whose discriminant has normal crossings (see Laufer [99], Lipman [110] or Popescu-Pampu [164] for details).

The weighted dual graphs of the minimal resolutions of cyclic quotient singularities may be described in the following way, starting from the defining pair (p, q):

Theorem 3.41. *The weighted dual graph of the cyclic quotient singularity $\mathcal{X}_{p,q}$ is illustrated in Figure 3.5. The sequence $(b_1, \ldots, b_s)$ is characterized by the conditions:*

$$\begin{cases} b_i \geq 2, \ \text{for all } i \in \{1, \ldots, s\}, \\ \dfrac{p}{q} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ldots - \cfrac{1}{b_s}}} \end{cases}.$$

The continued fraction expansion appearing in the previous theorem is called a **Hirzebruch-Jung expansion**. For the importance of such expansions in the study of the topology of normal surface singularities, one may consult my survey [160].

Rational surface singularities are the simplest surface singularities, if one takes the arithmetic genus as a measure of complexity. The next class in terms of this complexity contains therefore the singularities (X, x) with $p_a(X, x) = 1$. Wagreich [197] started their study and called them *elliptic singularities*, by analogy with elliptic curves, whose topological genus is 1. Unlike in the case of rational singularities, this class contains germs with arbitrary high geometric genus. Laufer discovered that there exists a subclass which is also defined topologically, and which has many properties in common with rational singularities. Namely, in [102, Theorems 3.4 and 3.10], he proved:

Theorem 3.42. *Let (X, x) be a normal surface singularity. Working with its minimal resolution, the following facts are equivalent:*

1. *One has $p_a(Z_{num}) = 1$ and $p_a(D) < 1$ for all $0 < D < Z_{num}$.*

2. *The fundamental and anti-canonical cycles are equal: $Z_{num} = Z_K$.*

3. *One has $p_a(Z_{num}) = 1$ and any connected proper subdivisor of E contracts to a rational singularity.*

4. *$p_g(X, x) = 1$ and (X, x) is Gorenstein.*

Laufer introduced a special name (making reference to condition (3)) for the singularities satisfying one of the previous conditions:

Definition 3.43. A normal surface singularity satisfying one of the equivalent conditions stated in Theorem 3.42 is called a **minimally elliptic** singularity.

One sees either from point (2) or from point (3) of Theorem 3.42 that the weighted dual graph of the minimal resolution of (X, x) determines whether this singularity is minimally elliptic or not. It may be shown that this weighted dual graph is determined by that of the minimal strict normal crossings resolution. Therefore, by Theorem 3.29, whether a normal surface singularity is minimally elliptic or not is determined by its topological type (that is, by its boundary, as introduced in Definition 3.23). Note that, by Theorem 3.42, all the singularities realising that topology are necessarily Gorenstein.

Concerning rational singularities, we saw in Theorem 3.39 that only the Kleinian ones are Gorenstein. Kleinian singularities are moreover taut (see Definition 3.47), which is not the case for all the minimally elliptic ones. Nevertheless, Laufer saw that the union of the class of Kleinian singularities and minimally elliptic singularities could be characterized in a subtle way using the property of being Gorenstein (see Laufer [102, Theorem 4.3]):

Theorem 3.44. *Let us fix a topological type of normal surface singularities. Then the singularities realising that type are generically Gorenstein if and only if the topological type corresponds either to a Kleinian singularity or to a minimally elliptic singularity.*

It is not clear a priori what means a *generic* property of the singularities with given topological type. Laufer gives the following meaning to it: a property is *generic* for a given topological type of singularities if, on the base of the miniversal space of deformations with fixed topological type, the singularities having that property form a dense open set.

If the minimal resolution of a rational singularity has a strict normal crossings exceptional divisor, this is not necessarily the case for minimally elliptic ones. But Laufer [102, Proposition 3.5] described completely the possible exceptions.

Let us introduce the following particular types of minimally elliptic singularities:

Definition 3.45. A normal surface singularity is a **simple elliptic** singularity if it is obtained by contracting a smooth elliptic curve with negative self-intersections embedded in a smooth surface. It is a **cusp** singularity if the weighted dual graph of its minimal strict normal crossings resolution is a circle and $\underline{p} = \underline{0}$.

Simple elliptic singularities were introduced by K. Saito [172] as the simplest elliptic singularities in the sense of Wagreich and cusp singularities received their name from the fact that they are the singularities obtained by compactifying the *cusps* of the Hilbert modular surfaces (see Hirzebruch [83]). They have the following common characterization with cyclic quotient singularities, proved by Neumann [142]:

Theorem 3.46. *If one changes the orientation of the boundary of a normal surface singularity, the resulting 3-manifold is no more orientation-preserving diffeomorphic to the boundary of an isolated surface singularity, excepted for* cyclic quotient singularities *and* cusp-singularities.
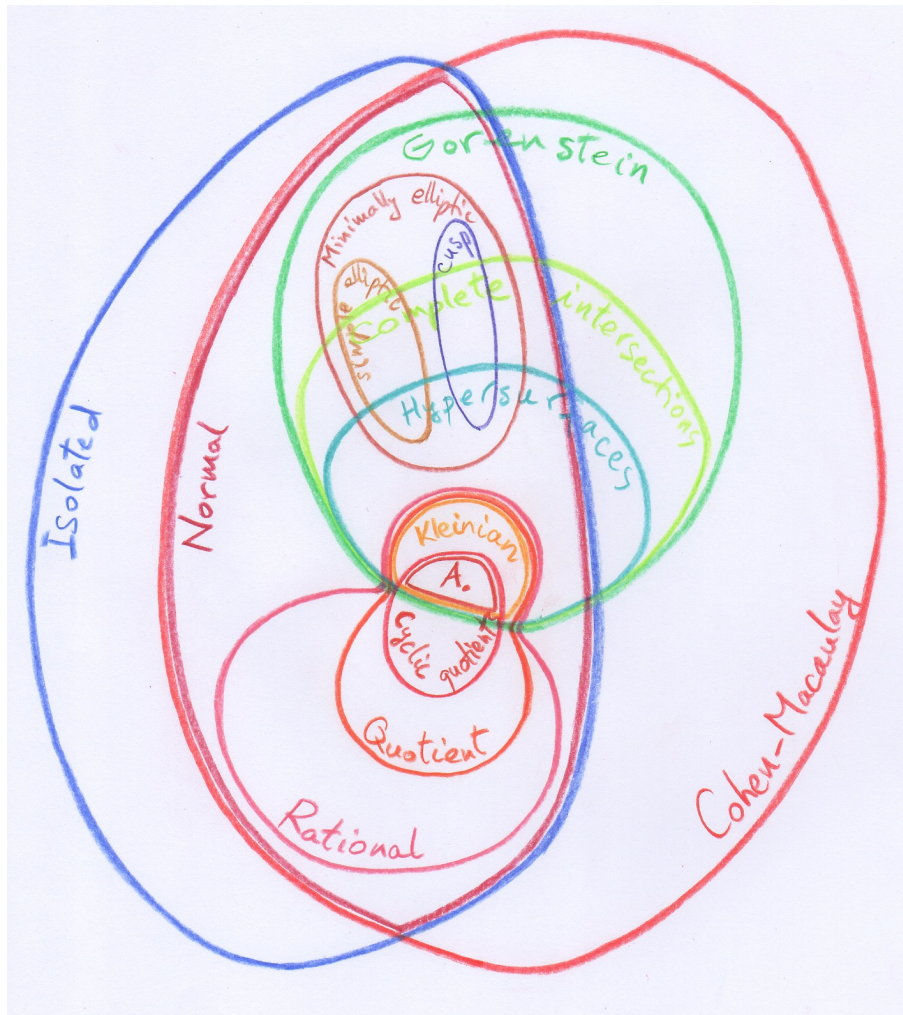
Figure 3.6. Inclusions between several classes of surface singularities

The previous two classes of singularities, as well as all Kleinian singularities have moreover the property that their topology determines their analytical type, that is, that they are *taut*:

Definition 3.47. A normal surface singularity or a weighted dual graph is called **taut** if its topological type determines its analytical type.

In [101], Laufer classified all the weighted dual graphs corresponding to taut singularities.

In order to help the reader get oriented among the various classes of surface singularities discussed in this text, I drew in Figure 3.6 an Euler-Venn diagram indicating the inclusion relations between them. The only exceptions are the classes of *taut* and *numerically Gorenstein* singularities, which don't play an important role in these notes.

For more details about Kleinian singularities, one may consult Hazewinkel et al. [76]**, Durfee** [39]**, Slodowy** [177]**, Cassens** & **Slodowy** [28]**, Brieskorn** [23]**.**

For more details about rational surface singularities, one may consult the original papers of Artin [5]**,** [6]**, Brieskorn** [21]**, Tjurina** [190]**, Lipman** [109]**, Laufer** [100]**, Lê And Tosun** [108]**, Okuma** [147]**, Stevens** [183]**, Némethi** [136]**, as well as the introductory texts of Behnke and Riemenschneider** [11]**, Reid** [168, Sections 4.12–4.15]**, Némethi** [132, Lecture 3]**, Bădescu** [10, Chapters 3-4] **and Ishii** [86, Section 7.3]**.**

For details about the classification of general normal surface singularities, I recommend Némethi's surveys [132], [134] as well as Reid [168] and Wall [201].

4. Smoothings of singularities and their Milnor fibers

4.1. A prototype: Milnor's study of hypersurface singularities

Mumford's theorem 3.25 characterizes the smooth points of normal surfaces as those for which the boundary of the associated singularity is diffeomorphic to a standard sphere. Around 1965, using recent work of Pham, Brieskorn proved that this is false in any higher dimension: he exhibited isolated hypersurface singularities defined by equations of the form:

$$z_0^{a_0} + \cdots + z_n^{a_n} = 0$$

(nowadays such singularities are called *Pham-Brieskorn singularities*, see Example 2.11) and whose boundaries are diffeomorphic to standard spheres.

Moreover, he showed that one could obtain like this also **exotic spheres**, that is, manifolds homeomorphic to standard spheres but not diffeomorphic to them. This was the first construction of such spheres as explicit real algebraic sets, after the discovery of their existence as smooth manifolds by Milnor [127]. Brieskorn's work intrigued a lot Milnor, who began to think about this discovery. His reflexions led to the book [129], which founded the topological theory of hypersurface singularities. See Brieskorn [23] and Durfee [41] for more details about the previous discoveries.

Let me describe briefly the main geometric actors introduced by Milnor [129].

Assume that $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is a germ of holomorphic function having an *isolated* critical point at 0. Then Milnor considered the following objects associated to it:

1. A sufficiently small Euclidean ball B_ϵ centered at the origin.

2. The intersection $M_\epsilon := f^{-1}(0) \cap \partial B_\epsilon$ of the critical level $f^{-1}(0)$ with the boundary of that ball.

3. The pieces $F_{\epsilon,\lambda}$ contained in B_ϵ of nearby regular levels $f^{-1}(\lambda)$, for $\lambda \in \mathbb{C}^*$ sufficiently small.

4. The intersections $B_\epsilon \cap f^{-1}(\mathbb{D}_\delta^2)$ for $\delta > 0$ sufficiently small, where $\mathbb{D}_\delta^2 \subset \mathbb{C}$ denotes the closed disc of radius δ.

5. The family $(F_{\epsilon,\lambda})_{|\lambda|=const}$ of such pieces, for a fixed non-zero absolute value of the level.

6. The map $\theta := \arg f : \partial B_\epsilon \setminus M_\epsilon \to \mathbb{S}^1$ defined by the argument of f.

Milnor proved in [129] that:

1. The radius $\epsilon > 0$ may be chosen such that the critical level $f^{-1}(0)$ intersects transversally all the spheres of radius $\leq \epsilon$ centered at the origin. One calls such a ball B_ϵ a **Milnor ball** and its boundary a **Milnor sphere** with respect to f.

2. If B_ϵ is a Milnor ball, then the pair $(\partial B_\epsilon, M_\epsilon)$ is independent of the choices, up to diffeomorphisms whose isotopy classes are well-defined. One calls it the **embedded link** of the critical point. Note that M_ϵ is then a representative of the *boundary* or *link* of the isolated hypersurface singularity $(f^{-1}(0), 0)$ (see Definition 3.23).

3. One may choose $\delta > 0$ such that the various sets $F_{\epsilon,\lambda}$ are diffeomorphic smooth manifolds-with-boundaries whose boundaries are diffeomorphic to M_ϵ, for each $\lambda \in \mathbb{C}^*$ such that $|\lambda| < \delta$.
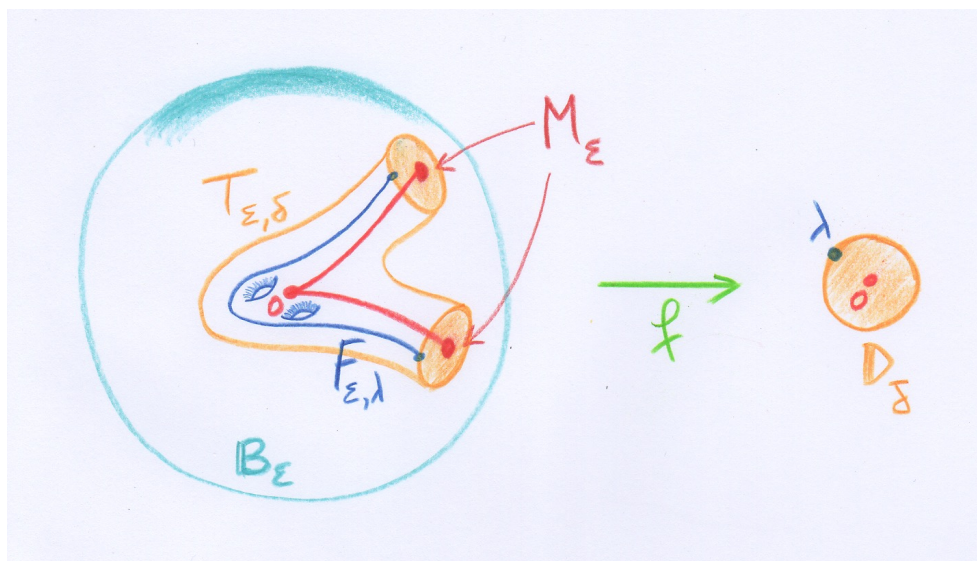
Figure 4.1. Milnor's view of an isolated critical point of holomorphic function

4. Whenever B_ϵ is a fixed Milnor ball, one may choose $\delta > 0$ sufficiently small such that $T_{\epsilon,\delta} :=$ $B_\epsilon \cap f^{-1}(\mathbb{D}_\delta^2)$ is homeomorphic to a ball and the restriction of f to $T_{\epsilon,\delta} \setminus F^{-1}(0)$ is a locally trivial fibration above $\mathbb{D}_\delta^2 \setminus \{0\}$. One calls the manifold with corners $T_{\epsilon,\delta}$ a **Milnor tube**.

5. The family $(F_{\epsilon,\lambda})_{|\lambda|=const}$ is a locally trivial fibration over the circle. It is called the **Milnor fibration** associated to the germ f and its fibers $F_{\epsilon,\lambda}$ are called the **Milnor fibers** of f.

6. (M_ϵ, θ) is an open book in ∂B_ϵ, whose associated fibration is isomorphic to the Milnor fibration. One calls it the **Milnor open book**.

The term "*open book*" was introduced later, by Winkelnkemper [205]. Here is its definition:

Definition 4.1. An **open book** on a closed manifold V is a pair (K, θ) consisting of:

1. a codimension 2 submanifold $K \subset W$, called the **binding**, with a trivialized normal bundle;

2. a fibration $\theta : W \setminus K \to \mathbb{S}^1$ which, in a trivialized tubular neighborhood $\mathbb{D}^2 \times K$ of K, is the normal angular coordinate (that is, the composition of the first projection $\mathbb{D}^2 \times K \to \mathbb{D}^2$ with the angular coordinate $\mathbb{D}^2 \setminus \{0\} \to \mathbb{S}^1$). The closures of the fibers of θ are called the **pages** of the open book.

In Figure 4.2 is represented a local view of an open book on a 3-dimensional manifold, in the neighborhood of a point of its binding. This view illustrates the fact that all the pages share the same boundary, which is the binding. As a simple example of open book, one may think about the system of meridians on a 2-dimensional sphere (parametrised by their longitudes), the binding being in this case the union of the two poles (see Figure 4.3). Note that this is not a Milnor open book, as the ambient manifold is even-dimensional.

A fundamental theorem of Milnor [129, Theorem 6.5] about the structure of Milnor fibers is:

Theorem 4.2. *The Milnor fibers of a germ of holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with isolated critical point is homotopically equivalent to a bouquet of a finite number of spheres of dimension n.*

The number of such spheres bears since then Milnor's name:
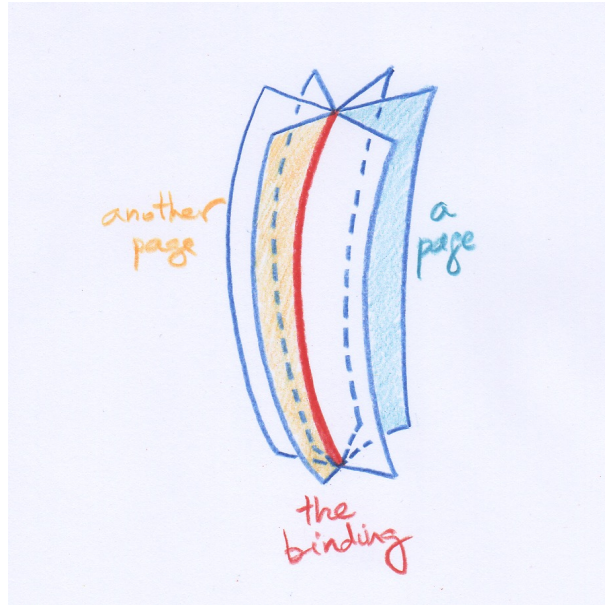
Figure 4.2. A local view of an open book near a point of its binding
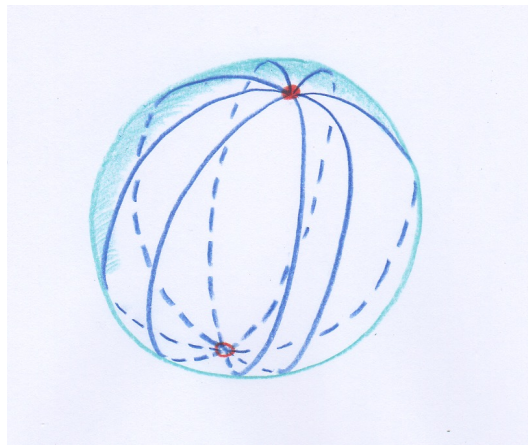
Figure 4.3. An open book on the 2-dimensional sphere

Definition 4.3. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ of holomorphic function with an isolated critical point. Then the number of n-spheres in a bouquet realization of the homotopy type of the Milnor fibers of f is called the **Milnor number** $\mu(f)$ of f.

The Milnor number has also an algebraic description:

Theorem 4.4. *The Milnor number of the germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with isolated critical point is equal to the dimension of the complex vector space*

$$\mathbb{C}\{z_1, \ldots, z_n\}/\left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right).$$

A proof of this theorem was sketched by Milnor in [129, Problem 3 of Appendix B]; another proof by Pham was sketched by Lê in [105, Page 175]. Detailed proofs may be found in Brieskorn [22] and in Orlik's survey [148].

The study of the structure of the Milnor fibration associated to an isolated critical point of holomorphic function is one of the mainstreams of singularity theory. Hundreds of papers have been dedicated to it. The main theme is to understand the action of the monodromy on the homology

of the fiber and its relations with other invariants, as asymptotic integrals and mixed Hodge structures. We won't pursue these themes here, but we will examine instead the way to associate Milnor fibers even to singularities which are not hypersurfaces.

For more details about Milnor fibrations, one may read Milnor's book [129]**, as well as the surveys of Arnold, Gussein-Zade and Varchenko** [2, Chapter I]**, Teissier** [188]**, Némethi** [133] **and Budur** [26]**. Vor various interpretations of the Milnor number, see Orlik's survey** [148]**.**

4.2. General facts about deformations and smoothings

Let us look a little differently at a holomorphic germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with an isolated critical point at the origin. After having done Milnor's construction explained at the beginning of subsection 4.1 and illustrated in Figure 4.1, we may think that the set of Milnor fibers $F_{\epsilon, \lambda} := f^{-1}(\lambda) \cap B_\epsilon$ parametrized by $\lambda \in \mathbb{D}_\delta^2 \setminus \{0\}$ is a family of smooth manifolds converging to the singular space $f^{-1}(0) \cap B_\epsilon$.

Is it possible to see analogously *any* complex analytic singularity as a limit of smooth spaces?

One needs a little care in the definition of such a limit process. The usual way in analytic geometry to conceptualize limits is to take *analytic families* of objects and to see how the nearby members approach a special member of the family. The constraint that the families be *analytic* means that the members of the family are requested to be the fibers of a complex analytic morphism.

Such a definition of *family* of analytic spaces is too general. For example, if one looks at the morphism of blow-up of the vertex 0 of a cone X of dimension at least 2 (see Example 2.34) and at the special fiber over 0, one would be forced to think at the projectivised cone $\mathbb{P}(X)$ as a limit of points, which is not desirable. In order to get a notion more proximate to the intuition, one would like to impose at least that all the fibers of the morphism be *equidimensional*. The algebraic notion of *flatness* ensures this property and in fact much more. That is why one restricts in the following way the notion of *analytic family*:

Definition 4.5. Let (X, x) be a germ of a not necessarily reduced complex analytic space. A **deformation** of (X, x) is a germ of *flat* morphism $\psi : (Y, y) \to (S, s)$ together with an isomorphism between (X, x) and the special fiber $(\psi^{-1}(s), y)$.

Let us explain also the notion of *flat* morphism, used in the previous definition:

Definition 4.6. Let A be a commutative ring. An A-module M is called **flat** if for any injective morphism of A-modules $N_1 \to N_2$, the induced morphism $N_1 \otimes M \to N_2 \otimes M$ is still injective.

A morphism $\psi : (Y, y) \to (S, s)$ of germs of analytic spaces is called **flat** if the corresponding morphism $\psi^* : \mathcal{O}_{S,s} \to \mathcal{O}_{Y,y}$ of local rings makes $\mathcal{O}_{Y,y}$ a flat $\mathcal{O}_{S,s}$-module.

It is difficult to explain more geometrically the meaning of this notion. For instance, Mumford said about it in [131, Section III.10] that "the concept of flatness is a riddle that comes out of algebra, but which technically is the answer to many prayers."

When the germ (X, x) is reduced, then a germ of holomorphic function $f \in \mathfrak{m}_{X,x}$ is flat as a morphism $f : (X, x) \to (\mathbb{C}, 0)$ if and only if f is not a divisor of zero. Such deformations over germs of smooth curves are called 1-**parameter deformations**. The simplest example is obtained when $(X, x) = (\mathbb{C}^n, 0)$ and f has an isolated critical point at $0 \in \mathbb{C}^n$. Then one gets the situation considered by Milnor and recalled in Subsection 4.1, in which now f is thought as a deformation of the singularity $(Z(f), 0)$.

In general, to think about a flat morphism as a deformation means to see it as a family of continuously varying fibers and to concentrate on a particular fiber. Of course, in order to speak about such fibers as well-defined topological spaces, one needs first to choose convenient representatives of the various germs, in a way which generalizes the one explained in Subsection 4.1 (see also Theorem 4.16 below).

If one starts from an analytic family, one gets new families by rearranging the fibers in the following way:

Definition 4.7. Let $\psi : (Y, y) \to (S, s)$ be a morphism of germs of analytic spaces. A second morphism $\psi_1 : (Y_1, y_1) \to (S_1, s_1)$ is obtained by **base change** from ψ if it may be included in a cartesian diagram of the form:

$$
\begin{array}{ccc}
(Y_1, y_1) & \longrightarrow & (Y, y) \\
\psi_1 \downarrow & & \downarrow \psi \\
(S_1, s_1) & \underset{\chi}{\longrightarrow} & (S, s)
\end{array}
$$

that is, if $\psi_1 = \chi^* \psi$. One says in this case that ψ_1 **is obtained from ψ through the base change** χ.

Note that the fibers of ψ_1 are precisely the fibers of ψ lying above the points of the image of the morphism χ. In particular, the special fiber is unchanged. Moreover, it is a basic property of flat morphisms to remain flat after base changes. For this reason, *by base changes the deformations of a singularity (X, x) are transformed into other such deformations.*

In order to compute the base change $\psi_1 = \chi^* \psi$, one has to write the equations defining the graph of the deformation ψ and to replace then the variables used to describe its base S using the equations defining the graph of the morphism χ.

Example 4.8. Consider the holomorphic germ $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ defined by $f(x, y, z) = x^2 + y^2 + z^2$. Look at it as a deformation of the isolated surface singularity $(Z(f), 0)$. If one denotes by t the variable of the base $(\mathbb{C}, 0)$, then the equation of the graph of f is $x^2 + y^2 + z^2 = t$. Consider the morphism $\chi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ defined by $\chi(u) = u^2$. Its graph has the equation $u^2 = t$. Replacing the variable t in the equation $x^2 + y^2 + z^2 = t$ using the equation $u^2 = t$, one gets the equation $x^2 + y^2 + z^2 = u^2$. It defines the total space Y_1 of the deformation f_1 obtained from f through the base change χ. This total space is a hypersurface in the affine space $\mathbb{C}^4$ of coordinates (x, y, z, u) and the morphism f_1 is simply the restriction to it of the natural projection to the axis of the coordinate u.

One is particularly interested in the situations in which there exist deformations generating all other deformations by base changes. The following definition is a reformulation of [68, Definition 1.8, page 234]:

Definition 4.9.

1. A deformation ψ of (X, x) is **complete** if any other deformation is obtainable from it by a base change.

2. A complete deformation ψ of (X, x) is called **versal** if for any other deformation over a base (T, t) and any identification of the induced deformation over a subgerm $(T', t) \hookrightarrow (T, t)$ with a morphism obtained from ψ by base change, one may extend this identification over all (T, t).

3. A versal deformation is **miniversal** if the embedding dimension of its base (S, s) is as small as possible.

Obviously, miniversal deformations are versal, and versal deformations are complete. Such deformations do not necessarily exist. But, whenever a miniversal deformation exists, it results easily from the definition that its base space is unique *up to non-unique isomorphism* (only the tangent map to the isomorphism is unique). For this reason, one does not speak about a *universal* deformation, and the word "miniversal" was coined, with the variant "semi-universal".

In many references, versal deformations are defined as the complete ones in Definition 4.9. Then is stated usually the theorem that the base of a versal deformation is isomorphic to the product of the base of a miniversal deformation and of a *smooth* germ. With this weaker definition, the

theorem is false. Indeed, starting from a complete deformation, by doing the product of its base with *any* germ (not necessarily smooth) and by taking the pull-back, we would get again a complete deformation. This shows that a complete deformation is not necessarily versal.

The following crucial existence theorem of versal deformations was proved by Schlessinger [174] in the formal category and by Grauert [63] in the holomorphic category (see also Tyurina [191], Hauser [74] and de Jong and Pfister [90, Section 10.3]):

Theorem 4.10. *Let* (X, x) *be an isolated singularity. Then it admits miniversal deformations, which are unique up to (non-unique) isomorphisms.*

Let us come back to the situation we were speaking about at the beginning of the section, where a singularity is seen as a limit of *smooth* spaces. The following vocabulary is used in this context:

Definition 4.11. A **smoothing** of a singularity (X, x) is a 1-parameter deformation $f : (Y, y) \to (\mathbb{C}, 0)$ of it whose general fiber is smooth in a neighborhood of y. A singularity is **smoothable** if it admits a smoothing. A **smoothing component** is an irreducible component of the reduced miniversal base space of (X, x), over which the generic fibers are smooth.

Isolated complete intersection singularities have very special miniversal deformations, as shown by the following result of Tyurina [191] (see also Looijenga [114, Chapter 6] or Greuel, Lossen and Shustin [68, Theorem 1.16]):

Theorem 4.12. *If* $\psi : (Y, y) \to (S, s)$ *is a miniversal deformation of an isolated complete intersection singularity, then both* (Y, y) *and* (S, s) *are smooth.*

The miniversal deformations of isolated complete intersections may be described concretely. Let us state this description in the case of isolated hypersurface singularities (as a particular case of [68, Theorem 1.16]):

Theorem 4.13. *Assume that* $f \in \mathbb{C}\{z_1, \ldots, z_n\}$ *with* $f(0, \ldots, 0) = 0$ *has an isolated critical point at* 0 *and that* $g_1, \ldots, g_\tau \in \mathbb{C}\{z_1, \ldots, z_n\}$ *descend to a basis of the complex vector space*

$$(4.1) \qquad \mathbb{C}\{z_1, \ldots, z_n\} / \left(f, \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n} \right).$$

Define $(Y, 0) \hookrightarrow (\mathbb{C}^n_{z_1, \ldots, z_n} \times \mathbb{C}^\tau_{t_1, \ldots, t_\tau}, 0)$ *as the zero-locus* $Z(f + t_1 g_1 + \cdots + t_\tau g_\tau)$. *Then the restriction to* $(Y, 0)$ *of the canonical projection* $(\mathbb{C}^n_{z_1, \ldots, z_n} \times \mathbb{C}^\tau_{t_1, \ldots, t_\tau}, 0) \to (\mathbb{C}^\tau_{t_1, \ldots, t_\tau}, 0)$ *is a miniversal deformation of the isolated hypersurface singularity* $(Z(f), 0)$.

In honor of Tyurina's work [191], the dimension $\tau(f)$ of the complex vector space (4.1) is called the **Tyurina number** of the function f with isolated critical point. It may be checked formally that it is an invariant of the isolated hypersurface singularity $(Z(f), 0)$. Theorem 4.13 gives a good reason for this invariance: τ is equal to the dimension of the miniversal base of $(Z(f), 0)$.

Note that Theorem 4.4 implies that the Tyurina number of f is not greater than its Milnor number: $\tau(f) \leq \mu(f)$. Kyoji Saito proved in [171] that one has the equality $\tau(f) = \mu(f)$ if and only if f may be transformed into a quasi-homogeneous polynomial by a holomorphic automorphism (change of variables) of $(\mathbb{C}^n, 0)$.

Example 4.14. Let us see an example of application of Theorem 4.13. Consider the function $f(x, y, z) = x^2 + y^2 + z^2$ of Example 4.8. Then $\left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = (x^2 + y^2 + z^2, 2x, 2y, 2z) = (x, y, z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$. Therefore, one has $\tau = \mu = 1$ and one may take $g_1 = 1$. The total space $(Y, 0) \hookrightarrow (\mathbb{C}^3_{x, y, z} \times \mathbb{C}_{t_1}, 0)$ is defined by the equation $x^2 + y^2 + z^2 + t_1 = 0$. Therefore, it is obtained from the deformation $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ of $(Z(f), 0)$ by the base change $t_1 = -t$. This base change being an analytic isomorphism, we see that f itself is in this case a miniversal deformation of $(Z(f), 0)$.

The miniversal base (S, s) of a general isolated singularity is not necessarily smooth or even reduced. Moreover, its reduction (S_{red}, s) may be reducible. The first example of this phenomenon was discovered by Pinkham [158, Chapter 8] (see also Behnke and Riemenschneider [11, Section 3.3]):

Proposition 4.15. *The germ at the origin of the cone over the rational normal curve of degree 4 in $\mathbb{P}^4$ has a reduced miniversal base space with two irreducible components. Both of them are smoothing components.*

I will give more details about Pinkham's example in Subsection 4.4 below. Let me mention also that, as was shown by Vakil [193], up to smooth factors, *any* germ of complex analytic space may occur as the base of the miniversal deformation of an isolated singularity of dimension 3.

Not all isolated singularities are smoothable. The most extreme case is attained with **rigid** singularities, which are not deformable at all in a non-trivial way. For example, quotient singularities of dimension ≥ 3 are rigid (see Schlessinger [175]). Let us note that the following conjecture (see [68, Conjecture II.1.1]), which goes back to the 1970s, is still open:

"There exist no rigid singular reduced curve singularities and no rigid singular normal surface singularities."

In [161, Proposition 4.5], I gave a purely topological obstruction to smoothability for singularities of dimension ≥ 3. In dimension 2 no such criterion is known in full generality. But for *Gorenstein* normal surface singularities, one has the constraint of Theorem 4.18 below.

Let us look now at the *topological structure* of the generic fibers above a smoothing component. We want to localize the study of the family in the same way as Milnor localized the study of a function on $\mathbb{C}^n$ near an isolated critical point. This is possible (see Looijenga [114, Section 2.8]):

Theorem 4.16. *Let (X, x) be an isolated singularity. Let $\psi : (Y, y) \rightarrow (S, s)$ be a miniversal deformation of it. Then, there exist (Milnor) representatives Y_{red} and S_{red} of the reduced total and base spaces of ψ such that the restriction $\psi : \partial Y_{red} \cap \psi^{-1}(S_{red}) \rightarrow S_{red}$ is a trivial C^∞-fibration. Moreover, one may choose such representatives such that over each smoothing component S_i, one gets a locally trivial C^∞-fibration $\psi : Y_{red} \cap \psi^{-1}(S_i) \rightarrow S_i$ outside a proper analytic subset.*

Hence, for each smoothing component S_i, the oriented diffeomorphism type of the oriented manifold with boundary $(Y_{red} \cap \psi^{-1}(s), \partial Y_{red} \cap \psi^{-1}(s))$ does not depend on the choice of the generic element $s \in S_i$: it is called the **Milnor fiber** of that component. Moreover, its boundary $\partial Y_{red} \cap \psi^{-1}(s)$ is canonically identified with the boundary $\partial(X, x)$ of (X, x) *up to isotopy*.

By looking at the various isolated singularities with the same topological type as a given singularity, one gets a set of Milnor fibers which is a topological invariant of the singularity. As explained in the introduction, the aim of this paper is to describe the classes of *surface* singularities for which the answer to the next question is known:

Given an isolated singularity (X, x), identify the Milnor fibers of all (normal) isolated singularities with the same topological type as (X, x) among the various oriented smooth manifolds having $\partial(X, x)$ as their boundary.

The cases in which the answer to the previous question is known are described in Subsection 6.2. All of them being surface singularities, let us see now some general results about the smoothings of surface singularities.

For more details about flatness, one may consult Mumford [131, Section III.10]**, Fischer** [55, Chapter 3]**, de Jong and Pfister** [90, Section 10.2]**. For more details on deformations and smoothings, one may consult Teissier** [187, Section 4]**, de Jong and Pfister** [90, Section 10.3]**, Stevens** [182]**, Greuel, Lossen and Shustin** [68] **as well as the papers of the collective volume** [140]**.**

4.3. General properties of smoothings of normal surface singularities

In dimension 2, no purely topological obstruction to smoothability for *all* normal singularities seems to be known, in contrast with higher dimensions (see Popescu-Pampu [161, Proposition 4.5]).

But there exist such obstructions for special *Gorenstein* (see Definition 2.25) normal surface singularities (see also [201]), as a consequence of Steenbrink's theorem 4.18 below.

In order to state that theorem, we need a new definition:

Definition 4.17. Let (X, x) be an isolated surface singularity and F a Milnor fiber of it associated to some smoothing component of its miniversal base space. The **Milnor number** μ of F is its second Betti number $\dim_{\mathbb{R}} H_2(F, \mathbb{R})$. One decomposes it as a sum of three terms:

- μ_0 is the dimension of the kernel of the intersection form on $H_2(F, \mathbb{R})$.

- μ_+ is the maximal dimension of a vector subspace of $H_2(F, \mathbb{R})$ on which the intersection form is positive definite.

- μ_- is the maximal dimension of a vector subspace of $H_2(F, \mathbb{R})$ on which the intersection form is negative definite.

That is, the triple (μ_0, μ_+, μ_-) is the inertia index of the intersection form on the real second homology with real coefficients of the Milnor fiber. It is an invariant of the chosen smoothing component.

Note that, when (X, x) is a hypersurface singularity, the notion of Milnor number introduced in Definition 4.17 coincides with that of Definition 4.3.

Here comes the announced theorem of Steenbrink [178], in which $p_g(X, x)$ denotes the geometric genus of (X, x) (see Definition 3.31) and $b_1(\partial(X, x))$ denotes the first Betti number of the boundary of (X, x):

Theorem 4.18. *Let (X, x) be a Gorenstein normal surface singularity. If it is smoothable, then all its Milnor fibers satisfy:*

(4.2) $$\mu_- = 10\, p_g(X, x) - b_1(\partial(X, x)) + (Z_K^2 + |I|).$$

In particular, if a Gorenstein normal surface singularity is smoothable, then:

$$10\, p_g(X, x) - b_1(\partial(X, x)) + (Z_K^2 + |I|) \geq 0.$$

Note that, by point (4) of Proposition 3.30, the first Betti number $b_1(\partial(X, x))$ may be computed from any strict normal crossings resolution with exceptional divisor $E = \sum_{i \in I} E_i$ as:

$$b_1(\partial(X, x)) = b_1(\Gamma) + 2 \sum_{i \in I} p_i,$$

where p_i denotes the genus of E_i and Γ denotes the dual graph of E. The term $Z_K^2 + |I|$ may also be computed using any normal crossings resolution, and is again a topological invariant of the singularity.

Theorem 4.18 gives non-trivial obstructions on the topology of smoothable normal Gorenstein surface singularities.

Example 4.19. For instance, Theorem 4.18 implies that among *simple elliptic* singularities (Definition 3.45), the smoothable ones have minimal resolutions whose exceptional divisor is an elliptic curve with self-intersection $\in \{-9, -8, \ldots, -1\}$. It is possible to prove that in fact all simple elliptic singularities whose resolution satisfies this constraint are smoothable.

One gets analogous constraints on the topology of smoothable *minimally elliptic* singularities (see Definition 3.43), a class determined by its topological type (see Theorem 3.42).

In what precedes, we have spoken only about μ_-. There is also a theorem concerning μ_0 and μ_+, proved first by Durfee [38] for isolated hypersurface singularities, then by Steenbrink [178] in this full generality:

Theorem 4.20. *Any Milnor fiber of a normal surface singularity (X, x) satisfies:*

$$\mu_0 + \mu_+ = 2\, p_g(X, x).$$

By adding the formulae of Theorems 4.18 and 4.20, one gets an expression of the Milnor number μ in terms of analytic invariants of the starting normal surface singularity, whenever it is Gorenstein. This generalizes Laufer's theorem [103, Theorem 1], proved for isolated hypersurface singularities in $\mathbb{C}^3$.

An immediate consequence of Theorem 4.20, of Definition 3.32 and of Theorem 4.24 below is:

Corollary 4.21. *Let (X, x) be a normal surface singularity. Then the following properties are equivalent:*

- *(X, x) is rational.*

- *(X, x) admits a smoothing for which the intersection form on the second homology of the Milnor fiber is negative definite.*

- *(X, x) is smoothable and all the Milnor fibers of (X, x) have negative definite intersection forms on their second homology groups.*

Theorem 4.20 shows that $\mu_0 + \mu_+$ is not a topological invariant of the singularity, but it is an analytical one (it does not depend on the smoothing component). In turn, μ_0 is topological, as was proved by Greuel and Steenbrink [67]:

Theorem 4.22. *Any Milnor fiber of a normal surface singularity (X, x) has vanishing first Betti number, which is equivalent to:*

$$\mu_0 = b_1(\partial(X, x)).$$

In fact, Greuel and Steenbrink proved in [67] the vanishing of the first Betti number (for homology with real coefficients) of a Milnor fiber of a *normal* isolated singularity of arbitrary dimension ≥ 2. Note that for hypersurfaces, this results from Milnor's theorem 4.2.

Combining Theorems 4.20 and 4.22, one gets the following constraint on the geometric genus, purely in terms of the topological type of (X, x):

Corollary 4.23. *For any normal surface singularity (X, x), one has:*

$$2\, p_g(X, x) \geq b_1(\partial(X, x)).$$

Let me explain the reason of the equivalence formulated in Theorem 4.22. One has the following analog of the exact sequence (3.5), which is proved exactly in the same way:

(4.3) $$H_2(F) \longrightarrow H^2(F) \longrightarrow H_1(\partial F) \longrightarrow H_1(F) \longrightarrow H^3(F).$$

Here F denotes the Milnor fiber under consideration and this time we work with *real* (co)homology. As the Milnor fiber F has the homotopy type of a CW-complex of dimension at most 2 (see Theorem 5.10 below), we deduce that $H^3(F) = 0$. As we work with real coefficients, we have moreover $H^2(F) \simeq H_2(F)^*$. We deduce therefore from (4.3) the following short exact sequence:

$$0 \longrightarrow H_2(F)^*/h(H_2(F)) \longrightarrow H_1(\partial F) \longrightarrow H_1(F) \longrightarrow 0,$$

in which $h : H_2(F) \longrightarrow H_2(F)^*$ is the map induced by the intersection form on the second homology of the Milnor fiber F. This short exact sequence shows that one has the equivalence:

$$H_1(F) = 0 \iff \dim_{\mathbb{R}} H_2(F)^*/h(H_2(F)) = \dim_{\mathbb{R}} H_1(\partial F).$$

This implies the equivalence formulated in Theorem 4.22.

As the only Gorenstein rational singularities are the Kleinian ones (see Theorem 3.39), we see that Theorem 4.18 does not help to decide whether a given rational surface singularity is smoothable or not. In fact, one has the following theorem of M. Artin [7]:

Theorem 4.24. *All rational surface singularities are smoothable. Moreover, any irreducible component of the reduced miniversal base space of such a singularity is a smoothing component.*

Among the components of the reduced miniversal base space S_{red} of a rational surface singularity, Artin showed that there is a distinguished one (called now the **Artin component**) which may be characterized in the following way:

Theorem 4.25. *Let (X, x) be a rational surface singularity and $\psi : (Y, y) \to (S, s)$ its miniversal deformation. Then the reduced base space S_{red} contains a unique irreducible component (T, s) such that the restriction $\psi_T : (Y_T, y) \to (T, s)$ of the miniversal deformation to T admits a simultaneous resolution after a finite base change.*

To say that $\psi_T : (Y_T, y) \to (T, s)$ **admits a simultaneous resolution after a finite base change** means that there exists a finite surjective morphism $\beta : (\overline{T}, \overline{s}) \to (T, s)$ such that the morphism $\psi_{\overline{T}} : (Y_{\overline{T}}, y) \to (\overline{T}, \overline{s})$ obtained from ψ through the base change β (see Definition 4.7) **admits a simultaneous resolution of its fibers**. In turn, this means that there exists a morphism $\pi : (Z, \tilde{X}) \to (Y_{\overline{T}}, y)$ which restricts to resolutions of singularities over every fiber of $\psi_{\overline{T}}$ in the neighborhood of x. That is, one has a commutative diagram

(4.4)
$$
\begin{array}{ccc}
(Z, \tilde{X}) & \xrightarrow{\quad \pi \quad} & (Y_{\overline{T}}, y) \\
& \gamma \searrow \qquad \swarrow \psi_{\overline{T}} & \\
& (\overline{T}, \overline{s}) &
\end{array}
$$

such that its restriction over each point t of (a representative of) $\overline{T}$ is a resolution of singularities of the surface $\psi_{\overline{T}}^{-1}(t)$.

All the fibers of the analytic morphism $(Z, \tilde{X}) \to (\overline{T}, \overline{s})$ being smooth, one may apply to a well-chosen representative of the previous morphism a version of Ehresmann's theorem adapted to proper morphisms between manifolds with boundary. One deduces from it that all those fibers are diffeomorphic to the total space of the given resolution $\tilde{X}$ of X.

Example 4.26. Consider again the holomorphic function $f(x, y, z) = x^2 + y^2 + z^2$ of Example 4.8. We saw in Example 4.14 that f is a miniversal deformation of the normal surface singularity $(Z(f), 0)$. We recognize it as the A_1 Kleinian singularity (see formulae (3.7)). Therefore, it is a rational surface singularity. As for all isolated complete intersection singularities, the miniversal base space is smooth (see Theorem 4.12), therefore irreducible. Theorem 4.25 implies therefore that there exists a finite base change of the deformation f of $(Z(f), 0)$ which admits a simultaneous resolution.

How to obtain it concretely? Well, the base change $t = u^2$ of Example 4.8 works! Indeed, the equation $x^2 + y^2 + z^2 = u^2$ of the total space Y_1 of the deformation f_1 obtained after this base change may be rewritten in the form $z_1 z_2 - z_3 z_4 = 0$, by the complex linear change of variables $x + iy = 2z_1, x - iy = 2z_2, u + z = 2z_3, u - z = 2z_4$. One recognizes the 3-dimensional isolated singularity of Example 2.42. We saw there that this singularity admits small resolutions. Consider one of those resolutions $\pi : \tilde{Y}_1 \to Y_1$. It may be shown by elementary computations in affine charts that

$$
\begin{array}{ccc}
(\tilde{Y}_1, \widetilde{Z(f)}) & \xrightarrow{\quad \pi \quad} & (Y_1, 0) \\
& f_1 \circ \pi \searrow \qquad \swarrow f_1 & \\
& (\mathbb{C}, 0) &
\end{array}
$$

is a diagram of simultaneous resolution, that is, that it has all the properties mentioned about the diagram (4.4). In order to do those computations, one needs only to know that the two small resolutions of $(Y_1, 0)$ may be obtained by starting either from $z_2/z_4 = z_3/z_1$ or from $z_2/z_3 = z_4/z_1$, seen as rational maps $Y_1 \cdots \to \mathbb{P}^1$.

As explained in [23], this example, worked out by Atiyah at the end of the 1950s, was the starting point of Brieskorn's reflections leading him to the discovery that the boundaries of certain Pham-Brieskorn singularities (see Example 2.11) are exotic spheres.

Let us consider again arbitrary rational surface singularities. It may be shown that the simultaneous resolution of Theorem 4.25 restricts to the minimal resolution above the special fiber, which

is the initial singularity. Moreover, it restricts to isomorphisms above the generic fibers. Using again Ehresmann's theorem, one gets:

Proposition 4.27. *The Milnor fibers associated to the Artin component of a rational surface singularity are diffeomorphic to a tubular neighborhood of the exceptional divisor of the minimal resolution of the singularity.*

Theorems 4.18 and 4.20 imply that all smoothings of Gorenstein normal surface singularities have the same Milnor number $\mu = \mu_0 + \mu_+ + \mu_-$. This is analogous to the following theorem of Buchweitz and Greuel [25], proved before by Milnor [129, Theorem 10.5] for *plane* curve singularities (that is, hypersurface singularities of dimension 1):

Theorem 4.28. *Let (X, x) be a smoothable reduced curve singularity with $r \geq 1$ irreducible components. Then all its Milnor fibers are connected and their first Betti number μ depends only on analytical invariants of (X, x):*

$$\mu = 2\,\delta(X, x) - r + 1.$$

For arbitrary normal surface singularities, the Milnor number may vary among the smoothing components, as illustrated by Pinkham's example described in Subsection 4.4. But its variation is determined by the variation of the dimension of the corresponding smoothing component in the miniversal base space. This results from Theorem 4.29 below, which is a consequence of results of Wahl [198, Theorem 3.13], Greuel and Looijenga [66] and Looijenga [115, Appendix].

In order to state it, we need to introduce the following analog of the geometric genus (see Definition 3.31):

$$\theta := \dim_{\mathbb{C}} R^1 \pi_* \Theta_{\tilde{X}}$$

where $\pi : \tilde{X} \to X$ is any resolution of singularities and $\Theta_{\tilde{X}}$ is the sheaf of germs of holomorphic vector fields on $\tilde{X}$. As for the geometric genus, it may be seen that this definition is independent of the resolution, and that $\theta \in \mathbb{Z}_+$.

Theorem 4.29. *Let X be a normal surface singularity. Consider a smoothing component of its reduced miniversal base space. Denote by $\beta > 0$ its dimension and by μ the Milnor number of a corresponding Milnor fiber. Then, working with the minimal resolution of (X, x):*

(4.5) $$2\,\mu - \beta = 14\,p_g - \theta - 2\,b_1(\partial(X, x)) + 2\,|I|.$$

Note that the right-hand side of equation (4.5) is an analytic invariant of (X, x). The specification that one works with the minimal resolution (not necessarily a normal crossings one!) allows to determine the value of $|I| = b_2(E)$.

Wahl proved a prototype of Theorem 4.29 in [198, Theorem 3.13]. He used another definition of the number β, still depending only on the smoothing component whose Milnor fiber is analysed. He used also the hypothesis that there exists a smoothing of the singularity (X, x) inside the given smoothing component which may be *globalized*, in the sense that it may be realized by a global deformation of a projective variety with only one singularity. Wahl conjectured that for an isolated singularity of arbitrary dimension, his definition of the number β could be interpreted as the dimension of the smoothing component under consideration ([198, Conjecture 4.2]). This conjecture was proved by Greuel and Looijenga in [66]. Moreover, Looijenga proved in [115, Appendix] that all smoothings of isolated singularities could be globalized. In this way one arrives at the statement of Theorem 4.29.

In fact, Wahl stated in the following way relation (4.5):

$$\beta = \theta - 14\,p_g + 2(\chi(F) - \chi(E)),$$

in which χ denotes the Euler-Poincaré characteristic and E denotes the exceptional divisor of the minimal resolution. We leave as an exercise to the reader to show that both relations are equivalent (indication: use Proposition 3.30 (4)).

For more details on Milnor fibers of smoothings of normal surface singularities, one may consult Wahl [198]**, Greuel and Looijenga** [66]**, Looijenga and Wahl** [116]**, Wahl** [199] **as well as the surveys** [11] **of Behnke and Riemenschneider and** [135] **of Némethi.**
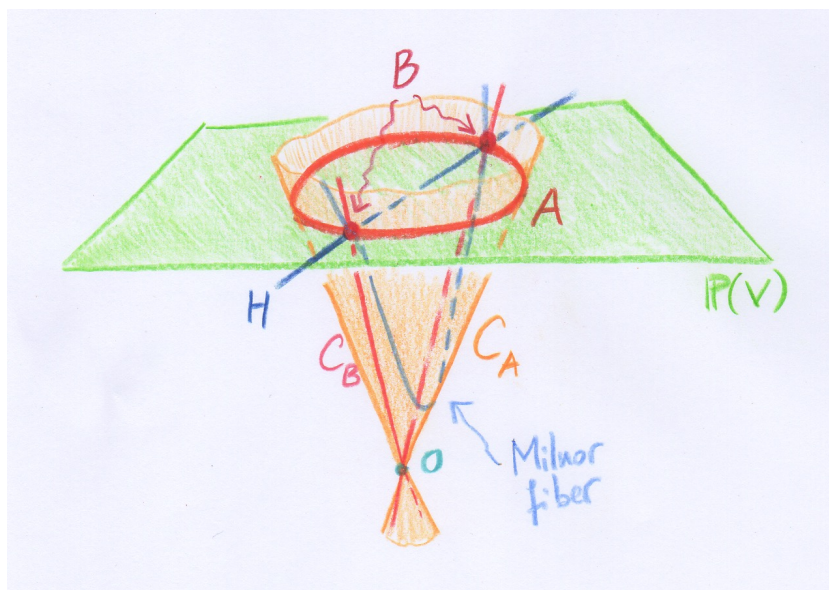
Figure 4.4. The method of sweeping out the cone with hyperplane sections

4.4. Pinkham's example with two smoothing components

Let us begin this section by explaining a general method for constructing smoothings, by "*sweeping out the cone with hyperplane sections*", in the words of Pinkham [158, Page 46]. It is probably the easiest way to construct smoothings, which explains why a drawing similar to Figure 4.4 was represented on the cover of Stevens' book [182]. My explanation follows the one I gave in [165, Section 4].

Let V be a complex vector space, whose projectivisation is denoted $\mathbb{P}(V)$: set-theoretically, it consists of the lines of V. Let A be a smooth subvariety of $\mathbb{P}(V)$. Denote by $C_A \hookrightarrow V$ the *affine cone* over it, and by $\overline{C_A} \hookrightarrow \overline{V}$ the associated *projective cone*. Here $\overline{V}$ denotes the projective space of the same dimension as V, obtained by adjoining $\mathbb{P}(V)$ to V as hyperplane at infinity. That is:

$$\overline{V} = \mathbb{P}(V \oplus \mathbb{C}) = V \cup \mathbb{P}(V).$$

The projective cone $\overline{C_A} = C_A \cup A$ is the Zariski closure of C_A in $\overline{V}$. The *vertex* of either cone is the origin O of V.

Assume now that $H \hookrightarrow \mathbb{P}(V)$ is a projective hyperplane which intersects A *transversally*. Denote by:

$$B := H \cap A$$

the corresponding hyperplane section of A. The affine cone C_H over H is the linear hyperplane of V whose projectivisation is H. The associated projective cone $\overline{C_H} \hookrightarrow \overline{V}$ is a projective hyperplane of $\overline{V}$.

Let L be the pencil of hyperplanes of $\overline{V}$ generated by $\mathbb{P}(V)$ and $\overline{C_H}$. That is, it is the pencil of hyperplanes of $\overline{V}$ passing through the "axis" H. In restriction to V, it consists in the levels of any linear form $f : V \to \mathbb{C}$ whose kernel is C_H. The 0-locus of $f|_{C_A}$ is the affine cone C_B over B.

As an immediate consequence of the fact that H intersects A transversally, we see that C_B has an isolated singularity at O and that all the non-zero levels of $f|_{C_A}$ are smooth. Moreover, f is not a zero-divisor in the local ring of C_A at its vertex. This shows that:

Lemma 4.30. *The map $f|_{C_A} : C_A \to \mathbb{C}$ gives a smoothing of the isolated singularity (C_B, O).*

Since the complement $C_A \setminus O$ of the vertex in the cone C_A is invariant under the natural $\mathbb{C}^*$-action by scalar multiplication on V, the Milnor fibers of $f|_{C_A} : (C_A, O) \to (\mathbb{C}, 0)$ are diffeomorphic

to the global (affine) fibers of $f|_{C_A} : C_A \to \mathbb{C}$. Those fibers are the complements $(W \cap \overline{C_A}) \setminus B$, for the members W of the pencil L different from $\overline{C_H}$ and $\mathbb{P}(V)$. But the only member of this pencil which intersects $\overline{C_A}$ non-transversally is $\overline{C_H}$, which shows that the pair $(W \cap \overline{C_A}, B)$ is diffeomorphic to $(\mathbb{P}(V) \cap \overline{C_A}, B) = (A, B)$. Therefore:

Proposition 4.31. *The Milnor fibers of the smoothing $f|_{C_A} : (C_A, O) \to (\mathbb{C}, 0)$ of the singularity (C_B, O) are diffeomorphic to the affine subvariety $A \setminus B$ of the affine space $\mathbb{P}(V) \setminus H$.*

The previous method may be applied to construct smoothings of germs of affine cones C_B at their vertices. In order to apply it, one has therefore to find another subvariety A of the same projective space, containing B, such that B is a section of A by a hyperplane intersecting it transversally. In general, this is a difficult problem.

An important point to be understood here is that, *even if (C_A, O) is normal, this is not necessarily the case for its hyperplane section (C_B, O)*. More generally, if (Y, y) is a *normal* isolated singularity and $f : (Y, y) \to (\mathbb{C}, 0)$ is a holomorphic function such that the germ $(f^{-1}(0), y)$ is reduced and with isolated singularity, this germ is not necessarily normal. Using this observation and the method of sweeping out the cone, I proved in [165] the following proposition which has to be contrasted with the fact that simple elliptic singularities are smoothable only for a finite number of topological types (see Example 4.19):

Proposition 4.32.

1. *For every simple elliptic singularity (X, x), there exists a smoothable isolated singularity whose normalization is (X, x). Moreover, one may realize it as the germ of a cone at its vertex.*

2. *For every normal surface singularity (X, x), there exists a smoothable isolated singularity whose normalization is (X, x)*

In [158], Pinkham developed a deformation theory for the singularities of normal cones at their vertices, and more generally for the singularities of normal complex affine varieties endowed with a $\mathbb{C}^*$-action. This class contains that of quotient singularities (see Definition 2.29) because the given linear action of the finite group on $\mathbb{C}^n$ commutes with the usual $\mathbb{C}^*$-action by scalar multiplication, which shows that this last action descends to the quotient of $\mathbb{C}^n$ by the finite group.

Pinkham looked in particular at the germs at their vertices of the affine cones over all the *rational normal curves*:

Definition 4.33. Consider an integer $p \in \mathbb{Z}_+^*$. A **rational normal curve of degree** p is a projective curve in $\mathbb{P}^p$ which is projectively equivalent to the image of the following **Veronese embedding**:

$$v_p : \quad \mathbb{P}^1 \quad \longrightarrow \quad \mathbb{P}^p$$
$$[x : y] \quad \longrightarrow \quad [x^p : x^{p-1}y : x^{p-2}y^2 : \cdots : y^p].$$

One may show that the rational normal curves in $\mathbb{P}^p$ are precisely the irreducible curves of degree p in $\mathbb{P}^p$ which are not contained in any hyperplane.

The singularities of the cones over the rational normal curves are special types of cyclic quotient singularities (see Definition 3.40):

Proposition 4.34. *The singularity of the cone over the rational normal curve of degree p is the cyclic quotient singularity $(\mathcal{X}_{p,1}, 0)$.*

Let me explain a proof of this proposition, as it illustrates the way one may arrive at defining systems of equations for *all* quotient surface singularities. This proof generalizes the arguments given in Example 2.31.

Let us start from the defining action

$$(\xi, (x, y)) \longrightarrow (\xi x, \xi y)$$

of the singularity $(\mathcal{X}_{p,1}, 0)$. In order to compute the quotient of the plane $\mathbb{C}^2$ with coordinates (x, y) by this action, one looks first at the subalgebra of $\mathbb{C}[x, y]$ containing the polynomials which are invariant under the action. An invariant polynomial is a sum of invariant monomials, because the monomials are eigenvectors of the action. As $(\xi, x^a y^b) \to (\xi x)^a (\xi y)^b = \xi^{a+b} x^a y^b$, we see that the monomial $x^a y^b$ is invariant if and only if $a + b \equiv 0 \pmod{p}$. One deduces easily from this fact that the subalgebra of invariant polynomials is:

$$\mathbb{C}[x^p, x^{p-1}y, x^{p-2}y^2, \ldots, y^p].$$

This is the algebra of restrictions of polynomials in $\mathbb{C}[z_0, z_1, z_2, \ldots, z_p]$ to the image of the polynomial morphism:

(4.6)
$$
\begin{array}{ccc}
\mathbb{C}^2 & \longrightarrow & \mathbb{C}^{p+1} \\
(x, y) & \longrightarrow & (x^p, x^{p-1}y, x^{p-2}y^2, \ldots, y^p)
\end{array} .
$$

Looking at Definition 4.33, one recognizes this image as the cone over a rational normal curve of degree p, which proves Proposition 4.34.

For the moment, this cone is described parametrically. One may show that a defining set of polynomials for it may be obtained by taking the 2×2 minors of the following matrix:

(4.7)
$$
\begin{pmatrix}
z_0 & z_1 & \cdots & z_{p-1} \\
z_1 & z_2 & \cdots & z_p
\end{pmatrix}.
$$

We will use this observation in a few moments.

Pinkham proved the following strengthened form of Proposition 4.15:

Proposition 4.35. *The singularities at the vertices of the cones over the rational normal curves have smooth miniversal base spaces, with the only exception of the singularity $(\mathcal{X}_{4,1}, 0)$, whose miniversal base space has two irreducible components which are both smooth, of dimensions 1 and 3, and intersect transversally.*

Among the two irreducible components of the miniversal base of $(\mathcal{X}_{4,1}, 0)$, the one of dimension 3 is the Artin component characterized in Theorem 4.25. Of course, for the other cones over the rational normal curves, the only component is also the Artin component. Pinkham proved that in all cases, the total space of the deformation of $(\mathcal{X}_{p,1}, 0)$ above the Artin component may be defined by taking the 2×2 minors of the following deformation of the matrix (4.7):

(4.8)
$$
\begin{pmatrix}
z_0 & z_1 + t_1 & \cdots & z_{p-1} + t_{p-1} \\
z_1 & z_2 & \cdots & z_p
\end{pmatrix}.
$$

The deformation morphism is obtained by restricting to this total space the projection of the complex space $\mathbb{C}^{2p}$ with coordinates $(z_0, \ldots, z_p, t_1, \ldots, t_{p-1})$ to the "parameter space" $\mathbb{C}^{p-1}$ with coordinates $(t_1, \ldots, t_{p-1})$.

The fact that in the case $p = 4$ one gets a second type of deformation originates in the possibility to represent in this case the collection of defining polynomials of $(\mathcal{X}_{4,1}, 0)$ as the set of 2×2 minors of the second matrix:

$$
\begin{pmatrix}
z_0 & z_1 & z_2 \\
z_1 & z_2 & z_3 \\
z_2 & z_3 & z_4
\end{pmatrix}.
$$

One may deform this system of equations by taking the 2×2 minors of the following deformed matrix:

$$
\begin{pmatrix}
z_0 & z_1 & z_2 \\
z_1 & z_2 + s & z_3 \\
z_2 & z_3 & z_4
\end{pmatrix}.
$$

Note that here one has one deformation parameter s, which parametrizes the component of dimension 1 in the miniversal base space of $(\mathcal{X}_{4,1}, 0)$.

In the affine space $\mathbb{C}^6$ with coordinates $(z_0, \ldots, z_4, s)$, consider the new coordinate (linear form) $z_5 := z_2 + s$. The total space Y of the deformation may therefore be defined by the vanishing of the 2×2 minors of the matrix:

$$\begin{pmatrix} z_0 & z_1 & z_2 \\ z_1 & z_5 & z_3 \\ z_2 & z_3 & z_4 \end{pmatrix}.$$

In the same way as the cone over the rational normal curve of degree p could be described parametrically by the morphism (4.6), one may also describe Y parametrically by the morphism:

$$V: \quad \begin{array}{ccc} \mathbb{C}^3 & \longrightarrow & \mathbb{C}^6 \\ (u, v, w) & \longrightarrow & (u^2, uw, uv, vw, v^2, w^2) \end{array} '$$

where the parameters of $\mathbb{C}^6$ are $(z_0, \ldots, z_4, z_5)$. The induced morphism $\mathbb{P}^2 \to \mathbb{P}^5$ is called also a **Veronese embedding**, this time of the projective plane as a surface of degree 2 in $\mathbb{P}^5$. Its image is the **Veronese surface** $A \hookrightarrow \mathbb{P}^5$. Therefore Y gets identified with the cone C_A over this projective surface.

The morphism V gets reexpressed in the following way if one passes to the initial coordinate system $(z_0, \ldots, z_4, s)$ on $\mathbb{C}^6$:

(4.9)
$$V: \quad \begin{array}{ccc} \mathbb{C}^3 & \longrightarrow & \mathbb{C}^6 \\ (u, v, w) & \longrightarrow & (u^2, uw, uv, vw, v^2, w^2 - uv) \end{array} .$$

This shows that the smoothing is obtained by the method of sweeping out the cone by the hyperplane section. The total space Y of the smoothing is the cone C_A over the Veronese surface A. The kernel of the linear form s which defines the deformation parameter intersects C_A along the cone C_B over a curve $B \hookrightarrow A$. Using the coordinates (u, v, w) on C_A and looking at the last coordinate of (4.9), we see that C_B is defined by the equation $w^2 - uv = 0$. Proposition 4.31 implies:

Proposition 4.36. *The Milnor fibers associated to the irreducible component of dimension* 1 *of the miniversal base of the cyclic quotient singularity* $(\mathcal{X}_{4,1}, 0)$ *are diffeomorphic to the complement of a smooth conic curve in the complex projective plane.*

What is the Milnor number of those Milnor fibers? One could use Theorem 4.29, which implies that:

(4.10)
$$2\mu_3 - 3 = 2\mu_1 - 1.$$

Here we denote by μ_k the Milnor number of the Milnor fiber associated to the component of dimension k. As the component of dimension 3 is the Artin component, we know that its Milnor fiber is diffeomorphic to a tubular neighborhood of the exceptional divisor in the minimal resolution. Therefore $\mu_3 = 1$, and the equality (4.10) implies that:

(4.11)
$$\mu_1 = 0.$$

In fact, it is possible to understand better the topological structure of this Milnor fiber, in such a way that the previous equality will appear obvious. Indeed, one has the following result of Lê, Seade and Verjovsky [106, Theorem 1.1] (see also Seade [176, Chapter V]), valid in arbitrary dimension:

Proposition 4.37. *The complement of a smooth quadric hypersurface in the complex projective space* $\mathbb{CP}^n$ *is diffeomorphic to the total space of the tangent bundle of the real projective space* $\mathbb{RP}^n$.

Let us see a proof of this proposition. The key point is to look at the special quadric hypersurface Q defined by the equation

(4.12)
$$z_0^2 + z_1^2 + \cdots + z_n^2 = 0$$

inside the complex projective space $\mathbb{CP}^n$ with homogeneous coordinates $[z_0 : z_1 : \cdots : z_n]$. The cone C_Q over Q is the affine hypersurface of $\mathbb{C}^{n+1}$ defined by the same equation (4.12). Let us consider

the defining quadratic form q of C_Q:

$$q : \quad \begin{array}{ccc} \mathbb{C}^{n+1} & \longrightarrow & \mathbb{C} \\ (z_0, z_1, \ldots, z_n) & \longrightarrow & z_0^2 + z_1^2 + \cdots + z_n^2 \end{array}$$

and its level 1:

$$F := q^{-1}(1).$$

The polynomial q being homogeneous, the affine manifold F is diffeomorphic with the Milnor fiber of q at the origin of $\mathbb{C}^{n+1}$.

Let us denote $Z := (z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$ and decompose it into its real and imaginary parts:

$$Z = X + iY, \; X, Y \in \mathbb{R}^{n+1}.$$

The defining equation $q = 1$ of F is equivalent to the system of equations:

(4.13)
$$\begin{cases} \|X\|^2 = \|Y\|^2 + 1 \\ X \cdot Y = 0 \end{cases}$$

in which $\| \cdot \|$ denotes the standard euclidean norm on $\mathbb{R}^{n+1}$, obtained by restricting the quadratic form q from $\mathbb{C}^{n+1}$ to $\mathbb{R}^{n+1}$, and $X \cdot Y$ denotes the associated scalar product.

Proposition 4.38. *The map:*

$$\Psi : \quad \begin{array}{ccc} F & \longrightarrow & \mathbb{S}^n \times \mathbb{R}^{n+1} \\ Z = X + iY & \longrightarrow & \left(\dfrac{X}{\|X\|}, Y \right) \end{array}$$

induces a diffeomorphism from F to the total space of the tangent bundle $T\mathbb{S}^n$, which sends the antipodal map:

$$A : \quad \begin{array}{ccc} F & \longrightarrow & F \\ Z & \longrightarrow & -Z \end{array}$$

into the differential of the antipodal map of $\mathbb{S}^n$.

Here $\mathbb{S}^n$ is the unit sphere in $\mathbb{R}^{n+1}$ for the norm $\| \cdot \|$. The total space $T\mathbb{S}^n$ of its tangent bundle is realized as the submanifold of $\mathbb{S}^n \times \mathbb{R}^{n+1}$ formed by the couples (X, Y) with Y orthogonal to X. The equations (4.13) show that the image of Ψ is included in $T\mathbb{S}^n$. They allow to construct explicitly an inverse, which may be seen to be smooth. This proves the first statement of the previous proposition. The property about antipodal maps is also easy to check directly using the equations (4.13).

Proposition 4.37 is now a direct consequence of Proposition 4.38. Indeed, by restricting to F the projectivisation map

$$\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n,$$

we get a Galois cover of degree 2 of the complement $\mathbb{CP}^n \setminus Q$. The Galois group acts precisely as the antipodal map A on F. Therefore, the complement $\mathbb{CP}^n \setminus Q$ is diffeomorphic to the quotient of F by this antipodal map. By Proposition 4.38, we see that $\mathbb{CP}^n \setminus Q$ is diffeomorphic to the quotient of $T\mathbb{S}^n$ by the diffeomorphism of the antipodal map on $\mathbb{S}^n$. But this last quotient is $T\mathbb{RP}^n$, which ends the proof of Proposition 4.37.

Returning to Pinkham's example, we may summarize the previous results as the following supplement of Proposition 4.15:

Proposition 4.39. *The Milnor fibers of the cyclic quotient singularity $(\mathcal{X}_{4,1}, 0)$ are diffeomorphic to:*

- *the total space of the complex line bundle of degree -4 over $\mathbb{CP}^1$, for the component of dimension 3 (the Artin component);*

- *the total space of the tangent bundle to the real projective plane $\mathbb{RP}^2$, for the component of dimension 1.*

	Affine geometry	**Complex geometry**
(1)	The model real line $\mathbb{R}$	The model complex line $\mathbb{C}$
(2)	An open subset $U \subset \mathbb{R}$	An open subset $U \subset \mathbb{C}$
(3)	An affine function $U \to \mathbb{R}$	A holomorphic function $U \to \mathbb{C}$
(4)	An affine function $U \to \mathbb{R}$	A harmonic function $U \to \mathbb{R}$
(5)	The relation $h\left(\dfrac{a+b}{2}\right) = \dfrac{h(a)+h(b)}{2}$	The relation $h(A) = \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{S}^1_{A,r}} h\, ds$
(6) ¿½	On a compact interval an affine function is determined by its boundary values	On a compact disc a harmonic function is determined by its boundary values
(7)	The relation $h'' = 0$	The relation $\Delta h = 0$
(8)	The condition $\rho'' \geq 0$	The condition $\Delta \rho \geq 0$
(9)	A (strictly) convex function $\rho : U \to \mathbb{R}$	A (strictly) subharmonic function $\rho : U \to \mathbb{R}$
(10)	An affine manifold of dimension 1 (a real affine curve)	A complex manifold of dimension 1 (a complex curve)
(11)	A real affine manifold	A complex manifold
(12)	An affine map from $U \subset \mathbb{R}$ to a real affine manifold	A holomorphic map from $U \subset \mathbb{C}$ to a complex manifold
(13)	An affine real-valued function on a real affine manifold	A holomorphic complex-valued function on a complex manifold
(14)	An affine real-valued function on a real affine manifold	A pluriharmonic real-valued function on a complex manifold
(15)	The condition $\mathrm{Hess}\,\rho \geq 0$ on a real affine manifold	The condition $-dd^c\rho \geq 0$ on a complex manifold
(16)	A (strictly) convex function on a real affine manifold	A (strictly) plurisubharmonic function on a complex manifold

Figure 5.1. Analogies between real affine geometry and complex geometry

As the total space of the tangent bundle $T\mathbb{RP}^2$ retracts by deformation onto $\mathbb{RP}^2$, they have the same second Betti number. As $\mathbb{RP}^2$ is a non-orientable surface, we see that $H_2(\mathbb{RP}^2, \mathbb{Z}) = 0$. This implies indeed that the Milnor number of the corresponding Milnor fibers is 0, as stated in relation (4.11).

Nowadays one knows handlebody presentations of the Milnor fibers of *all* cyclic quotient singularities. This is a consequence of the proof of a conjecture of Lisca [112, Page 16] by Némethi and myself [137]. This conjecture related the smoothings of cyclic quotient singularities with the *Stein fillings* of their *contact boundaries*. In order to explain this conjecture and the principle of its proof, let us pass now to the basics of contact topology.

A different way of analysing the structure of the Milnor fiber of $(\mathcal{X}_{4,1}, 0)$ **above the miniversal component of dimension** 1**, valid for more general singularities, was described by Wahl** [198, Sections 5.8, 5.9]**.**

5. Plurisubharmonic functions, Stein manifolds and contact manifolds

5.1. Basic analogies between affine geometry and complex geometry

When one is thinking about the geometry or topology of *complex manifolds*, it may be useful to look for analogies with the geometry of *real manifolds endowed with an affine structure*. In Figure 5.1 one may find a table of basic such analogies. Let us make a few comments about them.

- In lines (1) and (2) appear the basic building blocks allowing to construct manifolds of the two kinds, by performing cartesian products in order to augment the dimension and gluing of charts in order to enrich the topology.

- In lines (3) and (4) we state that both holomorphic functions $U \to \mathbb{C}$ and harmonic functions $U \to \mathbb{R}$ (where U is an open subset of $\mathbb{C}$) may be seen as analogs of affine functions $U \to \mathbb{R}$ (where U is an open subset of $\mathbb{R}$). In the first case one thinks of both types of maps as structure-preserving ones. In the second case one thinks about them as functions with the property that their values at the center of any ball (a segment, respectively a disc) is equal to the mean of their values on the boundary of the ball. Those relations are explicitly written for both cases in line (5). They imply that the value at the center of the ball is determined by the boundary values. More generally, one has the property of line (6). It is a good exercise to write the corresponding generalizations of the relations of line (5) and to understand in which sense the ingredients of those generalizations are analog of each other.

- On line (7), h'' denotes the second derivative of the function h defined on an open subset of $\mathbb{R}$ and $\Delta h := \left(\partial_x^2 + \partial_y^2 \right) h$ denotes the *Laplacian* of the function h defined on an open subset of $\mathbb{C} \simeq \mathbb{R}^2$. The relation $h'' = 0$ is identically true if and only if h is affine, and the relation $\Delta h = 0$ is identically true if and only if h is harmonic.

- The conditions written on line (8), generalizations of those of line (7), are definitions of the notions written in the same column on line (9) (for sufficiently regular functions). The strict inequalities correspond to the corresponding strict notions. Note that on a compact interval a convex function has a graph lying below the graph of the unique affine function with the same boundary values. One may show that the analogous property holds for a subharmonic function on a compact disc in $\mathbb{C}$: its graph lies below the graph of the unique harmonic extension of its boundary values to the whole disc. This is in fact the reason of the standard name "*subharmonic*". The analogous name for a convex function on a real interval would be "*subaffine*".

- Both types of curves on line (10) may be defined by gluing open sets from line (2) using homeomorphisms from line (3).

- Both types of manifolds on line (11) may be locally described as cartesian products of open sets from line (2). Then the gluing of charts is done using tuples of functions which restrict to functions from line (3) on each factor of such a product.

- In line (12) one has the notion of parametrized distinguished subobjects of dimension 1 in both geometries. As for the affine space $\mathbb{R}^n$ one gets simply the affinely parametrized lines, one may think about them as *geodesics*. In model charts, both types of geodesics may be described using tuples of maps from line (3).

- In lines (13) and (14) we mentioned that both holomorphic functions and pluriharmonic functions on a complex manifold may be seen as analogs of affine real-valued functions on a real affine manifold. In the first case we get in restriction to geodesics the types of functions from line (3) and in the second case we get the types of functions from line (4).

- The conditions of line (15) are generalizations to manifolds of arbitrary dimension of the conditions of line (8). In fact, in both cases a real-valued function ρ satisfies them identically if and only if it satisfies the corresponding condition (8) in restriction to any geodesic (map of the kind described in line (12)). The *Hessian* Hessρ is a field of quadratic forms on the tangent bundle of the affine manifold defined in the following way: for any point p and tangent vector V at p, extend canonically V by parallelism to a neighborhood of p (this extension is made possible by the presence of the affine structure) and define (Hess ρ)(V) as the second Lie

derivative $L_V^2 \rho$ of ρ in the direction of V. The object $-dd^c\rho$ is a real-valued differential form of degree 2, whose definition will be explained below.

- The functions on line (16) are definable by the corresponding condition on line (15).

We won't pursue these analogies here. We mentioned them only because we believe that real affine geometry is simpler to grasp in first instance than complex geometry.

The reader interested in developing her/his intuition about such analogies may consult the books [85] **of Hörmander and** [33] **of Cieliebak and Eliashberg.**

5.2. Plurisubharmonic functions

We have stated in the last line of Figure 5.1 that plurisubharmonic functions are analogs of convex functions on affine manifolds. Their precise definition is:

Definition 5.1. A smooth real-valued function ρ defined on a complex manifold M is **plurisubharmonic** (abbreviated **psh**) if $-dd^c\rho \geq 0$. It is called **strictly plurisubharmonic** (abbreviated **spsh**) if one has the stronger inequality $-dd^c\rho > 0$.

Here d denotes the operator of exterior differentiation of differential forms. Its definition needs only the underlying structure of differential manifold of the complex manifold M under consideration. By contrast, the operator d^c uses the complex structure:

$$(5.1) \qquad d^c\rho := d\rho \circ J,$$

where J is the field of multiplications by i on the complex tangent bundle of the manifold, identified as a real vector bundle with the real tangent bundle of the underlying differentiable manifold. In particular:

$$(5.2) \qquad J^2 = -Id_{TM}.$$

Introduce now the following real-valued differential forms on M:

$$(5.3) \qquad \begin{aligned} \lambda_\rho &:= -d^c\rho \\ \omega_\rho &:= d\lambda_\rho. \end{aligned}$$

By definition 5.1, the function $\rho : M \to \mathbb{R}$ is psh if and only if $\omega \geq 0$. What does it mean for a differential form of degree 2 to be non-negative ? This is again a concept which is defined using the complex structure:

Definition 5.2. A smooth real-valued differential 2-form ω on M is called **non-negative**, written $\omega \geq 0$ (respectively **positive**, written $\omega > 0$) if it is J-**invariant**, that is:

$$(5.4) \qquad \omega(JV, JW) = \omega(V, W)$$

for all tangent vectors V, W of M based at the same point and if:

$$(5.5) \qquad \omega(V, JV) \geq 0 \text{ (respectively } \omega(V, JV) > 0)$$

for all non-zero tangent vectors to M.

The **associated Riemannian metric** of a positive 2-form ω is defined by:

$$(5.6) \qquad g(V, W) := \omega(V, JW).$$

Note that the inequality $g(V, V) > 0$ is true for all non-zero tangent vectors, as an immediate reformulation of the strict inequality (5.5) and that the symmetry property $g(V, W) = g(W, V)$ results from (5.4), replacing first W by JW and using then the identity (5.2). Therefore, g is indeed a Riemannian metric for any positive 2-form ω.

It is simple to verify that the 2-form ω_ρ is J-invariant for any smooth real-valued function ρ defined on M. Therefore, in order to check whether ρ is psh or spsh, one has to check simply whether one has the corresponding inequality (5.5).

Let us restrict for the moment to the case where M is an open subset of $\mathbb{C}$. The next result shows that in this case the condition of line (15) of Figure 5.1 allowing to define (s)psh functions is equivalent to the (strict) condition of line (8), allowing to define (strictly) subharmonic functions. This explains the qualificative "*plurisubharmonic*": such a function is subharmonic in restriction to *all* the parametrized "geodesics" of the complex manifold.

Proposition 5.3. *Let $U \subset \mathbb{C} \simeq \mathbb{R}^2$ be open and let $\rho : U \to \mathbb{R}$ be smooth. Then:*

(5.7)
$$- dd^c \rho = (\Delta \rho)\, dx \wedge dy.$$

As a consequence, the condition $-dd^c\rho \geq 0$ is equivalent to the condition $\Delta\rho \geq 0$.

One may prove easily this proposition using the following relations:

$$z = x + iy$$
$$d\rho = (\partial_z \rho)\, dz + (\partial_{\overline{z}} \rho)\, d\overline{z}$$
$$d^c \rho = (i\partial_z \rho)\, dz - (i\partial_{\overline{z}} \rho)\, d\overline{z}$$
$$\partial_z = \tfrac{1}{2}(\partial_x - i\partial_y)$$
$$\partial_{\overline{z}} = \tfrac{1}{2}(\partial_x + i\partial_y)$$

Assume now that $\rho : M \to \mathbb{R}$ is strictly plurisubharmonic. Introduce also the following notations:

(5.8) $\qquad g_\rho :=$ the Riemannian metric associated to the positive form ω_ρ,

(5.9) $\qquad \Lambda_\rho :=$ the gradient vector field of ρ relative to g_ρ.

I want to explain now that this gradient vector field has also a *symplectic* interpretation. First of all, recall that a **symplectic form** on a smooth manifold is a closed 2-form ω which is non-degenerate at every point. This last condition means that the duality morphism associated to ω

(5.10)
$$\begin{array}{ccc} T_p M & \longrightarrow & T_p^* M \\ V & \longrightarrow & i_V \omega := \omega(V, \cdot) \end{array}$$

is an isomorphism at each point $p \in M$.

A positive 2-form is automatically non-degenerate and a 2-form of the type ω_ρ is automatically closed, because it is exact (see relations (5.3)). Therefore, ω_ρ *is a symplectic form whenever ρ is spsh*. By construction, the 1-form λ_ρ is a primitive of it. It is therefore a *Liouville form* for ω_ρ, and one may consider also its associated *Liouville vector field* in the following sense:

Definition 5.4. Let ω be an exact symplectic form on the smooth manifold M. A **Liouville form** of ω is a smooth primitive λ of it:

(5.11)
$$d\lambda = \omega.$$

A **Liouville vector field** of ω is a smooth vector field Λ such that the associated Lie derivative leaves ω invariant:

(5.12)
$$L_\Lambda \omega = \omega.$$

A Liouville vector field and a Liouville form of ω are **associated to each other** if they correspond by the isomorphism (5.10):

(5.13)
$$i_\Lambda \omega = \lambda.$$

In fact, a vector field Λ is Liouville if and only if the 1-form λ corresponding to it through equation (5.13) is Liouville. This is an immediate consequence of **Cartan's relation**

$$L_\Lambda = i_\Lambda \circ d + d \circ i_\Lambda$$

applied to ω.

The announced symplectic interpretation of the gradient vector field Λ_ρ is as a *Liouville vector field* of the symplectic form ω_ρ:

Proposition 5.5. *The gradient vector field Λ_ρ of the spsh function ρ relative to the Riemannian metric g_ρ is equal to the Liouville vector field of ω_ρ associated to its Liouville form λ_ρ.*

Indeed, the defining relation

$$d\rho(V) = g_\rho(\Lambda_\rho, V)$$

of the gradient vector field Λ_ρ (for all tangent vectors V) may be successively reformulated in the following ways:

$$
\begin{aligned}
&\qquad\qquad d\rho(V) = g_\rho(\Lambda_\rho, V) \\
&\overset{(5.6)}{\Longleftrightarrow}\quad d\rho(V) = \omega_\rho(\Lambda_\rho, JV) \\
&\overset{V \leftarrow JV}{\Longleftrightarrow}\quad d\rho(JV) = \omega_\rho(\Lambda_\rho, J^2 V) \\
&\overset{(5.1)+(5.2)}{\Longleftrightarrow}\quad d^c\rho(V) = \omega_\rho(\Lambda_\rho, -V) \\
&\overset{(5.3)}{\Longleftrightarrow}\quad \lambda_\rho(V) = \omega_\rho(\Lambda_\rho, V) \\
&\overset{\forall V}{\Longleftrightarrow}\qquad \lambda_\rho = i_{\Lambda_\rho}\omega_\rho.
\end{aligned}
$$

This last relation proves Proposition 5.5.

One of the simplest examples of spsh functions is the squared-distance function ρ_0 to the origin in $\mathbb{C}^n$ (see Equation (3.4)):

Proposition 5.6. *The function $\rho_0(z_1, \ldots, z_n) := |z_1|^2 + \cdots + |z_n|^2$ is spsh on $\mathbb{C}^n$. Its associated Riemannian metric g_{ρ_0} is the standard Euclidean metric on $\mathbb{C}^n = \mathbb{R}^{2n}$ and its associated Liouville vector field Λ_{ρ_0} is the radial vector field: $\Lambda_{\rho_0}(p) = p$ for all $p \in \mathbb{C}^n$.*

The restriction of a spsh function on a complex manifold to a submanifold is again spsh. For this reason, the restriction ρ of ρ_0 to any complex submanifold X of $\mathbb{C}^n$ is again spsh. If X is moreover a closed subset of $\mathbb{C}^n$, then ρ is *proper* and *bounded from below* (as a consequence of the fact that ρ_0 has these two properties). Therefore X is a *Stein manifold*, in the following sense:

Definition 5.7. A **Stein manifold** is a complex manifold which admits a proper spsh function bounded from below.

As we have seen, the closed complex submanifolds of some model complex space $\mathbb{C}^n$ are Stein. Grauert [61] proved the converse:

Theorem 5.8. *Any connected Stein manifold may be embedded as a closed complex submanifold of some complex affine space $\mathbb{C}^n$.*

One may see alternative characterizations of Stein manifolds in Cieliebak and Eliashberg [33, Section 5.3].

By arbitrarily C^2-small perturbations, any proper spsh function bounded from below on a Stein manifold X may be transformed into a function of the same kind which is moreover *Morse*. For instance, if the function is the restriction of ρ_0, given some embedding of X in $\mathbb{C}^n$, then one may take instead the squared-distance function to a generic point in some arbitrarily fixed neighborhood of the origin of $\mathbb{C}^n$.

Once one has a proper Morse function bounded from below on a smooth manifold, a fundamental theorem of Morse theory (see Milnor [128, Theorem 3.5]) implies that the manifold has the homotopy type of a CW-complex of dimension equal to the maximal index of the critical points of the function.

Strictly pluri-subharmonic functions have constrained indices:

Proposition 5.9. *The indices of the critical points of a Morse spsh function on a complex manifold of complex dimension n are at most equal to n.*

Therefore, one has the following fundamental topological property of Stein manifolds, discovered by Thom (see Milnor [128, Theorem 7.2]):

Theorem 5.10. *A Stein manifold of complex dimension n has the homotopy type of a CW-complex of dimension at most n. In particular, a Milnor fiber of a smoothable singularity of complex dimension n has the homotopy type of a CW-complex of dimension at most n.*

Let us see an intrinsic proof of Proposition 5.9, which uses the symplectic interpretation of the gradient vector field Λ_ρ stated in Proposition 5.5 (see details in Cieliebak and Eliashberg [33, Section 2.8]). Let $p \in X$ be a critical point of ρ and let D_p be the *stable cell* centered at it, formed as the union of the Λ_ρ-orbits which tend to p when the time goes to $+\infty$. Its dimension is by definition the index $\mathrm{ind}_p(\rho)$ of the function ρ at p. Assume by contradiction that this index is $> n$. Therefore the intersection $T_p D_p \cap J(T_p D_p)$ of the tangent space to the stable cell at p and of its image by the complex multiplication J is of positive dimension. Let $D_p^{\mathbb{C}}$ be the union of the Λ_ρ-orbits which tend to p tangentially to this intersection. It is a smooth ball of even positive dimension contained in D_p. As ω_ρ is symplectic on its tangent space at p, this will also be the case on some neighborhood $\rho^{-1}[\rho(p) - \epsilon, \rho(p)] \cap D_p^{\mathbb{C}}$ of it (where $\epsilon > 0$ is sufficiently small). But the positive flow of Λ_ρ shrinks such a neighborhood strictly, which contradicts the fact that this flow dilates ω_ρ exponentially (see Equation 5.12). This contradiction proves the desired inequality $\mathrm{ind}_p(\rho) \le n$.

For more details about Stein manifolds, one may consult Grauert and Remmert's book [64]**, Peternell's survey** [154] **and Cieliebak and Eliashberg's book** [33]**.**

5.3. Contact manifolds and their fillings

In the previous section we saw that a spsh function ρ on a complex manifold determines canonically a Riemannian metric, a symplectic structure, a vector field and a 1-form, which are related by various relations. We will examine now in which way those objects interact with the smooth levels of ρ. For simplicity, we will drop the dependency on ρ from our notations.

Let $M \hookrightarrow X$ be a regular level of ρ. It is a smooth hypersurface of X seen as a real manifold. Consider the restriction of the Liouville form λ of Equation (5.3) to M:

$$\alpha := \lambda|_M.$$

If V is a tangent vector to X at a point p, we denote by $V^\perp$ the real hyperplane of $T_p X$ which is g-orthogonal to V. One has the following properties, whose proofs we leave as an exercise to the reader (hint: use the equivalence $\lambda(V) = 0 \iff g(J\Lambda, V) = 0$):

Proposition 5.11. *At all the points of the smooth level manifold M of the spsh function ρ one has the following properties (see Figure 5.2):*

1. *$TM = \Lambda^\perp$.*

2. *$\ker \lambda = (J\Lambda)^\perp = J(TM)$.*

3. *$\ker \alpha = TM \cap J(TM)$.*

4. *$d\alpha$ is symplectic in restriction to $\ker \alpha$.*

Therefore, α is a *contact form* on the manifold M, in the following sense:

Definition 5.12. A **contact form** on a smooth manifold M is a smooth real-valued 1-form α such that $d\alpha$ is symplectic in restriction to $\ker \alpha$.

A basic fact about bilinear symplectic forms is that they exist only on even-dimensional vector spaces. As the kernel of a linear form on a vector space is either a hyperplane or the whole vector space, this shows that the field of kernels of a contact form on a manifold is necessarily of constant even dimension. The differential $d\alpha$ being symplectic, the form α cannot be identically 0, which shows that $\ker \alpha$ is a field of hyperplanes on M. This implies that M is of odd dimension and that $\ker \alpha$ is a *contact structure*, in the following sense:
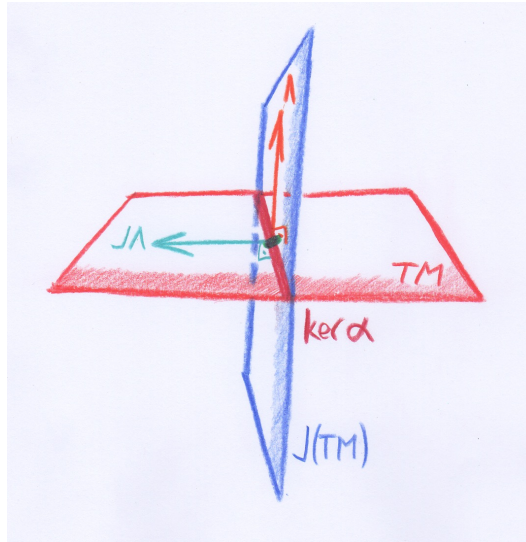
Figure 5.2. An illustration of Proposition 5.11
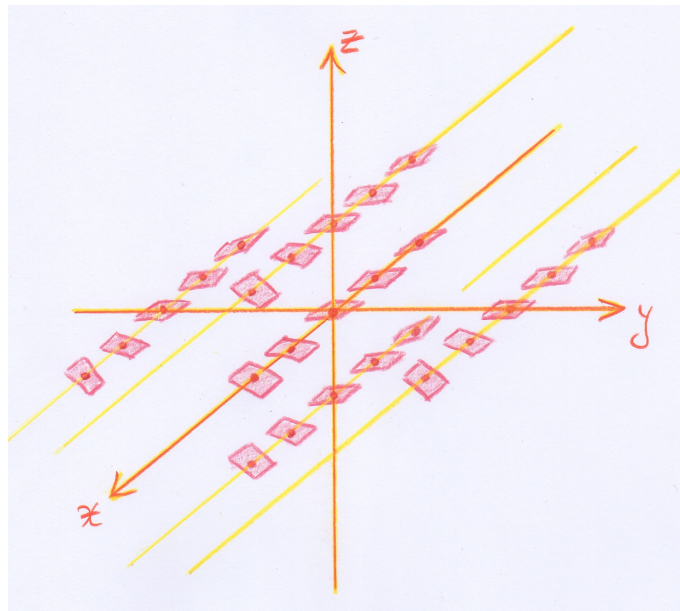
Figure 5.3. The contact structure $\ker(dz + x\,dy)$

Definition 5.13. A **contact structure** on a smooth manifold M is a smooth field ξ of hyperplanes which may be locally defined in the neighborhood of any point of M as the field of kernels of a contact form. A contact structure is called **oriented** if the vector bundle ξ is oriented and **cooriented** if its normal vector bundle TM/ξ is oriented. A manifold endowed with a contact structure is called a **contact manifold**.

Example 5.14. The 1-form $\alpha := dz + x\,dy$ is a contact form on $\mathbb{R}^3$. Its associated contact structure $\ker \alpha$ is represented in Figure 5.3. Note that it is invariant by translations parallel to the plane of coordinates (y, z). For this reason it is enough to understand how varies the plane field along the x-axis. I drew five of those planes, as well several of their translates.

The previous example allows in fact to understand the local structure of *all* contact structures on 3-manifolds. Indeed, in arbitrary dimensions contact forms have no local invariants (like complex

structures and unlike Riemannian structures) as emphasized by the following result of Darboux (see Geiges [57, Theorem 2.5.1]):

Theorem 5.15. *If α is a contact form on a $(2n+1)$-dimensional manifold M, then in the neighborhood of any point of M there exist local coordinates $(x_1, y_1, \ldots, x_n, y_n, z)$ such that:*

$$\alpha = dz + \sum_{k=1}^{n} x_k \, dy_k.$$

Globally the situation is distinct, due to the fact that there is a canonical vector field attached to any contact form: its so-called *Reeb vector field*, uniquely determined by the requirements to be in the kernel of $d\alpha$ and to have length 1 when measured by α. Then any dynamical invariants of the Reeb vector field are invariants of the contact form, which makes one feel that by deforming a form, the global structure may change drastically. In fact one can get subtle invariants from the study of Reeb vector fields. This is the subject of *contact homology*, but we won't speak about it here.

When one works with a contact structure instead of a contact form, the situation becomes completely different. Indeed, on a closed manifold there is an Ehresmann-type theorem, proved by Gray [65] (see also Geiges [57, Theorem 2.2.2]):

Theorem 5.16. *Two homotopic contact structures on a closed manifold are isotopic.*

The previous theorem shows that on closed manifolds, contact structures have no moduli, that is, that their classification up to isotopy is discrete. This is the reason why, when looking at the tangent distribution to a real hypersurface of a complex manifold, one does not keep the field of complex operators $J : \xi \to \xi$ as a supplementary structure. Indeed, then one would keep moduli, that is, the analog of Gray's theorem would not be true.

The first question which one asks in any classification problem is that of *existence* of the objects. In what concerns contact structures, the situation was settled first on closed 3-dimensional manifolds. Before explaining this, let us make a remark about contact structures in dimension 3:

Proposition 5.17. *Any contact structure on a smooth 3-manifold induces a canonical orientation. Therefore, any 3-manifold which may be endowed with a contact structure is orientable.*

In order to understand this fact, let us choose locally a defining contact 1-form α of the given contact structure ξ. As $d\alpha$ is by definition symplectic in restriction to ξ, this shows that the 3-form $\alpha \wedge d\alpha$ is everywhere non-zero on its domain. Therefore, it defines an orientation on this domain. The point is that this orientation is independent of the choice of defining 1-form. Indeed, a second such form α' may be written as $\alpha' = u\alpha$, where u is a smooth and non-vanishing function. Consequently, $\alpha' \wedge d\alpha' = u^2 \alpha \wedge d\alpha$, which shows that the orientations defined by $\alpha' \wedge d\alpha'$ and $\alpha \wedge d\alpha$ are the same.

Now, if a given 3-manifold is already oriented, one may compare a contact form on it with this orientation:

Definition 5.18. A contact form on an oriented 3-manifold is called **positive** if the orientation induced by it coincides with the given orientation.

We may state now the announced theorem about the existence of contact structures on 3-manifolds. The first statement was proved by Martinet [118] and the second one by Lutz [117] (see also Geiges [57, Theorems 4.1.1 and 4.3.1]):

Theorem 5.19. *Any closed oriented 3-manifold carries a positive contact structure. Moreover, one may find a positive contact structure in any homotopy class of cooriented plane fields.*

An analog of this theorem was recently proved in higher dimensions by Borman, Eliashberg and Murphy [19].

Once one knows that contact structures exist on a given closed manifold, it is natural to try to classify them. In view of Gray's Theorem 5.16, one has to classify them up to homotopy. Of course, if two contact structures are not homotopic as hyperplane fields, then they are nor homotopic as

contact structures. Therefore, one has to concentrate on the classification of contact structures in a given homotopy class of hyperplane fields. But how to show that two such contact structures are different? The first method to prove such a result was found by Bennequin [12], using the following notion:

Definition 5.20. An **overtwisted disk** in a contact 3-manifold is an embedded compact disk which is tangent to the contact structure along its boundary. A contact structure is called **overtwisted** if it contains an overtwisted disk and **tight** otherwise.

In fact, one finds several variants of the definition of overtwisted disks in the literature. But all of them lead to the same notions of overtwisted or tight contact 3-manifolds (see Geiges [57, Section 4.5]).

As in Proposition 5.6, let us denote by ρ_0 the squared-distance to the origin in $\mathbb{C}^2$. Its levels are the spheres centered at the origin. As ρ_0 is spsh, Proposition 5.11 implies that the fields of hyperplanes on them invariant by complex multiplication are contact structures. The homotheties centered at the origin identifies them all. We get therefore a well-defined contact structure on $\mathbb{S}^3$, which we will call the **natural contact structure**. It may be shown that its restriction to the complement of any point of $\mathbb{S}^3$ is isomorphic to the contact structure on $\mathbb{R}^3$ introduced in Example 5.14, which we will also call **natural**.

Bennequin proved in [12] that:

Theorem 5.21. *The natural contact structures on $\mathbb{S}^3$ and $\mathbb{R}^3$ are tight.*

By performing so-called *Lutz twists* (see Geiges [57, Section 4.3]), one may transform any contact structure on a 3-manifold into an overtwisted one, which is nevertheless contained in the same homotopy class of plane fields. This shows in particular that one may find on $\mathbb{S}^3$ two contact structures which are homotopic as plane fields, but not as contact structures. In fact, as proved by Eliashberg [45]:

Theorem 5.22. *Each homotopy class of plane fields on an oriented closed 3-manifold contains a unique overtwisted positive contact structure, up to isotopy.*

Therefore, the difficulty is to classify the *tight* contact structures on 3-manifolds. There exist oriented 3-manifolds which do not admit any tight contact structure (see point (1) of the enumeration of results at the end of this section), but it is still an open problem to characterize the 3-manifolds with this property.

Let us assume again that M is a regular level of a spsh function ρ on a complex manifold X. By Proposition 5.11, the hyperplane field $\ker\alpha$ is the field of hyperplanes invariant by complex multiplication and is a contact structure. It may be cooriented by the positive values of α and oriented by the restriction of J to $\ker\alpha = TM \cap J(TM)$ (see Proposition 5.11). Let us introduce the following terminology, which extends the one used above for $\mathbb{S}^3$:

Definition 5.23. Let M be a regular level of a spsh function on a complex manifold. Then the contact structure $TM \cap J(TM)$ oriented and cooriented as explained above is called the **natural contact structure** on M.

One may see that the natural contact structure depends only on the real hypersurface M of X, not on the defining strictly plurisubharmonic function.

Let us assume now that M is a regular level of a spsh function which is moreover proper and bounded from below. One has a special terminology for this situation:

Definition 5.24. Let $\rho : X \to \mathbb{R}$ be a proper spsh function bounded from below. Assume that $a \in \mathbb{R}$ is a regular value of ρ. Then the compact sublevel $X_{\rho \leq a} := \rho^{-1}(-\infty, a]$ is called a **Stein domain**, and a **Stein filling** of its boundary $M := \rho^{-1}(a)$ endowed with its natural contact structure. A contact manifold which is isomorphic to the boundary of a Stein domain endowed with its natural contact structure is called **Stein fillable**.

One has also the following related notion:

Definition 5.25. A closed oriented contact manifold is called **holomorphically fillable** if it is contactomorphic to the natural contact structure on a regular level $M = \rho^{-1}(a)$ of a smooth proper function $\rho : X \to \mathbb{R}$ bounded from below on a complex manifold X, which is spsh in the neighborhood of M. In this case, the sublevel $X_{\rho \leq a}$ is called a **holomorphic filling** of the contact manifold.

The difference with Definition 5.24 is that the function ρ is asked here to be strictly plurisubharmonic only in a neighborhood of the boundary M of the sublevel.

Bogomolov & de Oliveira proved in [18] that:

Theorem 5.26. *In dimension 3 all holomorphically fillable contact manifolds are also Stein fillable.*

This fact is false in higher odd dimensions (see Popescu-Pampu [161, Section 6]).

One may forget part of the previous structures and relations in order to arrive at purely symplectic concepts (which make no reference to an underlying complex structure):

Definition 5.27. Let (M, ξ) be a closed oriented and cooriented contact manifold.

A **strong symplectic filling** of (M, ξ) is a compact symplectic manifold (Y, ω) with boundary $\partial Y = M$ such that there exists a primitive α of ω *in a neighborhood of M* whose restriction to M is a defining form of ξ.

A **weak symplectic filling** of (M, ξ) is a compact symplectic manifold (Y, ω) with boundary $\partial Y = M$ such that the restriction of ω to ξ is a field of symplectic forms on ξ which define the given orientation of ξ.

A Stein filling of a contact manifold is obviously a strong symplectic filling and a strong symplectic filling is necessarily a weak symplectic filling. The systematic study of the previous notions of fillability in arbitrary dimensions was started by Eliashberg and Gromov [48]. Before that, they had proved in [70] and [46] the following generalization of Bennequin's theorem 5.21, specific to dimension 3:

Theorem 5.28. *A weakly symplectically fillable contact structure on a closed* 3-*manifold M is tight.*

In dimension 3, the three notions of fillability are in fact pairwise different and also different from the notion of *tightness*, as shown by the following results:

1. There exist oriented irreducible 3-manifolds which admit no positive tight contact structures: Etnyre & Honda [50] proved this for the Poincaré homology sphere with the orientation opposite to the one obtained as the boundary of the Kleinian complex surface singularity E_8 (see table 3.7). Therefore there exist reducible 3-manifolds which admit no tight contact structure at all (one simply takes the connected sum of two copies of the Poincaré homology sphere with both its orientations).

2. There exist tight contact manifolds which are not weakly symplectically fillable: Etnyre and Honda [51] constructed such structures on some Seifert manifolds.

3. There exist weakly symplectically fillable contact structures which are not strongly symplectically fillable: examples were constructed by Eliashberg [47] for $\mathbb{T}^3$ and by Ding and Geiges [37] for arbitrary torus bundles over the circle.

4. There exist strongly symplectically fillable contact manifolds which are not Stein fillable: Ghiggini [58] constructed such a structure on some small Seifert manifolds.

Those notions are also different in higher dimensions (see [161] and [119] for precise references).

Since 2000, a lot of effort was concentrated by various people on the problems of classification of 3-manifolds which admit one of the previous types of fillings, and for such manifolds, on the classification of those fillings up to diffeomorphism or homeomorphism. But those problems remain widely open.

The foundations of contact topology were described in Geiges' textbook [57]**. As an introduction to the problem of understanding the topology of fillings of contact** 3-**manifolds one may consult Ozbagci and Stipsicz' book** [150] **and Ozbagci's more recent survey** [149]**. For the same problem in higher dimensions, the basic reference is Cieliebak and Eliashberg's book** [33]**.**

6. Milnor fibers of surface singularities seen as Stein fillings

6.1. The contact boundary of an isolated singularity

One may constrain more the class of rug functions ρ used to define the notion of *boundary* of an isolated complex singularity (see the beginning of Section 3.3), by demanding them to be restrictions of squared-distances to the origin for some embedding in $(\mathbb{C}^n, 0)$ (see Equation (3.4)). As ρ_0 is spsh, its restrictions have the same property and their regular levels are endowed with natural contact structures (see Definition 5.23). Varchenko [196] proved using Gray's theorem 5.16, that those contact manifolds are independent of the choices of embedding and of sufficiently small level, up to contactomorphisms well-defined up to isotopy. This allows to introduce the following definition:

Definition 6.1. The oriented contact manifold associated in this way, up to contactomorphisms isotopic to the identity, to any isolated singularity (X, x), is called the **contact boundary** of (X, x), and is denoted $(\partial(X, x), \xi(X, x))$. An oriented contact manifold isomorphic to such a contact structure on the boundary of an isolated singularity is called **Milnor fillable**.

The name "Milnor fillable" was introduced in [30] in reference to the work [129] of Milnor recalled in Section 4.1. A Milnor fillable contact manifold (M, ξ) is holomorphically fillable (see Definition 5.25), as any resolution of a singularity whose contact boundary is contactomorphic to (M, ξ) gives a holomorphic filling of it.

Ustilovsky [192] proved the following property of a special class of Pham-Brieskorn singularities (see Definition 2.11), using the so-called *contact homology* (see also Kwon and van Koert [98]):

Theorem 6.2. *Let $m \geq 2$ be an integer. For varying $p \in \mathbb{Z}_+^*$ such that $p \equiv \pm 1 \,(\mathrm{mod}\, 8)$, the contact boundaries of the isolated hypersurface singularities defined by the equations*

$$z_0^p + z_1^2 + \cdots + z_{2m}^2 = 0$$

are pairwise non-isomorphic.

But Brieskorn [20] had proved that all the previous singularity boundaries are diffeomorphic to the standard sphere $\mathbb{S}^{4m+1}$ (see also Hirzebruch [82] or Hirzebruch and Mayer [84]). Therefore, the smooth structure on the boundary of an isolated singularity of dimension ≥ 3 does not determine the associated contact structure.

In complex dimension 2 the situation is radically different, as was proved by Caubel, Némethi and myself [30]:

Theorem 6.3. *Any Milnor fillable oriented 3-manifold admits a unique Milnor fillable contact structure up to contactomorphism. If the manifold is a rational homology sphere, then this contact structure is unique up to isotopy.*

Bhupal and Ozbagci [16, Proposition 4] proved that the second statement does not extend to the case where the Milnor fillable manifold is not a rational homology sphere.

Let us explain the principle of the proof of Theorem 6.3, as it applies Milnor's construction explained in Section 4.1 to a function with isolated critical point on an arbitrary irreducible complex isolated singularity (X, x).

A germ of holomorphic function $f \in \mathfrak{m}_{X,x}$ is said **to have an isolated critical point** at $x \in X$ if there is a representative of (X, f) such that f is regular outside x. By the general theorems of Lê and

Teissier [107] on limits of tangent hyperplanes to a germ of complex analytic space, one sees that, given (X, x), there are always such functions f with isolated critical points. Hamm [71] proved that f induces then again a **Milnor open book** on the boundary of (X, x), as in Milnor's initial situation where $(X, x) = (\mathbb{C}^n, 0)$ explained in Subsection 4.1.

One has then the following theorem of [30, Theorem 3.9], which was extended by Caubel [29] to the boundaries of the Milnor fibers of some non-isolated singularities:

Theorem 6.4. *Let (X, x) be an irreducible isolated singularity and let $f : (X, x) \to (\mathbb{C}, 0)$ be a germ of holomorphic function with isolated critical point. Then its Milnor open book carries the natural contact structure of the contact boundary of (X, x).*

Let us explain the meaning of the notion of *open book carrying a contact structure* (see Definition 4.1 for the notion of open book). This notion was introduced by Giroux [59]:

Definition 6.5. A positive contact structure ξ on a closed oriented manifold M is **carried by an open book** (N, θ) if it admits a defining contact form α which verifies the following:
- α induces a positive contact structure on N;
- $d\alpha$ induces a positive symplectic structure on each fiber of θ.

If a contact form α satisfies these conditions, then it is called **adapted** to (N, θ).

Giroux proved in [59] that:

Theorem 6.6. *On any 3-dimensional closed oriented manifold, any contact structure is carried by some open book and two positive contact structures carried by the same open book are isotopic.*

Giroux and Mohsen generalized this theorem to all dimensions (see a sketch of proof in [59]). Moreover, in dimension 3 Giroux [59] and Giroux and Goodman [60] described the relation existing between open books which carry the same contact structure.

Theorem 6.6 shows that in order to describe a positive contact structure on a 3-dimensional closed and oriented manifold, it is enough to describe an open book which carries it. This is the strategy adopted in [30] to prove Theorem 6.3. Namely, we combined Theorem 6.4, valid in arbitrary dimension, with the following result specific to complex dimension 2:

Corollary 6.7. *Let M be a closed connected oriented 3-manifold which is Milnor fillable. Then there exists an open book (N, θ) in M which can be completely characterized by the topology of M, such that, for any normal surface singularity (X, x) whose boundary $\partial(X, x)$ is orientation-preserving diffeomorphic to M, there exists a function $f \in m_{X,x}$ having an isolated critical point and whose Milnor open book is isomorphic to (N, θ).*

Before ending this section, let us mention two properties of Milnor fillable 3-manifolds which were proved since the publication of [30]:

1. Lekili and Ozbagci proved in [104] that the lift of a Milnor fillable contact structure to the universal cover of the manifold is still tight. When this universal cover is compact, this is obvious, as one gets again the contact boundary of a singularity. Their proof concentrates on the case when this universal cover is non-compact. Note that is still lacking in general (excepting some special classes) a characterization of the unique Milnor fillable contact structure among all possible positive Stein fillable and universally tight contact structures on a normal surface singularity boundary.

2. H. Park and Stpisicz proved in [153] that a configuration of symplectic surfaces in a 4-dimensional symplectic manifold which has the same weighted dual graph as that of a resolution of normal surface singularity (X, x) has a tubular neighborhood which is a strong symplectic filling of the contact boundary of (X, x). This ensures that the 4-manifold obtained by replacing such a tubular neighborhood by any Milnor fiber of (X, x) admits also a symplectic structure, which may be chosen to coincide with the starting one outside the given tubular neighborhood. This construction gives a lot more flexibility for performing "*symplectic surgeries*" than the prototypical "*rational blow-downs*" of Fintushel and Stern [54].

For more details about open books supporting contact structures, one may consult the foundational paper [59] **of Giroux, as well as the survey** [49] **of Etnyre, the paper** [60] **of Giroux and Goodman and Geiges' textbook** [57, Section 7.3]**. A survey of the properties of the contact boundaries of Pham-Brieskorn singularities was written by Kwon and van Koert in** [98]**.**

6.2. Cases when the Milnor fibers exhaust the Stein fillings

Assume that (X, x) is a smoothable isolated singularity, and let $f : (Y, y) \to (\mathbb{C}, 0)$ be one of its smoothings (see Definition 4.11). By choosing an embedding $(Y, y) \hookrightarrow (\mathbb{C}^n, 0)$ and restricting to Y the squared distance function ρ_0 to the origin (see Equation 3.4), one constructs representatives of the Milnor fibers of ψ which are Stein domains (see Definition 5.24) and whose contact boundaries are contactomorphic (by Gray's theorem 5.16) with the contact boundary of (X, x) introduced in Definition 6.1. Therefore:

Proposition 6.8. *Let (X, x) be a smoothable isolated singularity. Then the Stein representatives of its Milnor fibers constructed as before are Stein fillings of the contact boundary $(\partial(X, x), \xi(X, x))$.*

As a consequence, if one wishes to characterize the Milnor fibers of a given isolated singularity up to diffeomorphisms among the fillings of the boundary of the singularity, it is natural to restrict one's attention to the Stein fillings of the contact boundary. In complex dimension 2 the situation is special, due to Theorem 6.3. In this case, one is led to ask the following questions:

Is it possible to characterize the Milnor fibers of the various isolated surface singularities with a given topological type among the Stein fillings of the associated Milnor fillable contact 3-manifold? Are there situations in which one gets all the Stein fillings up to diffeomorphisms as such Milnor fibers?

Note that, because of the existence of *blow up operations* both in complex and in symplectic geometry (see McDuff [121] for the second case), from one holomorphic or strong symplectic filling of a contact 3-manifold one can get by successive blow ups an infinite number of pairwise non-homeomorphic such fillings. Therefore, the second question above has always a negative answer if one replaces in it the notion of *Stein filling* by one of those weaker notions of filling.

In full generality the previous questions are widely open. But there are answers known in particular cases. The first result going in this direction was obtained by Eliashberg [46] (see Cieliebak and Eliashberg [33, Theorem 16.6] for a strengthening):

Theorem 6.9. *Any Stein filling of the natural contact structure on $\mathbb{S}^3$ is diffeomorphic to the 4-dimensional compact ball.*

The "natural contact structure" on $\mathbb{S}^3$ being simply its natural contact structure when we see it as the unit sphere in $\mathbb{C}^2$ (see Definition 5.23), it is also the unique Milnor fillable contact structure on $\mathbb{S}^3$.

Theorem 6.9 was extended by Mc Duff [121] to all lens spaces of type $L(p, 1)$ (see Definition 3.40) endowed with their standard contact structure, which is again their unique Milnor fillable contact structure:

Theorem 6.10. *Let $p \geq 2$ be an integer. Consider the natural contact structure ξ_0 on the lens space $L(p, 1)$.*

1. *If $p \neq 4$, then $(L(p, 1), \xi_0)$ has only one Stein filling up to diffeomorphisms, which is the Milnor fiber of the unique smoothing of the cyclic quotient singularity $(\mathcal{X}_{p,1}, 0)$.*

2. *$(L(4, 1), \xi_0)$ has exactly two Stein fillings up to diffeomorphisms, which are the Milnor fibers of the two smoothings of the cyclic quotient singularity $(\mathcal{X}_{4,1}, 0)$.*
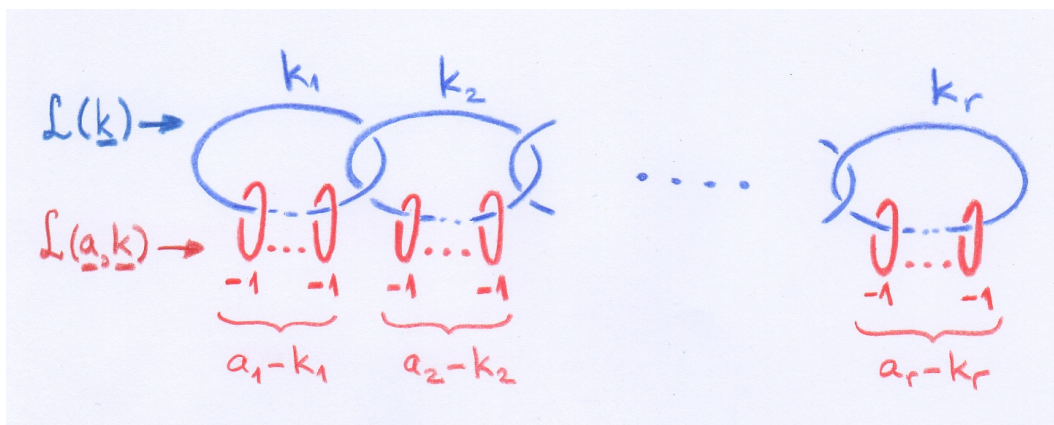
Figure 6.1. The framed links allowing to describe the Stein fillings of lens spaces

As explained by Cieliebak and Eliashberg [33, Theorem 16.10], this theorem was strengthened by Plamenevskaya and Van Horn Morris [159] and Hind [77].

Recall that we described the differentiable types of the two Milnor fibers of $(\mathcal{X}_{4,1}, 0)$ in Section 4.4 (see Proposition 4.39).

In [112] Lisca sketched a proof of a classification of the Stein fillings of *all* lens spaces endowed with their natural contact structures (again, the unique Milnor fillable one, obtained as the contact boundaries of the cyclic quotient singularities), and he described a detailed proof in [113]. In order to give an idea of the combinatorial complexity of his classification, let us explain his description of those fillings as smooth compact 4-manifolds with boundary.

Fix a lens space $L(p, q)$ (where $0 < q < p$ and p and q are coprime). Recall from Theorem 3.41 that the minimal resolution of the associated cyclic quotient singularity $(\mathcal{X}_{p,q}, 0)$ has a weighted dual graph which may be described using the Hirzebruch-Jung continued fraction expansion of $\dfrac{p}{q}$.

Let us introduce also the analogous expansion of $\dfrac{p}{p-q}$:

$$
\begin{cases}
a_i \geq 2, \ \text{ for all } i \in \{1, \dots, r\}, \\
\dfrac{p}{p-q} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\dots - \cfrac{1}{a_r}}}
\end{cases}.
$$

We will denote more concisely by $[a_1, a_2, \dots, a_r]$ the previous Hirzebruch-Jung continued fraction.

For each $r \geq 1$, consider the following finite subset of $\mathbb{N}^r$:

Definition 6.11. A sequence $\underline{k} = (k_1, \dots, k_r) \in \mathbb{Z}_+^r$ is called **admissible** if either $r = 1$ or $r \geq 2$, $\underline{k} \in (\mathbb{Z}_+^*)^r$, $[k_1, \dots, k_i] > 0$ for all $i \in \{1, \dots, r-1\}$ and $[k_1, \dots, k_r] \geq 0$. Let $\mathrm{adm}(\mathbb{Z}_+^r)$ be the set of admissible sequences from $\mathbb{Z}_+^r$. For $r \geq 1$, denote by:

(6.1)
$$
K_r := \{\underline{k} = (k_1, \dots, k_r) \in \mathrm{adm}(\mathbb{Z}_+^r) \mid [k_1, \dots, k_r] = 0\}
$$

the set of **admissible sequences which represent** 0.

The sequence $(a_1, \dots, a_r)$ determines the following subset of K_r:

Definition 6.12. Let $\underline{a} \in \mathbb{Z}_+^r$ be fixed with $a_i \geq 2$ for all $i \in \{1, \dots, r\}$. Then

(6.2)
$$
K_r(\underline{a}) := \{\underline{k} \in K_r \mid k_i \leq a_i, \text{ for all } i \in \{1, \dots, r\}\}
$$

is the set of $\underline{a}$-**admissible sequences which represent** 0.

Let us fix a $\underline{a}$-admissible sequence $\underline{k} = (k_1, \ldots, k_r)$ which represents 0. Associate to it the framed link in the oriented 3-dimensional sphere $\mathbb{S}^3$ drawn in Figure 6.1. This link is a disjoint union of the two links $\mathcal{L}(\underline{k})$ (a chain of unknoted circles, any two consecutive ones forming a Hopf link) and $\mathcal{L}(\underline{a}, \underline{k})$ (consisting of k packets of ear-rings for the composing knots of $\mathcal{L}(\underline{k})$).

Consider the closed and oriented 3-manifold obtained by surgery on $\mathbb{S}^3$ along the framed link $\mathcal{L}(\underline{k})$. It is possible to show that there exists an orientation-preserving diffeomorphism identifying it with $\mathbb{S}^1 \times \mathbb{S}^2$. Keep calling $\mathcal{L}(\underline{a}, \underline{k})$ the image inside $\mathbb{S}^1 \times \mathbb{S}^2$ of the initial framed link with the same name in $\mathbb{S}^3$.

Definition 6.13. Let $W_{p,q}(\underline{k})$ be the smooth oriented 4-manifold with boundary obtained by attaching index 2 handles to $\mathbb{S}^1 \times \mathbb{B}^3$ along the framed link $\mathcal{L}(\underline{a}, \underline{k})$.

The manifold $W_{p,q}(\underline{k})$ is therefore obtained by attaching index 2 handles to the 4-ball along the whole framed link described in Figure 6.1, and replacing the sublevel of a corresponding Morse function which contains the ball and the handles attached along $\mathcal{L}(\underline{k})$ with the manifold $\mathbb{S}^1 \times \mathbb{B}^3$. Lisca showed that this construction does not depend on the choice of the orientation-preserving diffeomorphism identifying the surgered sphere with $\mathbb{S}^1 \times \mathbb{S}^2$.

Lisca's classification theorem may be stated briefly like this:

Theorem 6.14. *The manifolds $W_{p,q}(\underline{k})$ exhaust the Stein fillings of the standard contact structure on $L(p, q)$ and are pairwise non-diffeomorphic by diffeomorphisms fixing the boundary.*

Therefore, the Stein fillings of the lens space $L(p, q)$ endowed with its Milnor fillable contact structure are parametrized by the $\underline{a}$-admissible sequences which represent 0. The same objects had appeared in the classification of the irreducible components of the reduced miniversal base space of the cyclic quotient singularity $(\mathcal{X}_{p,q}, 0)$, conjectured and partially proved by Christophersen [32]. The proof of the following theorem was completed by Stevens [180], using deep results of Kollár and Shepherd-Barron [96]:

Theorem 6.15. *Assume that $\dfrac{p}{p-q} = [a_1, \ldots, a_r]$. Then there is a bijection from the set $K_r(\underline{a})$ of $\underline{a}$-admissible sequences representing 0 to the set of irreducible components of the reduced base space of the miniversal deformation of $(\mathcal{X}_{p,q}, 0)$.*

Christophersen and Stevens gave moreover equations describing the restriction of the miniversal deformation to each such component (generalizing Pinkham's equations described in Section 4.4 and Riemenschneider's equations for the deformation over the Artin component given in [169]). Némethi and myself used those equations in [137] in order to prove the following theorem, which answers affirmatively a conjecture of Lisca [113], first formulated in [112]:

Theorem 6.16. *The Milnor fiber associated to the component of the miniversal base space of $(\mathcal{X}_{p,q}, 0)$ which is parametrized by $\underline{k} \in K_r(\underline{a})$ in the Christophersen-Stevens correspondence is diffeomorphic to Lisca's manifold $W_{p,q}(\underline{k})$. Therefore, the Milnor fibers of a cyclic quotient singularity exhaust the Stein fillings of its contact boundary up to diffeomorphism.*

In the paper [137] we got a second proof of the last statement by using the work [89] of de Jong and van Straten on *sandwiched surface singularities*, which form a class of rational surface singularities containing the cyclic quotients. We extended partially our results to all sandwiched singularities in [138]. Nevertheless, the question to know if the analog of the last statement of Theorem 6.16 extends to all sandwiched surface singularities remains open. Note that sandwiched surface singularities are not in general taut (see Definition 3.47), therefore the question is: *do the Milnor fibers of all the sandwiched surface singularities with a fixed topological type exhaust the Stein fillings of their common contact boundary?*

The last statement of Theorem 6.16 was extended by Park, Park, Shin and Urzúa [151] to all *quotient* surface singularities. They concentrated on the cases not covered by Theorem 6.16. Using Stevens' description from [181] of the irreducible components of their miniversal base space Bhupal and Ono's classification from [14] of the Stein fillings of their contact boundaries up to diffeomorphisms, they proved:

Theorem 6.17. *There is a bijective correspondence between the set of diffeomorphism classes fixing the boundary of Stein fillings of the contact boundary of a quotient surface singularity and the set of irreducible components of its miniversal base space. This correspondence identifies each Stein filling with the Milnor fiber of the associated component.*

The particular case of *Kleinian* surface singularities was proved before by Ohta and Ono [145], who had also treated in [144] the case of *simple elliptic* singularities:

Theorem 6.18. *The Stein fillings of the contact boundary of a simple elliptic singularity is diffeomorphic either to the tubular neighborhood of the exceptional divisor in the minimal resolution, or to the Milnor fiber of the unique smoothing component, when the singularity is smoothable.*

Recall from Example 4.19 that simple elliptic singularities are smoothable only for a finite number of topological types. Anyway, the previous theorem is different in spirit from Theorems 6.16 and 6.17, as the differentiable types of Stein fillings of the contact boundary are not obtainable only as Milnor fibers. This is due in fact to the constraint of looking only at *normal* representatives of this topological type. As shown in Proposition 4.32, if one drops this constraint, then Ohta and Ono's theorem may be reformulated in the following way:

Theorem 6.19. *The Stein fillings of the contact boundary of a simple elliptic singularity are realized by the Milnor fibers of the smoothings of the isolated surface singularities whose normalization is simple elliptic.*

Note that one gets *all* the Stein filling from any simple elliptic singularity and the isolated singularities which have it as normalization. This is not the case for general non-taut singularities. For instance, Laufer [103, Page 48] gave an example of two isolated hypersurface singularities (whose equations are $x^2 + y^7 + z^{14} = 0$ and $x^3 + y^4 + z^{12} = 0$) with the same topological type, therefore with the same contact boundaries, by Theorem 6.3, but with non-homeomorphic Milnor fibers (an immediate application of Theorem 4.4 shows that their Milnor numbers are different). Note that, being hypersurface singularities, by Tyurina's theorem 4.12 both of them have a single Milnor fiber, up to diffeomorphisms fixing the boundary. Note also that by point (3.30) of Proposition 3.30, their boundaries are not rational homology spheres, as their minimal resolutions have smooth exceptional divisors of genus 3.

For the moment, no examples of such pairs with rational homology sphere boundaries are known. In fact, Mendris and Némethi conjectured in [124] that:

"If the link of an isolated hypersurface singularity is a rational homology 3-sphere, then it determines the equisingularity type, the embedded topological type, the equivariant Hodge numbers and the multiplicity of the singularity."

Let us finish this sections with several remarks about related directions of research:

1. If the isolated surface singularity (X, x) is fixed, the existence of a miniversal deformation shows that, up to diffeomorphisms, there is only a finite number of Stein fillings of its contact boundary which appear as Milnor fibers of its smoothings. For the classes of singularities considered till now in this subsection, there is also a finite number of Stein fillings and even of strong symplectic fillings. This fact is not general. Ohta and Ono [146] showed that there exist Milnor fillable contact 3-manifolds which admit an infinite number of minimal strong symplectic fillings, pairwise not homotopically equivalent. Later, Akhmedov and Ozbagci [1] proved that there exist Milnor fillable contact 3-manifolds which admit even an infinite number of Stein fillings pairwise non-diffeomorphic, but homeomorphic. Moreover, by varying the contact 3-manifold, the fundamental groups of such fillings exhaust all finitely presented groups. For details one may consult Ozbagci's survey [149].

2. In [9] Baykur and Van Horn-Morris construct many contact 3-manifolds which admit infinitely many Stein fillings with arbitrarily large Euler characteristics and arbitrarily small signature, disproving in this way a conjecture of Ozbagci and Stipsicz.

3. Another direction of research concentrated on the question of classification of those normal surface singularities admitting Milnor fibers which are rational homology balls. Such Milnor fibers were used for performing the operation of *rational blow-down* introduced by Fintushel and Stern [54] and generalized by Stipsicz, Szabó, Wahl [185]. Due to the efforts of several researchers, the *normal* surface singularities which have smoothings whose Milnor fibers are rational homology balls are now completely classified. See Park, Shin and Stipsicz [152], Bhupal and Stipsicz [17] and Fowler [56] for details about this direction of research.

For details about the structure of the miniversal deformation of cyclic quotient singularities, one may consult Behnke and Riemenschneider [11]**, Hamm** [72]**, Riemenschneider** [170] **and Stevens** [184]**.**

For an introduction to the techniques of study of the Stein fillings of contact 3-**manifolds, one may consult Ozbagci and Stipsicz' book** [150]**, and the surveys** [17] **of Bhupal and Stipsicz and** [149] **of Ozbagci. For a sketch of the proof of Theorem 6.16 and of its partial extension to the class of sandwiched surface singularities, one may consult Némethi's survey** [135]**.**

7. Open questions

Let me conclude with a list of open questions about the contact topological aspects of singularities. Even if my text concentrated on isolated singularities, note that some of the questions concern non-isolated singularities and their smoothings.

1. *Characterize Milnor fillable oriented contact structures among all contact structures on the boundary of a normal surface singularity.* As mentioned in Remark (1) at the end of Subsection 6.1, the Milnor fillable contact structures are necessarily Stein fillable and universally tight.

2. *Characterize the topological types of normal surface singularities whose set of isolated but not necessarily normal representatives produces a finite number of Milnor fibers, up to diffeomorphisms fixing the boundary.* Of course, as a consequence of Grauert's theorem 4.10, a fixed isolated singularity has a finite number of Milnor fibers up to diffeomorphisms fixing the boundary. As stated in Proposition 4.32, non-normal isolated singularities can produce different Milnor fibers than their normalizations. Therefore it is not a priori clear that even the topological types of taut singularities (see Definition 3.47) produce a finite number of Milnor fibers. Nevertheless, one may show that this is the case for the topological types of taut and *rational* singularities, as a consequence of results of Kollár (see [165, Remark 5.10]).

3. *Describe the topologies of Milnor fibers for a given topology of normal surface singularity.* As explained in Section 4.4, this was done till now for all quotient surface singularities and for simple elliptic singularities.

4. *For which topological types, the Milnor fibers give all the Stein fillings up to diffeomorphisms fixing the boundary?* By Theorems 6.16 and 6.17, this is the case for all quotient surface singularities. It is also true for simple elliptic singularities, by combining Theorem 6.18 and Proposition 4.32.

5. *Are there isolated singularities of dimension at least* 2 *with exotic pairs of Milnor fibers (i.e. homeomorphic but non-diffeomorphic)?* In this question I use the standard vocabulary of differential topology, in which a manifold *A* is called an "exotic" version of a manifold *B* if it is homeomorphic but not diffeomorphic to *B*.

6. *Characterize the topological types of oriented 3-manifolds appearing as boundaries of Milnor fibers of smoothings of not necessarily isolated surface singularities.* By the works of Michel and Pichon [125], [126] and Némethi and Szilard [139] (for smooth total spaces of the smoothings) and Fernández de Bobadilla and Menegon Neto [53] (for total spaces with isolated singularity), one knows that such boundaries are graph-manifolds, similarly to the boundaries of isolated surface singularities. The present question asks to get an analog of Theorem 3.27.

7. *Which oriented contact structures on oriented 3-manifolds does one obtain as contact boundaries of such Milnor fibers?* This question is an extension of question (1) above.

8. *Is it possible to read on the contact boundary of an isolated complex singularity of dimension at least 3 whether it is isomorphic to the contact boundary of an isolated Cohen-Macaulay singularity?* Note that there are constraints on the finitely presented groups which may occur as fundamental groups of boundaries of such singularities (see Kollár [95, Theorem 4]).

9. *Does the contact boundary of an isolated and irreducible complex singularity of dimension at least 3 determine the simple homotopy type of the dual complex of the exceptional divisor of a simple normal crossings resolution of the singularity?* As shown independently by Stepanov [179] and Kontsevich and Soibelman [97, Section A.4], this simple homotopy type is independent of the chosen resolution. The fact that its homotopy type has this property is a consequence of the previous work of Danilov [36]. I restrict the question to dimensions ≥ 3 because Neumann's theorem 3.29 shows that the answer is affirmative for dimension 2.

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Université Lille 1, UFR de Maths., Bâtiment M2, Cité Scientifique, 59655, Villeneuve d'Ascq Cedex, France. •
patrick.popescu@math.univ-lille1.fr