



## Winter Braids Lecture Notes

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**Johnson-Morita theory in mapping class groups and monoids of homology cobordisms of surfaces**

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# Johnson-Morita theory in mapping class groups and monoids of homology cobordisms of surfaces

TAKUYA SAKASAI

## Abstract

This article is the notes of a series of lectures in the workshop “Winter Braids VI”, Lille, in February 2016. We begin by recalling fundamental facts on mapping class groups of surfaces and overview the theory of Johnson homomorphisms developed by Johnson himself and Morita. Then we see how this theory is extended as invariants of homology cobordisms of surfaces and discuss an application to knot theory.

## 1. Introduction

The mapping class group of a compact oriented surface is defined as the group of all isotopy classes of self-diffeomorphisms of the surface. The role of this group and its subgroup named the Torelli group in low dimensional topology is widely accepted to be important. The structures of these mysterious groups have been studied for a long time. In 1980's, Dennis Johnson introduced a homomorphism, now called the Johnson homomorphism, from the Torelli group to a certain symplectic module. He used it to prove several fundamental but deep facts on the Torelli group. After that, Shigeyuki Morita clarified and extended this theory. In a series of his works, he revealed close relationships to invariants of 3-dimensional manifolds and cohomology of the mapping class group. Since then, this theory has been further extended by many people in various directions.

This article is based on the author's lectures to PhD students and postdocs titled “Johnson-Morita theory” in the workshop “Winter Braids VI”, Lille, in February 2016. It also includes some supplemental results and related problems. In the workshop, we started from the review of works of Johnson and Morita together with fundamental facts on mapping class groups, and then discussed recent developments of the theory of Johnson homomorphisms and its applications:

Talk 1: Mapping class group, Torelli group and the first Johnson homomorphism,

Talk 2: Higher Johnson homomorphisms,

Talk 3: Extension to homology cobordisms of surfaces and an application to knot theory.

Talk 1 corresponds to Sections 2 and 3, Talk 2 to Section 4 and Talk 3 to Sections 6 and 7. Section 5 collects known problems on Johnson homomorphisms for mapping class groups, which were not mentioned in the workshop.

**Caution 1.1.** For easier access to the theory of Johnson homomorphisms and to keep the talks in introductory level, in the workshop the author did not discuss the most recent descriptions of the theory

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due to Massuyeau [62] and Kawazumi-Kuno [53], which are sophisticated but need more preliminaries on algebra, and explained arguments near to the original (classical) ones. The present article also follows this line.

References are far from complete and the arguments in several places do not follow the chronological order. For more details on mapping class groups, we refer to books of Birman [7] and Farb-Margalit [24], and survey papers of Ivanov [44] and Morita [75].

**Notation 1.2.** All maps act on elements from the left. Homology groups are assumed to be with coefficients in the ring of integers  $\mathbb{Z}$ . For two subgroups  $K$  and  $L$  in a group  $G$ , we denote by  $[K, L]$  the commutator subgroup, which is generated by elements  $[k, l] := klk^{-1}l^{-1}$  for  $k \in K$  and  $l \in L$ . All manifolds and maps between them are assumed to be smooth.

## 2. Mapping class groups and Torelli groups

### 2.1. Mapping class groups

Let  $\Sigma_g$  be a closed connected oriented surface of genus  $g$ . We denote by  $\text{Diff}_+(\Sigma_g)$  the group of orientation-preserving self-diffeomorphisms of  $\Sigma_g$  endowed with  $C^\infty$ -topology. This group is known to be a Fréchet Lie group so that it is a huge but not so bad topological space. For example, it has the same homotopy type as a countable CW-complex. Let

$$\mathcal{M}_g := \pi_0(\text{Diff}_+(\Sigma_g))$$

be the group of its path-connected components, which is by definition the *diffeotopy group* of  $\Sigma_g$ . That is,  $\mathcal{M}_g$  is the group of all isotopy classes of orientation-preserving self-diffeomorphisms of  $\Sigma_g$ . We customarily call this group the *mapping class group*. Generally, the word “mapping class group” stands for the *homeotopy group*, the group of all homotopy classes of (orientation-preserving) self-homeomorphisms. However, the classical surface topology admits us to identify the diffeotopy group and the homeotopy group for the surface  $\Sigma_g$ . (The proof of this fact seems to diverge to many classical papers. A good reference is Boldsen’s preprint [10].) In what follows, we call an element of  $\mathcal{M}_g$  a *mapping class*.

The role of  $\mathcal{M}_g$  in topology is very wide and important. By definition, it governs the symmetry of a surface from a topological (homotopical) point of view. In the 3-dimensional case, every element of  $\mathcal{M}_g$  is used to construct a 3-manifold through the methods of Heegaard decompositions, mapping tori and open book decompositions (fibered knots). In the 4-dimensional case, the group  $\mathcal{M}_g$ , in particular relations of elements of  $\mathcal{M}_g$  are used to construct Lefschetz fibrations. More generally, elements of  $\mathcal{M}_g$  serve as the holonomy of oriented  $\Sigma_g$ -bundles. In fact, Earle-Eells [21] and Gramain [34] showed in different ways that each connected component of the Fréchet Lie group  $\text{Diff}_+(\Sigma_g)$  is contractible if  $g \geq 2$ , which implies that the classifying space  $B\text{Diff}_+(\Sigma_g)$  of  $\text{Diff}_+(\Sigma_g)$  is homotopy equivalent to the Eilenberg-MacLane complex  $K(\mathcal{M}_g, 1)$  of  $\mathcal{M}_g$  (see Remark 2.2 below for this complex). Therefore the group cohomology of  $\mathcal{M}_g$  is nothing other than the module of all characteristic classes of oriented  $\Sigma_g$ -bundles (see Morita [70] for details).

The group  $\mathcal{M}_g$  is important also in complex analysis, differential geometry, algebraic geometry and mathematical physics through the geometry of the Teichmüller space and the moduli space of Riemann surfaces.

So we now want to understand the structure of  $\mathcal{M}_g$ . In a few low genus cases, the group is completely understood. When  $g = 0$ , it was shown by Munkres [78] that  $\text{Diff}_+(S^2)$  is path-connected, that is,  $\mathcal{M}_0$  is trivial. (If you work in  $C^0$ -category, the corresponding fact that every orientation-preserving homeomorphism of  $S^2$  is homotopic, in fact isotopic, to the identity follows from Schönflies’ theorem and the Alexander trick, both of which are classical results in topology. See Birman’s book [7, Chapter 4].) Soon after the result of Munkres, Smale [94] showed that the rotation group  $\text{SO}(3)$  is a strong deformation retract of  $\text{Diff}_+(S^2)$  (see also the above cited papers by Earle-Eells and Gramain, where it is also shown that each connected component of the Fréchet Lie group  $\text{Diff}_+(\Sigma_1)$  is homotopy equivalent to  $\Sigma_1 = T^2$  regarded as the space of all parallel translations of  $T^2$ ).

In positive genus cases,  $\mathcal{M}_g$  give non-trivial groups. To see it, let us recall the well-known facts on a generating system of  $\mathcal{M}_g$ . Let  $c$  be a simple closed curve in  $\Sigma_g$ . A *Dehn twist*  $T_c \in \mathcal{M}_g$  along  $c$  is the

isotopy class of the self-diffeomorphism of  $\Sigma_g$  which twists a regular neighborhood of  $c$  as in Figure 2.1. That is, we first cut the surface along  $c$  and twist once smoothly to the right direction (the direction depends on the orientation of  $\Sigma_g$ ) with support in the neighborhood, then we reglue the surface. Of course, such a diffeomorphism has several kinds of ambiguity such as the choice of a regular neighborhood and the speed of twisting, but determines a unique element in  $\mathcal{M}_g$ . Moreover, the mapping class  $T_c$  depends only on the homotopy class of  $c$ .

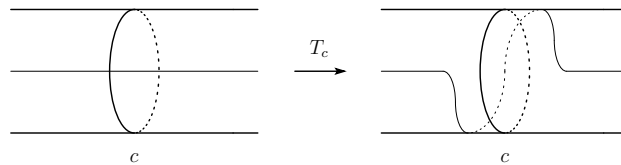


Figure 2.1: Dehn twist  $T_c$  along a simple closed curve  $c$

One of the most fundamental results on  $\mathcal{M}_g$  is that it is generated by *Dehn twists*. Furthermore, we have the following:

**Theorem 2.1** (Dehn, Lickorish). *The mapping class group  $\mathcal{M}_g$  is generated by  $(3g - 1)$  Dehn twists  $\{T_{c_i}\}_{i=1}^{3g-1}$ , where the simple closed curves  $c_1, c_2, \dots, c_{3g-1}$  are given as in Figure 2.2.*

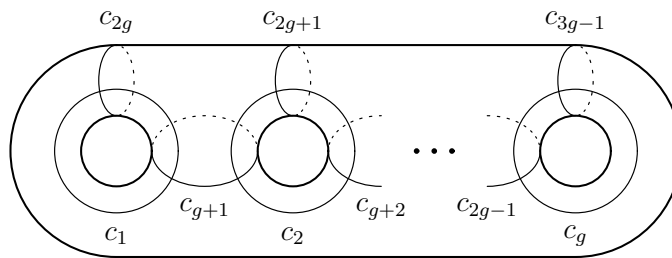


Figure 2.2: The Lickorish generators

Indeed, more is known. Birman and Hilden gave a finite presentation of  $\mathcal{M}_2$  in [9] using a covering argument. Then, following results of McCool, Hatcher-Thurston on the existence, Harer and Wajnryb gave finite presentations of  $\mathcal{M}_g$  for  $g \geq 3$  (see Farb-Margalit [24, Section 5.2] for the precise statement). From the presentation, we can immediately compute the abelianization  $H_1(\mathcal{M}_g) = \mathcal{M}_g/[\mathcal{M}_g, \mathcal{M}_g]$  (this way of computation does not follow the chronological order):

$$H_1(\mathcal{M}_g) \cong \begin{cases} 0 & (g = 0) \\ \mathbb{Z}/12\mathbb{Z} & (g = 1, \text{ folklore as mentioned below}) \\ \mathbb{Z}/10\mathbb{Z} & (g = 2, \text{ Mumford [77]}) \\ 0 & (g \geq 3, \text{ Powell [86]}) \end{cases}.$$

**Remark 2.2** (Homology of groups). The abelianization  $G/[G, G]$  of a group  $G$  coincides with the first homology group  $H_1(G)$  of  $G$ . One definition of the homology group  $H_*(G)$  of a (discrete) group  $G$  is given by

$$H_*(G) := H_*(K(G, 1)),$$

where the right hand side is the (usual cellular) homology group of the *Eilenberg-MacLane complex*  $K(G, 1)$ . The CW-complex  $K(G, 1)$  is characterized up to homotopy equivalence by its homotopy groups:

$\pi_1(K(G, 1)) \cong G$  and  $\pi_n(K(G, 1)) = 0$  for  $n \neq 1$ . For example,  $H_*(F_n) \cong H_*(\vee_n S^1)$  for a free group  $F_n$  of rank  $n$  and  $H_*(\pi_1(\Sigma_g)) \cong H_*(\Sigma_g)$  for  $g \geq 1$ . Taking the homology of groups satisfies the functorial properties. That is, a group homomorphism  $f : G_1 \rightarrow G_2$  induces a homomorphism  $f_* : H_*(G_1) \rightarrow H_*(G_2)$ , which is compatible with the composition of homomorphisms, and the identity map  $\text{id}_G$  of  $G$  induces the identity map of  $H_*(G)$ . For a CW-complex  $X$ , we always have a continuous map  $\eta_X : X \rightarrow K(\pi_1(X), 1)$  inducing the identity on  $\pi_1$ . Indeed,  $K(\pi_1(X), 1)$  is obtained by adding 3-, 4-, ... cells to  $X$  to eliminate higher homotopy groups and the map  $\eta_X : X \rightarrow K(\pi_1(X), 1)$  is taken to be the inclusion. From the construction, we see that  $\eta_X$  induces an isomorphism on  $H_1$  and a surjection on  $H_2$ . In what follows, we shall use homology of groups up to degree 3. For more details on (co)homology of groups, we refer to Brown's book [11]. A paper [56] of Korkmaz and Stipsicz is a good reference for a systematic computation of the first and second homology groups of  $\mathcal{M}_g$ .

## 2.2. Torelli groups

As seen above, the abelianization of  $\mathcal{M}_g$  is small or trivial. This means that it is difficult to extract some information on  $\mathcal{M}_g$  from it. Hence, to understand the structure of  $\mathcal{M}_g$ , we are required to find homomorphisms from  $\mathcal{M}_g$  to other groups (or vice versa). One good way to do so is to consider actions of  $\mathcal{M}_g$  on other objects. The most fundamental one is the natural action of  $\mathcal{M}_g$  on the first homology group  $H := H_1(\Sigma_g) \cong \mathbb{Z}^{2g}$  of the surface  $\Sigma_g$ . The module  $H$  has a natural non-degenerate anti-symmetric bilinear form

$$\mu : H \otimes H \longrightarrow \mathbb{Z}$$

called the *intersection pairing*, which reflects the Poincaré duality of  $\Sigma_g$ . Using this, we identify  $H$  with its dual  $H^* = H^1(\Sigma_g)$ , the first cohomology group. We take a symplectic basis  $\{a_1, a_2, \dots, a_g, b_1, b_2, \dots, b_g\}$  of  $H$  as in Figure 2.3. That is, the elements of the basis satisfy

$$\mu(a_i, a_j) = \mu(b_i, b_j) = 0, \quad \mu(a_i, b_j) = \delta_{i,j}.$$

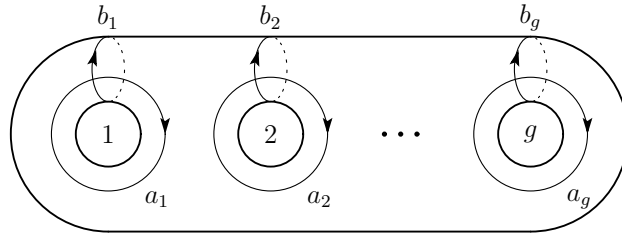


Figure 2.3: A symplectic basis of  $H_1(\Sigma_g)$

Note that the action of  $\mathcal{M}_g$  on  $H$  preserves the intersection pairing  $\mu$ . In terms of group homomorphisms, this action is given by a homomorphism

$$\sigma : \mathcal{M}_g \longrightarrow \text{Aut}(H, \mu) = \text{Sp}(H).$$

Here the target is the symplectic (i.e.  $\mu$ -preserving) automorphism group of  $H$  and it is isomorphic to the symplectic matrix group  $\text{Sp}(2g, \mathbb{Z})$  under the above basis. Using an elementary matrix transformation technique, we see that  $\sigma$  is surjective.

Using this representation, it is easy to see that  $\mathcal{M}_g$  is non-trivial for  $g \geq 1$ . In fact, we can show by a topological argument on the torus  $\Sigma_1 = T^2$  that when  $g = 1$ , the map  $\sigma : \mathcal{M}_1 \rightarrow \text{Sp}(H) \cong \text{Sp}(2, \mathbb{Z}) = \text{SL}(2, \mathbb{Z})$  gives an isomorphism.

The kernel of the action  $\sigma$  is called the *Torelli group* denoted by  $\mathcal{I}_g$ . That is,  $\mathcal{I}_g$  consists of all isotopy classes of self-diffeomorphisms of  $\Sigma_g$  which act trivially on  $H$ . From the above discussion, we see soon that  $\mathcal{I}_0$  and  $\mathcal{I}_1$  are trivial. However,  $\mathcal{I}_g$  for  $g \geq 2$  are highly non-trivial and seem to be more complicated

and mysterious than  $\mathcal{M}_g$ . At present, only the case when  $g = 2$  is clarified. Mess [68] showed that  $\mathcal{I}_2$  is an infinitely generated free group.

Prior to Mess' result, a generating system of  $\mathcal{I}_g$  was given by Birman and Powell. A *BP-map* of genus  $h$  is the product  $T_{\gamma_1} \circ T_{\gamma_2}^{-1}$  of two Dehn twists with distinct sign, where  $\{\gamma_1, \gamma_2\}$  is a pair of disjoint, homologous, non-bounding (non-separating) simple closed curves which bound a subsurface of genus  $h$  with two boundary components (see Figure 2.4). Such a pair is called a Boundary Pair. On the other hand, a *BSCC-map* of genus  $h$  is a Dehn twist along a Bounding (separating) Simple Closed Curve (see Figure 2.5) which bounds a subsurface of genus  $h$  with connected boundary. It is easy to see that BP-maps and BSCC maps are elements in  $\mathcal{I}_g$ . On the other hand, using Birman's former result, Powell [86] showed that for  $g \geq 3$ ,  $\mathcal{I}_g$  is generated by all BP-maps of genus 1 and BSCC-maps of genus 1 and 2. When  $g = 2$ , BP-maps are all trivial and BSCC maps generate  $\mathcal{I}_2$ . Note that Hatcher and Margalit [41] gave a new self-contained proof of these facts.

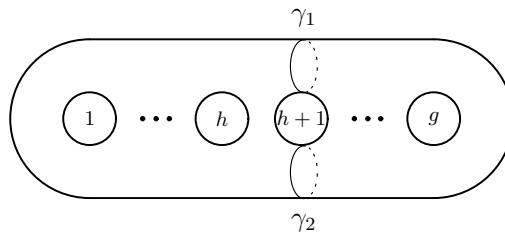


Figure 2.4: a BP-map  $T_{\gamma_1} \circ T_{\gamma_2}^{-1}$  of genus  $h$

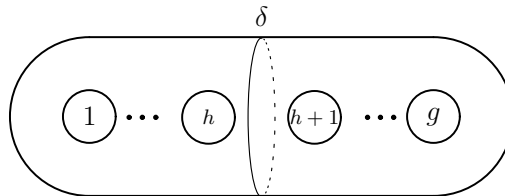


Figure 2.5: a BSCC-map  $T_\delta$  of genus  $h$

### 2.3. Mapping class groups and 3-manifolds

We finish this section by mentioning a relationship of  $\mathcal{M}_g$  and  $\mathcal{I}_g$  to 3-dimensional topology. There are several methods for constructing an oriented closed 3-manifold from a mapping class.

One of the methods is given by gluing two copies of handlebodies  $H_{(g)}$  of genus  $g$  along their boundaries  $\partial H_{(g)} \cong \Sigma_g$  by using an element in  $\mathcal{M}_g$ , so that the resulting 3-manifold is oriented and closed. More precisely, we fix an orientation of  $H_{(g)}$  and denote by  $-H_{(g)}$  the  $H_{(g)}$  with opposite orientation. There exists an orientation-reversing self-diffeomorphism  $\iota_g$  of  $\Sigma_g$  such that the oriented closed 3-manifold  $H_{(g)} \cup_{\iota_g} (-H_{(g)})$  obtained by identifying each  $x \in \partial(-H_{(g)})$  with  $\iota_g(x) \in \partial H_{(g)}$  is the 3-sphere. We fix such a diffeomorphism  $\iota_g$ . Then, for a given mapping class  $[f] \in \mathcal{M}_g$  with  $f \in \text{Diff}_+(\Sigma_g)$ , we construct

$$M_f := H_{(g)} \cup_{\iota_g \circ f} (-H_{(g)}),$$

by identifying each  $x \in \partial(-H_{(g)})$  with  $\iota_g \circ f(x) \in \partial H_{(g)}$ . It is easy to see that the resulting manifold depends only on the mapping class of  $f$ . Consequently, we obtain an oriented closed 3-manifold  $M$  with a decomposition  $M = H_{(g)} \cup_{\partial \Sigma_g} (-H_{(g)})$  with an identification map between  $\partial H_{(g)}$  and  $\partial(-H_{(g)})$ . Such a

decomposition is called a *Heegaard decomposition* of genus  $g$  for  $M$ . If  $[f]$  is in  $\mathcal{I}_g$ , then  $M_f$  has the same homology group as  $S^3 = M_{\text{id}}$ . That is,  $M_f$  is an integral homology 3-sphere. It is a classical fact that for any oriented closed 3-manifold  $M$ , there exists  $[f] \in \mathcal{M}_g$  for some  $g$  such that  $M \cong M_f$ . For any oriented integral homology 3-sphere  $M$ ,  $[f]$  can be taken from  $\mathcal{I}_g$ . In summary, we have the diagram:

$$(2.1) \quad \begin{array}{ccc} \bigsqcup_{g \geq 0} \mathcal{M}_g & \xrightarrow{\sqcup M_\bullet} & \{\text{oriented closed 3-manifolds}\} \\ \uparrow & & \uparrow \\ \bigsqcup_{g \geq 0} \mathcal{I}_g & \xrightarrow{\sqcup M_\bullet} & \{\text{oriented integral homology 3-spheres}\} \end{array}$$

Another method for obtaining an oriented closed 3-manifold from a mapping class is given by constructing the *mapping torus*. For  $f \in \text{Diff}_+(\Sigma_g)$ , we construct a  $\Sigma_g$ -bundle  $T_f$  over  $S^1$  by

$$T_f = \Sigma_g \times [0, 1] / ((x, 1) \sim (f(x), 0), x \in \Sigma_g).$$

It is easy to see that the 3-manifold  $T_f$  depends only on the conjugacy class of the mapping class of  $f$ .

### 3. Johnson's results on the Torelli group

#### 3.1. Johnson's remarkable results

In the first half of 1980's, Johnson gave seminal results on the structure of the Torelli group in a series of his papers. In [45], he showed that only the BP-maps of genus 1 are needed for the generation of  $\mathcal{I}_g$  for  $g \geq 3$ . Since two BP-maps of the same genus are conjugate, his result means that  $\mathcal{I}_g$  is normally generated by *one* BP-map of genus 1. Moreover, he showed the following surprising fact by a tough investigation of combinatorics of BP-maps:

**Theorem 3.1** (Johnson [49]). *For  $g \geq 3$ , there exists a finite set of BP-maps (of genus not necessarily equal to 1) which generates  $\mathcal{I}_g$ .*

The next surprising result of Johnson is the determination of the abelianization  $H_1(\mathcal{I}_g)$ . The (*first*) *Johnson homomorphism*, which is our main concern, appears in it and gives the free part of  $H_1(\mathcal{I}_g)$ . In the following,  $\wedge^3 H$  denotes the third exterior power of  $H$ , in which  $H$  is embedded by the map  $x \mapsto x \wedge \omega$  with  $\omega := \sum_{i=1}^g a_i \wedge b_i$ , the symplectic element.

**Theorem 3.2** (Johnson [50, 51, 48]). (1) *For  $g \geq 3$ , there exists an  $\mathcal{M}_g$ -equivariant surjective homomorphism*

$$\tau_1 : \mathcal{I}_g \longrightarrow \wedge^3 H/H$$

*whose kernel coincides with the subgroup  $\mathcal{K}_g$  generated by the BSCC maps. Here  $\mathcal{M}_g$  acts on  $\mathcal{I}_g$  by the conjugation and on  $\wedge^3 H$  diagonally through  $\sigma : \mathcal{M}_g \rightarrow \text{Sp}(H)$ .*

(2) *For  $g \geq 3$ , the homomorphism  $\tau_1$  gives the free part of  $H_1(\mathcal{I}_g)$ . More precisely,*

$$H_1(\mathcal{I}_g) \cong \wedge^3 H/H \oplus (2\text{-torsions}).$$

**Remark 3.3.** The precise description of  $H_1(\mathcal{I}_g)$  is given in [51]. In fact, the 2-torsion part of  $H_1(\mathcal{I}_g)$  is governed by the *Birman-Craggs-Johnson homomorphism* (see papers of Birman-Craggs [8] and Johnson [47]), which unifies the Rochlin invariants of homology 3-spheres associated with each element of  $\mathcal{I}_g$  via various Heegaard embeddings of  $\Sigma_g$  into  $S^3$ .

Remaining well-known problems are the following:

- Problem 3.4.** (1) Determine whether  $\mathcal{I}_g$  is finitely presentable or not for  $g \geq 3$ .  
 (2) Determine whether  $\mathcal{K}_g$  is finitely generated or not.

It might be a good way of attacking the problem (1) to look for infinite index subgroups of  $\mathcal{M}_g$  which are finitely presentable and include  $\mathcal{I}_g$ . As for (2), it was shown by Dimca and Papadima [20] that  $H_1(\mathcal{K}_g) \otimes \mathbb{Q}$  is finite dimensional over  $\mathbb{Q}$  for  $g \geq 4$ .

[Addendum: In April 2017, the problem (2) was affirmatively solved by Ershov-He and Church-Putman independently. It was announced that their joint paper will appear shortly.]

### 3.2. Mapping class groups of surfaces with connected boundary

To explain the definition of the Johnson homomorphism  $\tau_1$ , it is useful to introduce a variant of  $\mathcal{M}_g$ . Let  $\Sigma_{g,1}$  be the surface obtained from  $\Sigma_g$  by removing an open disk. We define the mapping class group  $\mathcal{M}_{g,1}$  of  $\Sigma_{g,1}$  to be the group of path-connected components of the Fréchet Lie group consisting of all self-diffeomorphisms of  $\Sigma_{g,1}$  fixing (a neighborhood of)  $\partial\Sigma_{g,1}$  pointwise. Removing an open disk from  $\Sigma_g$  does not affect on  $H_1$ , so that we identify  $H_1(\Sigma_{g,1})$  with  $H$ . The Torelli group  $\mathcal{I}_{g,1}$  for  $\mathcal{M}_{g,1}$  is defined as the kernel of the natural action  $\sigma : \mathcal{M}_{g,1} \rightarrow \text{Sp}(H)$ . It is also normally generated by one BP-map of genus 1 for  $g \geq 3$  (see [45]). Here, the genus of a BP-map (resp. BSCC-map) of  $\mathcal{I}_{g,1}$  is defined by the genus of the subsurface bounded by the BP (resp. BSCC) not having the original boundary of  $\Sigma_{g,1}$ .

We have natural surjective homomorphisms  $\mathcal{M}_{g,1} \rightarrow \mathcal{M}_g$  and  $\mathcal{I}_{g,1} \rightarrow \mathcal{I}_g$  by extending each mapping class of  $\Sigma_{g,1}$  by the identity on the disk  $\Sigma_g - \Sigma_{g,1}$  to get a mapping class of  $\Sigma_g$ . The kernels of these homomorphisms are common and known to be isomorphic to the fundamental group of the unit tangent bundle of  $\Sigma_g$  (see Johnson's paper [49, Section 3], for instance). Basically, this fact follows from the homotopy exact sequence of the locally trivial fibration obtained by applying the fibering theorem of Cerf [12] and Palais-Hirsch [84] to the Fréchet Lie groups of diffeomorphisms of  $\Sigma_{g,1}$  (fixing a neighborhood of  $\partial\Sigma_{g,1}$ ) and  $\Sigma_g$ .

The fundamental group  $\pi := \pi_1(\Sigma_{g,1}, *)$  of  $\Sigma_{g,1}$  with respect to a base point  $*$  on the boundary is a free group of rank  $2g$ . Since the self-diffeomorphisms of  $\Sigma_{g,1}$  we are considering fix the base point  $*$ , we have the natural action of  $\mathcal{M}_{g,1}$  on  $\pi$ . This is the reason we have introduced  $\mathcal{M}_{g,1}$ . In fact, this action characterizes mapping classes.

**Theorem 3.5** (Dehn, Nielsen, Baer, Epstein, Zieschang, et al.). (1) *The action  $\mathcal{M}_{g,1} \rightarrow \text{Aut}(\pi)$  is injective and the image is given by*

$$\{\varphi \in \text{Aut}(\pi) \mid \varphi(\zeta) = \zeta\},$$

where  $\zeta \in \pi$  is the (oriented) loop along the boundary.

(2) *The above action induces an injection  $\mathcal{M}_g \hookrightarrow \text{Out}(\pi_1(\Sigma_g))$  whose image consists of the outer automorphism classes of  $\pi_1(\Sigma_g) \cong \pi / \langle\langle \zeta \rangle\rangle$  inducing the identity map on  $H_2(\pi_1(\Sigma_g)) \cong H_2(\Sigma_g) \cong \mathbb{Z}$ . Here,  $\langle\langle \zeta \rangle\rangle$  denotes the normal closure of  $\zeta$  in  $\pi$ .*

Recall that the *outer automorphism group*  $\text{Out}(G)$  of a group  $G$  is defined as the quotient group of  $\text{Aut}(G)$  with respect to the normal subgroup of the inner (conjugation) automorphisms.

For the reader's convenience, the abelianization of  $H_1(\mathcal{M}_{g,1})$  is as follows (see Korkmaz-Stipsicz [56, Section 5]):

$$H_1(\mathcal{M}_{g,1}) \cong \begin{cases} 0 & (g = 0) \\ \mathbb{Z} & (g = 1) \\ \mathbb{Z}/10\mathbb{Z} & (g = 2) \\ 0 & (g \geq 3) \end{cases}.$$

The proof of Theorem 3.2 is given by showing first the corresponding statements for  $\mathcal{I}_{g,1}$ . The actions of  $\mathcal{M}_{g,1}$  on  $\mathcal{I}_{g,1}$  and  $\wedge^3 H$  are similar to those of  $\mathcal{M}_g$ .

**Theorem 3.6** (Johnson [50, 51, 48]). (1) *For  $g \geq 2$ , there exists an  $\mathcal{M}_{g,1}$ -equivariant surjective homomorphism*

$$\tau_1 : \mathcal{I}_{g,1} \longrightarrow \wedge^3 H$$

whose kernel coincides with the subgroup  $\mathcal{K}_{g,1}$  generated by the BSCC maps.



(2) For  $g \geq 3$ , the homomorphism  $\tau_1$  gives the free part of  $H_1(\mathcal{I}_{g,1})$ . More precisely,

$$H_1(\mathcal{I}_{g,1}) \cong \wedge^3 H \oplus (2\text{-torsions}).$$

Once we show this theorem, it is easy to get the results for  $\mathcal{I}_g$ .

### 3.3. The first Johnson homomorphism

The idea of the definition of the first Johnson homomorphism  $\tau_1$  for  $\mathcal{I}_{g,1}$  is to consider the natural action of  $\mathcal{M}_{g,1}$  on the 2-step nilpotent group  $\pi/[\pi, [\pi, \pi]]$  instead of the action on  $H = \pi/[\pi, \pi]$ . Johnson defined  $\tau_1$  as in the following way.

First, consider the map

$$\tau : \mathcal{I}_{g,1} \longrightarrow \text{Hom}(\pi, [\pi, \pi]/[\pi, [\pi, \pi]])$$

assigning to  $[f] \in \mathcal{I}_{g,1}$  a homomorphism sending  $\gamma \in \pi$  to  $f_*(\gamma)\gamma^{-1}$ , which is in  $[\pi, \pi]$  since  $\mathcal{I}_{g,1}$  acts trivially on  $H = \pi/[\pi, \pi]$ . We identify  $[\pi, \pi]/[\pi, [\pi, \pi]]$  with  $\wedge^2 H$ , the second exterior power of  $H$ , by the map  $[\gamma_1, \gamma_2] \mapsto \overline{\gamma_1} \wedge \overline{\gamma_2}$  for  $\gamma_1, \gamma_2 \in \pi$  and their homology classes  $\overline{\gamma_1}, \overline{\gamma_2} \in H$  (see the isomorphism (4.1) in the next section for a more general statement). Then using the identification  $H = H^*$ , we have

$$\text{Hom}(\pi, [\pi, \pi]/[\pi, [\pi, \pi]]) \cong \text{Hom}(H, \wedge^2 H) = H^* \otimes \wedge^2 H = H \otimes \wedge^2 H.$$

It is easy to check that the resulting map  $\tau : \mathcal{I}_{g,1} \rightarrow H \otimes \wedge^2 H$  is an  $\mathcal{M}_{g,1}$ -equivariant homomorphism, where  $\mathcal{M}_{g,1}$  acts on  $\mathcal{I}_{g,1}$  by conjugation and on  $H \otimes \wedge^2 H$  diagonally through  $\sigma : \mathcal{M}_{g,1} \rightarrow \text{Sp}(H)$ . We observe from the following explicit computations for generators that the image of  $\tau$  coincides with the  $\text{Sp}(H)$ -submodule  $\wedge^3 H$  of  $H \otimes \wedge^2 H$ , where  $x \wedge y \wedge z \in \wedge^3 H$  corresponds to  $x \otimes (y \wedge z) + y \otimes (z \wedge x) + z \otimes (x \wedge y) \in H \otimes \wedge^2 H$ .

**Example 3.7.** (1) In Figure 2.4, we remove an open disk from the subsurface bounded by  $\gamma_1$  and  $\gamma_2$  including  $g$ -th hole to get a surface  $\Sigma_{g,1}$ . Then the BP-map  $T_{\gamma_1} \circ T_{\gamma_2}^{-1}$  of genus  $h$  is in  $\mathcal{I}_{g,1}$ . For  $1 \leq h \leq g-1$ , we have

$$\tau(T_{\gamma_1} \circ T_{\gamma_2}^{-1}) = \left( \sum_{i=1}^h a_i \wedge b_i \right) \wedge b_{h+1} \in \wedge^3 H \subset H \otimes \wedge^2 H,$$

where we regard  $\{a_1, \dots, a_g, b_1, \dots, b_g\}$  as the lift of the symplectic basis in Figure 2.3 to  $H_1(\Sigma_{g,1}) = H$ .

(2) Similarly, the BP-map  $T_\delta \in \mathcal{I}_{g,1}$  of genus  $h$  is obtained from Figure 2.5. For this map, we have  $\tau(T_\delta) = 0$ .

Consequently, we obtain an  $\mathcal{M}_{g,1}$ -equivariant homomorphism  $\mathcal{I}_{g,1} \rightarrow \wedge^3 H$ , which we denote by  $\tau_1$ .

Note that this is only the starting point for the Johnson's remarkable results. To prove Theorems 3.2 and 3.6, we need further tough arguments as in his papers. By definition, the kernel of  $\tau_1$  coincides with the kernel of the natural action  $\mathcal{M}_{g,1} \rightarrow \text{Aut}(\pi/[\pi, [\pi, \pi]])$ . Theorem 3.6 (1) says that it is just  $\mathcal{K}_{g,1}$ , which is a highly non-trivial fact.

**Remark 3.8.** After over 20 years of the publication of Johnson's papers, alternative proofs of many of his results were given by Putman [87, 88] (see also his lecture notes [89]) in a more conceptual and generalized way.

## 4. Higher Johnson homomorphisms

### 4.1. Definition of higher Johnson homomorphisms

In [48], Johnson mentioned about a generalization of his homomorphism  $\tau_1$ . After that, Morita [71] gave an improvement of Johnson's formulation and it is now commonly used. Hereafter, we discuss only generalizations to subgroups of  $\mathcal{M}_{g,1}$  for simplicity.

**Notation 4.1.** The *lower central series*  $\{\Gamma_k G\}_{k \geq 1}$  of a group  $G$  is defined inductively by

$$\Gamma_1 G := G, \quad \Gamma_k G := [G, \Gamma_{k-1} G] \quad (k \geq 2).$$

The  $k$ -th nilpotent quotient  $N_k(G)$  of  $G$  is defined by  $N_k(G) := G/\Gamma_k G$ . We put  $N_k := N_k(\pi)$ , the  $k$ -th free nilpotent quotient of rank  $2g$ .

The idea of the definition of higher Johnson homomorphisms  $\tau_k$  for  $k \geq 2$  is to consider the natural actions

$$\sigma_k : \mathcal{M}_{g,1} \longrightarrow \text{Aut}(N_k)$$

of  $\mathcal{M}_{g,1}$  on the nilpotent quotients  $N_k$ , where  $\sigma_2 = \sigma$  and we have used  $\sigma_3$  to define  $\tau_1$ .

From the definition, we have a central extension

$$0 \longrightarrow \Gamma_k \pi / \Gamma_{k+1} \pi \longrightarrow N_{k+1} \longrightarrow N_k \longrightarrow 1$$

for  $k \geq 2$ . It is classically known in combinatorial group theory (see Magnus-Karrass-Solitar [59] for instance) that there is an  $\text{Aut}(\pi)$ -equivariant isomorphism

$$(4.1) \quad \Gamma_k \pi / \Gamma_{k+1} \pi \cong \mathcal{L}_k,$$

where  $\mathcal{L}_k$  is the degree  $k$ -part of the free Lie algebra  $\mathcal{L}_* := \bigoplus_{j \geq 1} \mathcal{L}_j$  generated by  $H$ . The isomorphism is explicitly given by

$$\Gamma_k \pi / \Gamma_{k+1} \pi \ni [\gamma_1, [\gamma_2, \dots, [\gamma_{k-1}, \gamma_k] \dots]] \longmapsto [\bar{\gamma}_1, [\bar{\gamma}_2, \dots, [\bar{\gamma}_{k-1}, \bar{\gamma}_k] \dots]] \in \mathcal{L}_k$$

for  $\gamma_i \in \pi$  and its homology class  $\bar{\gamma}_i \in H$ . In fact, the direct sum  $\bigoplus_{j \geq 1} (\Gamma_j \pi / \Gamma_{j+1} \pi)$  is endowed with a Lie algebra structure by commutators, and the isomorphisms for  $k \geq 1$  yield the natural identification  $\bigoplus_{j \geq 1} (\Gamma_j \pi / \Gamma_{j+1} \pi) \cong \mathcal{L}_*$  as Lie algebras.

**Example 4.2.** We have  $\mathcal{L}_1 = H$  by definition. The anti-symmetry relation gives  $\mathcal{L}_2 \cong \wedge^2 H$ . We have the well-defined isomorphism  $\mathcal{L}_3 \xrightarrow{\cong} (H \otimes \wedge^2 H) / \wedge^3 H$  sending  $[x, [y, z]] \in \mathcal{L}_3$  with  $x, y, z \in H$  to  $x \otimes (y \wedge z) \in (H \otimes \wedge^2 H) / \wedge^3 H$ , where  $\wedge^3 H$  is embedded in  $H \otimes \wedge^2 H$  by the map  $x \wedge y \wedge z \mapsto x \otimes (y \wedge z) + y \otimes (z \wedge x) + z \otimes (x \wedge y)$  and embodies the Jacobi identity. It is difficult to give an explicit description for components of  $\mathcal{L}_*$  in higher degrees.

The domain groups of higher Johnson homomorphisms are given by the following:

**Definition 4.3.** The Johnson filtration  $\{\mathcal{M}_{g,1}[k]\}_{k \geq 0}$  of  $\mathcal{M}_{g,1}$  is defined by

$$\mathcal{M}_{g,1}[0] := \mathcal{M}_{g,1}, \quad \mathcal{M}_{g,1}[k] := \text{Ker}(\sigma_{k+1} : \mathcal{M}_{g,1} \rightarrow \text{Aut}(N_{k+1})) \quad (k \geq 1).$$

We have  $\mathcal{M}_{g,1}[1] = \mathcal{I}_{g,1}$  by definition, and Johnson's result (Theorem 3.6 (1)) says that  $\mathcal{M}_{g,1}[2] = \mathcal{K}_{g,1}$ . Since the intersection  $\bigcap_{k \geq 1} \Gamma_k \pi$  is known to be trivial for the free group  $\pi$  (i.e.  $\pi$  is residually nilpotent, see [59, Section 5.5] for example), Theorem 3.5 (1) implies that the intersection  $\bigcap_{k \geq 0} \mathcal{M}_{g,1}[k]$  is trivial.

We now consider the gap between  $\sigma_{k+2}$  and  $\sigma_{k+1}$ .

**Theorem 4.4** (Andreadakis [3], Morita [72]). For  $k \geq 1$ , we have an exact sequence

$$0 \longrightarrow \text{Hom}(\pi, \Gamma_{k+1} \pi / \Gamma_{k+2} \pi) \longrightarrow \text{Aut}(N_{k+2}) \longrightarrow \text{Aut}(N_{k+1}) \longrightarrow 1,$$

where the inclusion  $\text{Hom}(\pi, \Gamma_{k+1} \pi / \Gamma_{k+2} \pi) \hookrightarrow \text{Aut}(N_{k+2})$  is given by assigning to  $\phi \in \text{Hom}(\pi, \Gamma_{k+1} \pi / \Gamma_{k+2} \pi)$  the automorphism of  $N_{k+2}$  induced from the map  $\pi \ni \gamma \mapsto \phi(\gamma)\gamma \in N_{k+2} = \pi / \Gamma_{k+2} \pi$ .

Since the image of the restriction of  $\sigma_{k+2}$  to  $\mathcal{M}_{g,1}[k]$  is sent to the trivial group by the map  $\text{Aut}(N_{k+2}) \rightarrow \text{Aut}(N_{k+1})$ , we may consider the target of the restricted map to be  $\text{Hom}(\pi, \Gamma_{k+1} \pi / \Gamma_{k+2} \pi)$ . Now we have

$$\text{Hom}(\pi, \Gamma_{k+1} \pi / \Gamma_{k+2} \pi) \cong \text{Hom}(H, \mathcal{L}_{k+1}) = H^* \otimes \mathcal{L}_{k+1} = H \otimes \mathcal{L}_{k+1}.$$

Consequently, we get to the definition of the  $k$ -th Johnson homomorphism

$$\tau_k := \sigma_{k+2}|_{\mathcal{M}_{g,1}[k]} : \mathcal{M}_{g,1}[k] \longrightarrow H \otimes \mathcal{L}_{k+1}$$

for  $k \geq 1$ . When  $k = 1$ , this definition coincides with the one in the previous section. The homomorphism  $\tau_k$  is  $\mathcal{M}_{g,1}$ -equivariant, where  $\mathcal{M}_{g,1}$  acts on  $\mathcal{M}_{g,1}[k]$  by conjugation and on  $H \otimes \mathcal{L}_{k+1}$  diagonally through  $\sigma : \mathcal{M}_{g,1} \rightarrow \text{Sp}(H)$ . The kernel of  $\tau_k$  is  $\mathcal{M}_{g,1}[k+1]$ .

**Example 4.5.** In [71], Morita derived the formula of  $\tau_2$  for BSCC-maps, generators of  $\mathcal{K}_{g,1}$ . For the BSCC  $T_\delta$  of genus  $h$  in Example 3.7 (2), we have

$$\tau_2(T_\delta) = \sum_{i,j=1}^h (a_i \otimes b_j - b_i \otimes a_j) \otimes b_j \wedge a_j \in H \otimes \mathcal{L}_3 = H \otimes ((H \otimes \wedge^2 H) / \wedge^3 H).$$

**Remark 4.6.** Prior to works of Johnson and Morita on  $\{\mathcal{M}_{g,1}[k]\}_{k \geq 0}$ , Andreadakis [3] studied the automorphism group  $\text{Aut}(F_n)$  of free groups  $F_n$  of rank  $n \geq 2$  by using its action on the nilpotent quotients  $N_k(F_n)$ . In fact, he defined a filtration of  $\text{Aut}(F_n)$  as in the way we have discussed. By using the exact sequence of Theorem 4.4, we can define Johnson homomorphisms for the subgroups in the filtration of  $\text{Aut}(F_n)$ . For more details, see Satoh's survey paper [93].

**Remark 4.7.** Kawazumi [52] gave a unified description of Johnson homomorphisms by using Magnus expansions of free groups. See also Massuyeau [62] and Kawazumi-Kuno [53] for the most recent descriptions of Johnson homomorphisms and their extensions.

## 4.2. Refinement of higher Johnson homomorphisms

In [73], Morita defined a refinement

$$\tilde{\tau}_k : \mathcal{M}_{g,1}[k] \longrightarrow H_3(N_{k+1})$$

of the  $k$ -th Johnson homomorphism  $\tau_k : \mathcal{M}_{g,1}[k] \rightarrow H \otimes \mathcal{L}_{k+1}$ . It turns out to give a strong constraint on the image of  $\tau_k$  in  $H \otimes \mathcal{L}_{k+1}$  by recovering  $\tau_k$  from  $\tilde{\tau}_k$ . For that, we recall the central extension

$$0 \longrightarrow \mathcal{L}_{k+1} \longrightarrow N_{k+2} \longrightarrow N_{k+1} \longrightarrow 1.$$

The Lyndon-Hochschild-Serre spectral sequence (see Brown's book [11, Chapter VII-6] for details) associated with this central extension is of the form

$$E_{p,q}^2 = H_p(N_{k+1}) \otimes H_q(\mathcal{L}_{k+1}) \implies H_*(N_{k+2}).$$

By observing this spectral sequence, we obtain an exact sequence

$$(4.2) \quad H_3(N_{k+2}) \longrightarrow (E_{3,0}^2 = H_3(N_{k+1})) \xrightarrow{d_{3,0}^2} (E_{1,1}^2 = H \otimes \mathcal{L}_{k+1}) \xrightarrow{[\cdot, \cdot]} (E_{1,1}^\infty = \mathcal{L}_{k+2}) \longrightarrow 0,$$

where the first map is induced from the projection  $N_{k+2} \rightarrow N_{k+1}$ , the second map is the differential  $d_{3,0}^2$  in the spectral sequence and the third map is just the bracket map in the free Lie algebra  $\mathcal{L}_* = \bigoplus_{j \geq 1} \mathcal{L}_j$ .

**Theorem 4.8** (Morita [73]). *For  $k \geq 1$ , the composition  $d_{3,0}^2 \circ \tilde{\tau}_k$  coincides with  $\tau_k$ . Consequently, the image of  $\tau_k$  is in the  $\text{Sp}(H)$ -submodule*

$$\mathfrak{h}_{g,1}(k) := \text{Ker}([\cdot, \cdot] : H \otimes \mathcal{L}_{k+1} \rightarrow \mathcal{L}_{k+2}).$$

See also Garoufalidis-Levine [28, Proposition 2.5] for a direct proof that the image of  $\tau_k$  is in  $\mathfrak{h}_{g,1}(k)$ .

**Example 4.9.** (1) When  $k = 1$ , it is easy to see from Example 4.2 that  $\mathfrak{h}_{g,1}(1) \cong \wedge^3 H$ . We observe that  $H_3(N_2) = H_3(H) \cong \wedge^3 H$  and that the differential  $d_{3,0}^2$  is just the inclusion  $\wedge^3 H \hookrightarrow H \otimes \mathcal{L}_2$ . That is, we have  $\tilde{\tau}_1 = \tau_1$ .

(2) Using Morita's formula of  $\tau_2$  mentioned in Example 4.5, Yokomizo [98] determined the full image of  $\tau_2$  in  $\mathfrak{h}_{g,1}(2)$ . He showed that

$$\mathfrak{h}_{g,1}(2) / \text{Im } \tau_2 \cong (\mathbb{Z}/2\mathbb{Z})^{(g-1)(2g+1)}$$

for  $g \geq 2$  with a specific basis.

Since, at present, we do not have any generating system of  $\mathcal{M}_{g,1}[k]$  for  $k \geq 3$  which is explicitly written, it is a difficult problem to determine the full image of higher Johnson homomorphisms.

Morita's definition of  $\tilde{\tau}_k$  is purely algebraic and it is not easy to grasp its topological meaning. Later, Heap [42] gave a topological definition of this homomorphism, which we see in the next subsection.

**Remark 4.10.** The Johnson filtration  $\{\mathcal{M}_g[k]\}_{k \geq 0}$  of  $\mathcal{M}_g$  is obtained from the natural outer action  $\sigma_{k+1} : \mathcal{M}_g \rightarrow \text{Out}(N_{k+1}(\pi_1(\Sigma_g)))$  in a similar way to that of  $\mathcal{M}_{g,1}$ . The  $k$ -th Johnson homomorphism  $\tau_k$  for  $\mathcal{M}_g[k]$  is naturally induced from  $\tau_k : \mathcal{M}_{g,1} \rightarrow \mathfrak{h}_{g,1}(k)$  by taking an appropriate quotient of the target  $\mathfrak{h}_{g,1}(k)$ . The restriction of the map  $\mathcal{M}_{g,1} \rightarrow \mathcal{M}_g$  to  $\mathcal{M}_{g,1}[k]$  is surjective onto  $\mathcal{M}_g[k]$ .

### 4.3. Topological construction of the refinement of higher Johnson homomorphisms

Heap's topological description of the refinement  $\tilde{\tau}_k$  uses the open book construction (decomposition) of closed 3-manifolds. For a given mapping class  $[f] \in \mathcal{M}_{g,1}$ , we construct an oriented and closed 3-manifold  $C_f$  by

$$C_f := (\Sigma_{g,1} \times [0, 1]) / \left( \begin{array}{ll} (x, 1) \sim (f(x), 0) & (x \in \Sigma_{g,1}) \\ (y, 0) \sim (y, t) & (y \in \partial\Sigma_{g,1}, t \in [0, 1]) \end{array} \right).$$

The 3-manifold  $C_f$  depends only on the conjugacy class of the mapping class of  $f$ . This description of the 3-manifold  $C_f$  is called the *open book decomposition* associated with  $[f] \in \mathcal{M}_{g,1}$ . The image of  $\partial\Sigma_{g,1} \times \{0\}$  by the canonical projection  $\rho : \Sigma_{g,1} \times [0, 1] \rightarrow C_f$  gives a knot in  $C_f$  called the *binding* of the open book. It is also regarded as a *fibred knot* in  $C_f$  with a Seifert surface given by the projection of  $\Sigma_{g,1} \times \{0\}$ .

Prior to Heap's work, Johnson [48] stated that  $\tau_k$  can be described via Massey products of mapping tori and Kitano [54] gave the full proof. The following construction by Heap can be regarded as a refinement of the description of Johnson and Kitano.

For a given mapping class  $[f] \in \mathcal{M}_{g,1}[k]$ , we construct the oriented closed 3-manifold  $C_f$  as above. The fundamental group of  $C_f$  is given by

$$\pi_1(C_f) \cong \pi / \langle\langle f_*(\gamma)\gamma^{-1} \mid \gamma \in \pi \rangle\rangle.$$

Since  $f_* : N_{k+1} \rightarrow N_{k+1}$  is trivial, we have a commutative diagram

$$\begin{array}{ccc} \pi_1(\Sigma_{g,1}) = \pi_1(\Sigma_{g,1} \times [0, 1]) & \xrightarrow{p_*} & \pi_1(C_f) \\ \downarrow & & \downarrow \\ N_{k+1} & \xrightarrow[\rho_*]{\cong} & N_{k+1}(\pi_1(C_f)) \end{array}$$

whose bottom horizontal map is an isomorphism. Let  $B_f$  be the composition

$$B_f : C_f \xrightarrow{\eta_{C_f}} K(\pi_1(C_f), 1) \longrightarrow K(N_{k+1}(\pi_1(C_f)), 1) \xrightarrow{(p_*^{-1})^\dagger} K(N_{k+1}, 1)$$

of the natural continuous maps where  $\eta_{C_f}$  is the map mentioned in Remark 2.2 and the others are induced from the homomorphisms in the above diagram. (In general, a group homomorphism  $f : G_1 \rightarrow G_2$  induces a unique continuous map  $f_\dagger : K(G_1, 1) \rightarrow K(G_2, 1)$  with  $(f_\dagger)_* = f$  up to homotopy.) Define a map

$$\theta_k : \mathcal{M}_{g,1}[k] \longrightarrow H_3(K(N_{k+1}, 1)) = H_3(N_{k+1})$$

by assigning to  $[f] \in \mathcal{M}_{g,1}[k]$  the image  $(B_f)_*([C_f])$  of the fundamental class  $[C_f] \in H_3(C_f)$  by  $(B_f)_*$ .

**Theorem 4.11** (Heap [42]). *The map  $\theta_k : \mathcal{M}_{g,1}[k] \rightarrow H_3(N_k)$  is a homomorphism. It coincides with  $\tilde{\tau}_k$  and the kernel of  $\theta_k = \tilde{\tau}_k$  is  $\mathcal{M}_{g,1}[2k]$ .*

The additivity of the map  $\theta_k$ , which can be checked directly without showing the coincidence with  $\tilde{\tau}_k$ , will be mentioned in Section 6 in a generalized form. To see that the kernel of  $\theta_k = \tilde{\tau}_k$  is  $\mathcal{M}_{g,1}[2k]$ , we recall  $H_*(N_{k+1})$  up to degree 3. The abelianization  $H_1(N_{k+1})$  of  $N_{k+1}$  is  $N_2 = H$ . By applying Hopf's general formula (see Brown's book [11, Chapter II-5]) for the computation of  $H_2$  to the presentation  $N_{k+1} \cong \pi / \Gamma_{k+1}\pi$ , we have

$$H_2(N_{k+1}) \cong (\Gamma_{k+1}\pi \cap [\pi, \pi]) / [\pi, \Gamma_{k+1}\pi] = \Gamma_{k+1}\pi / \Gamma_{k+2}\pi \cong \mathcal{L}_{k+1}.$$

The structure of  $H_3(N_{k+1})$  was determined by Igusa and Orr as follows. For  $i \geq j$ , let  $\varphi_{i,j} : H_3(N_i) \rightarrow H_3(N_j)$  be the map induced from the natural projection  $N_i \rightarrow N_j$ . Define a filtration

$$H_3(N_{k+1}) = \mathcal{F}^{k+1}H_3(N_{k+1}) \supset \mathcal{F}^{k+2}H_3(N_{k+1}) \supset \dots$$

of  $H_3(N_{k+1})$  by  $\mathcal{F}^l H_3(N_{k+1}) := \text{Im}(\varphi_{l,k+1} : H_3(N_l) \rightarrow H_3(N_{k+1}))$ .

**Theorem 4.12** (Igusa-Orr [43]). *The filtration  $\{\mathcal{F}^l H_3(N_{k+1})\}_{k+1 \leq l \leq 2k+1}$  of  $H_3(N_{k+1})$  satisfies the following:*

1.  $\mathcal{F}^{2k+1} H_3(N_{k+1}) = 0$ , namely  $\varphi_{2k+1,k+1} : H_3(N_{2k+1}) \rightarrow H_3(N_{k+1})$  is the 0-map.
2. For  $k+1 \leq l \leq 2k$ , the quotient  $\mathcal{G}^l H_3(N_{k+1}) := \mathcal{F}^l H_3(N_{k+1}) / \mathcal{F}^{l+1} H_3(N_{k+1})$  is a finitely generated free abelian group whose rank depends only on  $l$ .
3. For  $k+1 \leq l \leq 2k+1$ , we have  $\text{Ker } \varphi_{l,k+1} = \text{Im } \varphi_{2k+1,l} \subset H_3(N_l)$ .

From this theorem, we see that  $H_3(N_{k+1})$  is obtained from the finitely generated free abelian group  $\mathcal{F}^{2k} H_3(N_{k+1}) = \mathcal{G}^{2k} H_3(N_{k+1})$  by extending one after another by the finitely generated free abelian groups  $\mathcal{G}^{2k-1} H_3(N_{k+1}), \dots, \mathcal{G}^{k+1} H_3(N_{k+1})$ . Therefore we have a non-canonical direct sum decomposition

$$H_3(N_{k+1}) \cong \mathcal{G}^{k+1} H_3(N_{k+1}) \oplus \dots \oplus \mathcal{G}^{2k} H_3(N_{k+1}).$$

Note that in this direct sum decomposition, only the projection to the *first* summand is canonical. From the exact sequence (4.2), it follows that the summand  $\mathcal{G}^{k+1} H_3(N_{k+1})$  coincides with  $\mathfrak{h}_{g,1}(k)$ . Hence the composition of  $\theta_k$  with the first projection recovers the homomorphism  $\tau_k$ .

In summary,  $H_3(N_j)$  is described as in the following table ([99]):

$j$						
$\vdots$						
$k+1$			<u><math>k+1</math></u>	$\dots$	$\dots$	<u><math>2k</math></u>
$\vdots$			$\ddots$			$\ddots$
$5$			<u><math>5</math></u>	<u><math>6</math></u>	$\dots$	<u><math>8</math></u>
$4$			<u><math>4</math></u>	<u><math>5</math></u>	<u><math>6</math></u>	
$3$	<u><math>3</math></u>	<u><math>4</math></u>				
$2$	<u><math>2</math></u>					

The direct sum decomposition of  $H_3(N_j)$

Here, we denote  $\mathcal{G}^l H_3(N_j)$  by  $\underline{l}$ . The modules aligned vertically are mapped by  $\varphi_{i_1, i_2}$  isomorphically. We have a non-canonical direct sum decomposition

$$(4.3) \quad H_3(N_{k+1}) \cong \mathfrak{h}_{g,1}(k) \oplus \dots \oplus \mathfrak{h}_{g,1}(2k-1).$$

From this isomorphism, we see that there is a non-canonical direct sum decomposition

$$\text{Im } \tilde{\tau}_k \cong \text{Im } \tau_k \oplus \dots \oplus \text{Im } \tau_{2k-1},$$

namely the refinement  $\tilde{\tau}_k$  gives a bigger free abelian quotient of  $\mathcal{M}_{g,1}[k]$  than  $\tau_k$  for  $k \geq 2$ .

## 5. Further topics and problems on Johnson homomorphisms

In this section, we collect known problems on Johnson homomorphisms for mapping class groups to show that they are related to many interesting subjects. The discussions here are more advanced than the other parts of paper, so that the readers who study Johnson homomorphisms for the first time can skip this section.

### 5.1. Johnson homomorphisms and the symplectic derivation Lie algebra

The  $k$ -th Johnson homomorphism  $\tau_k : \mathcal{M}_{g,1}[k] \rightarrow \mathfrak{h}_{g,1}(k)$  can be regarded as an embedding of  $\mathcal{M}_{g,1}[k]/\mathcal{M}_{g,1}[k+1]$  into  $\mathfrak{h}_{g,1}(k) \subset \text{Hom}(H, \mathcal{L}_{k+1})$ . By assembling the Johnson homomorphisms  $\tau_k$ , we have an embedding

$$\tau := \bigoplus_{k \geq 1} \tau_k : \bigoplus_{k \geq 1} \mathcal{M}_{g,1}[k]/\mathcal{M}_{g,1}[k+1] \longrightarrow \mathfrak{h}_{g,1} := \bigoplus_{k \geq 1} \mathfrak{h}_{g,1}(k)$$

as a graded module. The Johnson filtration  $\{\mathcal{M}_{g,1}[k]\}_{k \geq 1}$  starting from  $\mathcal{M}_{g,1}[1] = \mathcal{I}_{g,1}$  is central (i.e.  $[\mathcal{M}_{g,1}[1], \mathcal{M}_{g,1}[k]] \subset \mathcal{M}_{g,1}[k+1]$  holds for all  $k \geq 1$ ), for all  $\tau_k$  are  $\mathcal{M}_{g,1}$ -equivariant. Hence the module  $\bigoplus_{k \geq 1} \mathcal{M}_{g,1}[k]/\mathcal{M}_{g,1}[k+1]$  has a natural structure of a Lie algebra (over  $\mathbb{Z}$ ) whose bracket operation is induced from taking commutators. It turns out that  $\tau$  becomes a graded Lie algebra embedding under the following Lie algebra structure of  $\mathfrak{h}_{g,1}$ .

The module  $\text{End}(\mathcal{L}_*)$  of all endomorphisms of  $\mathcal{L}_*$  is a Lie algebra whose bracket operation is given by  $[A, B] = AB - BA$  for  $A, B \in \text{End}(\mathcal{L}_*)$ . We take the submodule  $\text{Der}(\mathcal{L}_*)$  of all derivations of  $\mathcal{L}_*$ . Here  $D \in \text{End}(\mathcal{L}_*)$  is a derivation if

$$(5.1) \quad D([X, Y]) = [D(X), Y] + [X, D(Y)]$$

holds for any  $X, Y \in \mathcal{L}_*$ . It is easy to check that  $\text{Der}(\mathcal{L}_*)$  is a Lie subalgebra. By the equality (5.1), a derivation  $D$  is characterized by its action on  $\mathcal{L}_1 = H$ . Hence we have

$$\text{Der}(\mathcal{L}_*) \cong \text{Hom}(H, \bigoplus_{j \geq 1} \mathcal{L}_j) \cong \bigoplus_{k \geq 0} \text{Hom}(H, \mathcal{L}_{k+1}) = \bigoplus_{k \geq 0} H \otimes \mathcal{L}_{k+1}.$$

If we assign to  $H \otimes \mathcal{L}_{k+1}$  degree  $k$ ,  $\text{Der}(\mathcal{L}_*)$  becomes a graded Lie algebra. By a straightforward calculation, we have the following.

**Proposition 5.1.**  $D \in H \otimes \mathcal{L}_{k+1}$  is in  $\mathfrak{h}_{g,1}(k)$  if and only if  $D$  is a symplectic derivation, namely  $D(\omega) = 0$  holds for  $\omega = \sum_{i=1}^g [a_i, b_i] \in \mathcal{L}_2$ .

**Theorem 5.2** (Morita [73]).

$$\tau = \bigoplus_{k \geq 1} \tau_k : \bigoplus_{k \geq 1} \mathcal{M}_{g,1}[k]/\mathcal{M}_{g,1}[k+1] \longrightarrow \mathfrak{h}_{g,1}$$

is an  $\mathcal{M}_{g,1}$ -equivariant Lie algebra embedding. In fact, the restriction of the action to  $\mathcal{I}_{g,1}$  is trivial, so that  $\tau$  is  $\text{Sp}(H)$ -equivariant.

The determination of  $\text{Im } \tau$  is a difficult problem as is already mentioned. To make the situation simpler, we take tensor products with  $\mathbb{Q}$  and consider the embedding

$$\tau^{\mathbb{Q}} = \bigoplus_{k \geq 1} \tau_k^{\mathbb{Q}} : \bigoplus_{k \geq 1} (\mathcal{M}_{g,1}[k]/\mathcal{M}_{g,1}[k+1]) \otimes \mathbb{Q} \longrightarrow \mathfrak{h}_{g,1} \otimes \mathbb{Q}.$$

From the discussion in the previous sections,  $\tau_1^{\mathbb{Q}}$  and  $\tau_2^{\mathbb{Q}}$  are surjective. By using an argument by Asada-Nakamura [4, Lemma 2.2.8], we can see that  $\tau^{\mathbb{Q}}$  is  $\text{Sp}(H \otimes \mathbb{Q}) \cong \text{Sp}(2g, \mathbb{Q})$ -equivariant. This means that we may describe  $\text{Im } \tau^{\mathbb{Q}}$  by using representation theory of symplectic groups. In fact, Asada and Nakamura determined  $\text{Im } \tau_3^{\mathbb{Q}}$  in terms of  $\text{Sp}(2g, \mathbb{Q})$ -representations.

In general, Morita showed that  $\text{Im } \tau_k^{\mathbb{Q}}$  is smaller than  $\mathfrak{h}_{g,1}(k) \otimes \mathbb{Q}$ . He defined an  $\text{Sp}(H)$ -equivariant homomorphism

$$\begin{aligned} \text{Tr}_{2k+1} : \mathfrak{h}_{g,1}(2k+1) \subset H \otimes \mathcal{L}_{2k+2} &\hookrightarrow H \otimes H^{\otimes(2k+2)} = H^{\otimes(2k+3)} \\ &\xrightarrow{\mu \otimes \text{id}_{H^{\otimes(2k+1)}}} H^{\otimes(2k+1)} \xrightarrow{\text{proj.}} S^{2k+1}H, \end{aligned}$$

where  $S^{2k+1}H$  is the  $(2k+1)$ -st symmetric product of  $H$ . The map  $\text{Tr}_{2k+1}$  is called the *trace map*. By an easy calculation, we see that  $\text{Tr}_{2k+1}$  is non-trivial.

**Theorem 5.3** (Morita [73]). *The homomorphism*

$$\tau_{2k+1}^{\mathbb{Q}} : (\mathcal{M}_{g,1}[2k+1]/\mathcal{M}_{g,1}[2k+2]) \otimes \mathbb{Q} \longrightarrow \mathfrak{h}_{g,1}(2k+1) \otimes \mathbb{Q}$$

is not surjective for any  $k \geq 1$ . In fact,

$$\text{Im } \tau_{2k+1} \subset \text{Ker}(\text{Tr}_{2k+1} : \mathfrak{h}_{g,1}(2k+1) \rightarrow S^{2k+1}H)$$

holds for  $k \geq 1$ .

Generalizing the trace maps, Enomoto and Satoh [23] found more components in the cokernel of  $\tau^{\mathbb{Q}}$ . Different approaches to the cokernel were given by Conant-Kassabov-Vogtmann [17] and Kawazumi-Kuno [53].

Based on Theorem 5.3, Morita posed the following:

**Problem 5.4.** Find topological meanings of the cokernel of  $\tau$  and the trace maps.

In [74], he suggests several approaches to attack this problem.

## 5.2. Integral homology 3-spheres

As mentioned in Section 2.3, any integral homology 3-sphere can be represented as  $M_f = H_{(g)} \cup_{\iota_g \circ f} (-H_{(g)})$  for some  $g \geq 0$  and  $f \in \mathcal{I}_g$ . On the other hand, we see by a topological observation that  $M_f \cong M_{\iota_g^{-1} \circ h_1 \circ \iota_g \circ f \circ h_2}$  holds for any diffeomorphisms  $h_1, h_2 \in \mathcal{N}_g$ , where  $\mathcal{N}_g$  is the subgroup of  $\mathcal{M}_g$  consisting of all isotopy classes of diffeomorphisms of  $\Sigma_g = \partial H_{(g)}$  which can be extended to diffeomorphisms of  $H_{(g)}$ .

In [71], Morita showed that for any integral homology 3-sphere  $M_f$  with  $f \in \mathcal{I}_g$ , there exist  $h_1, h_2 \in \mathcal{N}_g \cap \mathcal{I}_g$  such that  $\tau_1(\iota_g^{-1} \circ h_1 \circ \iota_g \circ f \circ h_2) = 0$ , which implies that any integral homology 3-sphere can be represented as  $M_{f'}$  for some  $g \geq 0$  and  $f' \in \mathcal{K}_g$ . Precisely speaking, he gave this argument in  $\mathcal{I}_{g,1}$ . However, it can be applied to  $\mathcal{I}_g$  almost verbatim.

Applying this argument to the next filter  $\mathcal{K}_g = \mathcal{M}_g[2]$ , Pitsch [85] proved that we may further take such  $f'$  from  $\mathcal{M}_g[3]$ . Now we have the following well-known problem.

**Problem 5.5.** Let  $k \geq 4$  be an integer. Can any integral homology 3-sphere be represented as  $M_f$  for some  $g \geq 0$  and  $f \in \mathcal{M}_g[k]$ ?

In the same paper, Morita proved that assigning to  $f \in \mathcal{K}_g$  the Casson invariant  $\lambda(M_f)$  of the integral 3-sphere  $M_f$  gives a homomorphism  $\lambda : \mathcal{K}_g \rightarrow \mathbb{Z}$ . Moreover, he showed that there exists an  $\mathcal{M}_{g,1}$ -invariant homomorphism  $d : \mathcal{K}_{g,1} \rightarrow \mathbb{Z}$  which sends any BSCC map of genus  $h$  to  $4h(h-1)$ , and gave a relationship between  $d$  and  $\lambda$ .

## 5.3. Hain's works on the Torelli Lie algebra

Finally, we mention deep results of Hain about a Lie algebra associated with the Torelli group. The *Torelli Lie algebra* is the graded Lie algebra defined by

$$\mathfrak{t}_g = \bigoplus_{k \geq 1} \mathfrak{t}_g(k), \quad \mathfrak{t}_g(k) := (\Gamma_k \mathcal{I}_g / \Gamma_{k+1} \mathcal{I}_g) \otimes \mathbb{Q}.$$

It plays an important role in the theory of finite type invariants for integral homology 3-spheres (see Garoufalidis-Levine [27] and Habiro [37] for details). Since  $\Gamma_k \mathcal{I}_g \subset \mathcal{M}_g[k]$ , a natural homomorphism  $\mathfrak{t}_g \rightarrow \text{Im } \tau^{\mathbb{Q}}$  is induced.

In [40], Hain gave an explicit presentation of the Lie algebra  $\mathfrak{t}_g$  for  $g \geq 3$ . The proof requires deep knowledge of algebraic geometry including the theory of relative Malcev completions and the mixed Hodge theory, and hence it is beyond the scope of this article. Giving a topological proof of the presentation of  $\mathfrak{t}_g$  is a problem posed by Hain. We have also a presentation for the Lie algebra of  $\mathcal{I}_{g,1}$  (see Habegger-Sorger [36]).

In the same paper, Hain further proved the following important facts.

**Theorem 5.6** (Hain [40]). (1)  $\mathfrak{t}_g(2) = \text{Im } \tau_2^{\mathbb{Q}} \oplus \mathbb{Q}$  for  $g \geq 6$ . The summand  $\mathbb{Q}$  corresponds to the Casson invariant homomorphism  $\lambda$  described by Morita.

(2) The natural homomorphism  $\mathfrak{t}_g \rightarrow \text{Im } \tau^{\mathbb{Q}}$  is surjective for  $g \geq 3$ . In particular, the Lie algebra  $\text{Im } \tau^{\mathbb{Q}}$  is generated by its degree 1 part.

(3) For any  $k \geq 1$ , there exists a mapping class  $f \in \mathcal{M}_{g,1}[k]$  such that  $d(M_f) \neq 0$ .

The statement (2) and Enomoto-Satoh's result mentioned in Section 5.1 provide powerful tools to determine the rational Johnson image  $\text{Im } \tau^{\mathbb{Q}}$ . Up to now,  $\text{Im } \tau_k^{\mathbb{Q}}$  is determined for  $k \leq 6$ . The result is given without proof in a paper [76, Table 1] of Morita, Suzuki and the author. The details will appear elsewhere.

## 6. Monoids of homology cobordisms of surfaces

### 6.1. Homology cobordisms of surfaces

Monoids of homology cobordisms were introduced by Goussarov [33] and Habiro [37] in their study of finite type invariants of 3-manifolds using their clover and clasper surgery theory. From our context, these monoids are considered to be enlargements of the Johnson filtration of the mapping class group. A remarkable point is that it is possible to extend Johnson homomorphisms to the monoids, which implies that the theory of Johnson homomorphisms can be applied to wider objects in 3-dimensional topology. We will see in Corollary 6.7 that considering monoids of homology cobordisms gives one answer to Morita's question (Problem 5.4).

The following definition is due to Garoufalidis and Levine [28]:

**Definition 6.1.** A *homology cobordism over  $\Sigma_{g,1}$*  consists of an oriented compact 3-manifold  $M$  with two embeddings  $i_+, i_- : \Sigma_{g,1} \hookrightarrow \partial M$ , called the *markings*, such that:

1.  $i_+$  is orientation-preserving and  $i_-$  is orientation-reversing;
2.  $i_+|_{\partial\Sigma_{g,1}} = i_-|_{\partial\Sigma_{g,1}}$ ;
3.  $\partial M = i_+(\Sigma_{g,1}) \cup i_-(\Sigma_{g,1})$  and  $i_+(\Sigma_{g,1}) \cap i_-(\Sigma_{g,1}) = i_+(\partial\Sigma_{g,1}) = i_-(\partial\Sigma_{g,1})$ , namely  $\partial M$  is diffeomorphic to the double of  $\Sigma_{g,1}$ ;
4. The induced maps  $(i_+)_*, (i_-)_* : H_*(\Sigma_{g,1}) \rightarrow H_*(M)$  are isomorphisms, namely  $M$  is a *homology product* over  $\Sigma_{g,1}$ .

We denote a homology cobordism by  $(M, i_+, i_-)$ .

Two homology cobordisms  $(M, i_+, i_-)$  and  $(N, j_+, j_-)$  over  $\Sigma_{g,1}$  are said to be *isomorphic* if there exists an orientation-preserving diffeomorphism  $f : M \xrightarrow{\cong} N$  satisfying  $j_+ = f \circ i_+$  and  $j_- = f \circ i_-$ . We denote by  $\mathcal{C}_{g,1}$  the set of all isomorphism classes of homology cylinders over  $\Sigma_{g,1}$ . We define a product operation on  $\mathcal{C}_{g,1}$  called *stacking* by

$$(M, i_+, i_-) \cdot (N, j_+, j_-) := (M \cup_{i_- \circ (j_+)^{-1}} N, i_+, j_-)$$

for  $(M, i_+, i_-), (N, j_+, j_-) \in \mathcal{C}_{g,1}$ , which endows  $\mathcal{C}_{g,1}$  with a monoid structure (see Figure 6.1). The unit is given by

$$1_{g,1} := (\Sigma_{g,1} \times [0, 1], \text{id} \times 1, \text{id} \times 0),$$

where collars of  $i_+(\Sigma_{g,1}) = (\text{id} \times 1)(\Sigma_{g,1})$  and  $i_-(\Sigma_{g,1}) = (\text{id} \times 0)(\Sigma_{g,1})$  are stretched half-way along  $(\partial\Sigma_{g,1}) \times [0, 1]$  so that  $i_+(\partial\Sigma_{g,1}) = i_-(\partial\Sigma_{g,1})$ .

**Example 6.2.** (1) For each self-diffeomorphism  $f$  of  $\Sigma_{g,1}$  which fixes  $\partial\Sigma_{g,1}$  pointwise, we can construct a homology cobordism by setting

$$(\Sigma_{g,1} \times [0, 1], \text{id} \times 1, f \times 0)$$

with the same treatment of the boundary as above. It is easily checked that the isomorphism class of  $(\Sigma_{g,1} \times [0, 1], \text{id} \times 1, f \times 0)$  depends only on the (boundary fixing) isotopy class of  $f$  and that this construction gives a monoid homomorphism from the mapping class group  $\mathcal{M}_{g,1}$  to  $\mathcal{C}_{g,1}$ . In fact, it is an injective homomorphism (see Garoufalidis-Levine [28, Section 2.4], Levine [57, Section 2.1], Habiro-Massuyeau [39] and [29, Proposition 2.3]).



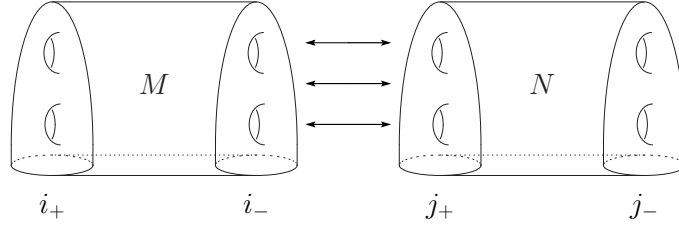


Figure 6.1: Stacking of homology cobordisms

(2) When  $g = 0$ , the monoid  $\mathcal{C}_{0,1}$  is naturally identified with the monoid  $\theta_{\mathbb{Z}}^3$  of integral homology 3-spheres whose product operation is given by the connected sum. Indeed, filling the boundary of a homology cobordism over  $\Sigma_{0,1}$  by a 3-ball induces the identification of the monoids. Moreover, for an integral homology 3-sphere  $X$ , we have a homology cobordism  $((\Sigma_{g,1} \times [0, 1]) \# X, \text{id} \times 1, \text{id} \times 0)$ . This construction gives a monoid embedding  $\theta_{\mathbb{Z}}^3 \hookrightarrow \mathcal{C}_{g,1}$ . Indeed, we have a retraction  $\mathcal{C}_{g,1} \rightarrow \theta_{\mathbb{Z}}^3$  by using prime decompositions of 3-manifolds (see [31, Proposition 2.1]).

To a homology cobordism  $(M, i_+, i_-) \in \mathcal{C}_{g,1}$ , we assign the automorphism

$$\sigma(M, i_+, i_-) := (i_+)_*^{-1} \circ (i_-)_*$$

of  $H$ . It is easy to check that  $\sigma(M, i_+, i_-)$  is in  $\text{Sp}(H)$  and this map defines a monoid homomorphism

$$\sigma : \mathcal{C}_{g,1} \longrightarrow \text{Sp}(H)$$

extending the action of  $\mathcal{M}_{g,1}$  on  $H$ .

**Definition 6.3.** A homology cobordism  $(M, i_+, i_-)$  over  $\Sigma_{g,1}$  belonging to  $\text{Ker } \sigma$  is called a *homology cylinder over  $\Sigma_{g,1}$* . We denote by  $\mathcal{IC}_{g,1}$  the submonoid of  $\mathcal{C}_{g,1}$  consisting of homology cylinders over  $\Sigma_{g,1}$ .

## 6.2. Johnson homomorphisms for homology cobordisms

One of the interesting things about  $\mathcal{C}_{g,1}$  is that we can extend the Johnson homomorphisms. Since the fundamental group  $\pi_1(M)$  of  $(M, i_+, i_-) \in \mathcal{C}_{g,1}$  is generally *not* isomorphic to  $\pi = \pi_1(\Sigma_{g,1})$ , there is little hope of the existence of the natural action of  $\mathcal{C}_{g,1}$  on  $\pi$ . However, what we need for the definition of Johnson homomorphisms is not the action on  $\pi$  but the action on the nilpotent quotients  $N_k = N_k(\pi)$ . Our conditions for homology cobordisms are sufficient to obtain such actions as we see below.

A group homomorphism  $\varphi : G_1 \rightarrow G_2$  is said to be *2-connected* if  $\varphi$  induces an isomorphism  $\varphi_* : H_1(G_1) \xrightarrow{\cong} H_1(G_2)$  and an epimorphism  $\varphi_* : H_2(G_1) \twoheadrightarrow H_2(G_2)$ .

**Theorem 6.4** (Stallings [97]). *If a homomorphism  $\varphi : G_1 \rightarrow G_2$  is 2-connected, then it induces an isomorphism  $\varphi_* : N_k(G_1) \xrightarrow{\cong} N_k(G_2)$  of the  $k$ -th nilpotent quotients for every  $k \geq 2$ .*

For any  $(M, i_+, i_-) \in \mathcal{C}_{g,1}$ , the homomorphisms  $(i_+)_*, (i_-)_* : \pi \rightarrow \pi_1(M)$  are 2-connected. Therefore they induce isomorphisms  $(i_+)_*, (i_-)_* : N_k \xrightarrow{\cong} N_k(\pi_1(M))$  for every  $k \geq 2$ . Define a map

$$\sigma_k : \mathcal{C}_{g,1} \longrightarrow \text{Aut}(N_k)$$

by  $\sigma_k(M, i_+, i_-) = (i_+)_*^{-1} \circ (i_-)_*$ . We see that  $\sigma_k$  is a monoid homomorphism extending the action of  $\mathcal{M}_{g,1}$  on  $N_k$ .

**Definition 6.5.** The *Johnson filtration*  $\{\mathcal{C}_{g,1}[k]\}_{k \geq 0}$  of  $\mathcal{C}_{g,1}$  is defined by

$$\mathcal{C}_{g,1}[0] := \mathcal{C}_{g,1}, \quad \mathcal{C}_{g,1}[k] := \text{Ker}(\sigma_{k+1} : \mathcal{C}_{g,1} \rightarrow \text{Aut}(N_{k+1})) \quad (k \geq 1).$$

We have  $\mathcal{C}_{g,1}[1] = \mathcal{IC}_{g,1}$  by definition. Differently from the case of  $\mathcal{M}_{g,1}$ , the intersection  $\bigcap_{k \geq 0} \mathcal{C}_{g,1}[k]$  is not trivial since homology cobordisms obtained as in Example 6.2 (2) have trivial images by all  $\sigma_k$ .

In [28], Garoufalidis and Levine introduced the subgroup

$$\text{Aut}_0(N_k) := \left\{ \varphi \in \text{Aut}(N_k) \mid \begin{array}{l} \text{There exists an endomorphism } \tilde{\varphi} \text{ of } \pi \text{ lifting } \varphi \\ \text{and satisfying } \tilde{\varphi}(\zeta) \equiv \zeta \pmod{\Gamma_{k+1}\pi}. \end{array} \right\}.$$

of  $\text{Aut}(N_k)$  and observed that the image of  $\sigma_k$  is included in it. Moreover, they showed that there exists an exact sequence

$$1 \longrightarrow \mathfrak{h}_{g,1}(k) \longrightarrow \text{Aut}_0(N_{k+2}) \longrightarrow \text{Aut}_0(N_{k+1}) \longrightarrow 1,$$

as a restriction of the sequence in Theorem 4.4. Using this, we define the  $k$ -th Johnson homomorphism

$$\tau_k : \mathcal{C}_{g,1}[k] \longrightarrow \mathfrak{h}_{g,1}(k)$$

for homology cobordisms by restricting  $\sigma_{k+2}$  to  $\mathcal{C}_{g,1}[k]$ . Clearly, this is an extension of  $\tau_k$  for  $\mathcal{M}_{g,1}[k]$  and in particular, the latter half of Theorem 4.8 automatically follows.

As for the image of  $\tau_k$  for homology cobordisms, the following remarkable fact holds:

**Theorem 6.6** (Garoufalidis-Levine [28], Habegger [35]). *For  $k \geq 2$ , the image of  $\sigma_k$  coincides with  $\text{Aut}_0(N_k)$ .*

**Corollary 6.7.** *The  $k$ -th Johnson homomorphism  $\tau_k : \mathcal{C}_{g,1}[k] \rightarrow \mathfrak{h}_{g,1}(k)$  is surjective for any  $k \geq 1$ .*

The proof by Garoufalidis and Levine uses a homological version of a classical surgery technique, what we call “the middle homotopy group elimination”. Habegger’s proof uses more specific techniques in low dimensional topology. It is interesting to compare these proofs.

### 6.3. Borromean surgery and closures of homology cobordisms

The most flexible construction of a homology cobordism is to use Borromean surgery introduced by Matveev [66]. His theory was generalized to the theory of clover and clasper surgeries due to Goussarov [33] (see also Garoufalidis-Goussarov-Polyak [26]) ) and Habiro [37] independently. In their theory, Borromean surgeries serve as the elementary moves.

Consider a handlebody  $H_{(3)}$  of genus 3 in the standard position in  $S^3$  including the 6 component link  $B$  each of whose components is given the 0-framing as in Figure 6.2.

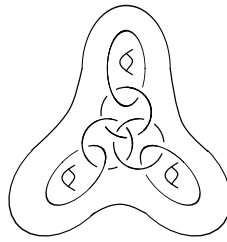


Figure 6.2: The handlebody  $H_{(3)}$  with the 6 component link  $B$

For an oriented 3-manifold  $M$ , we take an embedding  $C$  of the handlebody  $H_{(3)}$  with the link  $B$  into  $M$  and give a simultaneous surgery along the embedded framed link. The resulting 3-manifold  $M_C$  is said to be obtained from  $M$  by a *Borromean surgery* along  $C$ . The equivalence relation generated by Borromean surgeries is called the *Borromean surgery equivalence*. Matveev showed that this surgery preserves the homology and its torsion linking form, and moreover that two oriented closed 3-manifolds  $M_1$  and  $M_2$  can be obtained from one another by Borromean surgeries if and only if there exists an isomorphism  $\varphi : H_1(M_1) \xrightarrow{\cong} H_1(M_2)$  preserving their torsion linking forms. In [60, Section 1], Massuyeau proved that a Borromean surgery along an embedding  $C$  is equivalent to a *Torelli surgery* along the boundary  $\Sigma$  of the

embedded handlebody; once cut  $M$  along a surface  $\Sigma \cong \Sigma_g$  (of any genus  $g$ ) and reglue it by an element of  $\mathcal{I}_g$ .

Returning to homology cobordisms over  $\Sigma_{g,1}$ , we see that Borromean (Torelli) surgeries preserve the conditions in Definition 6.1.

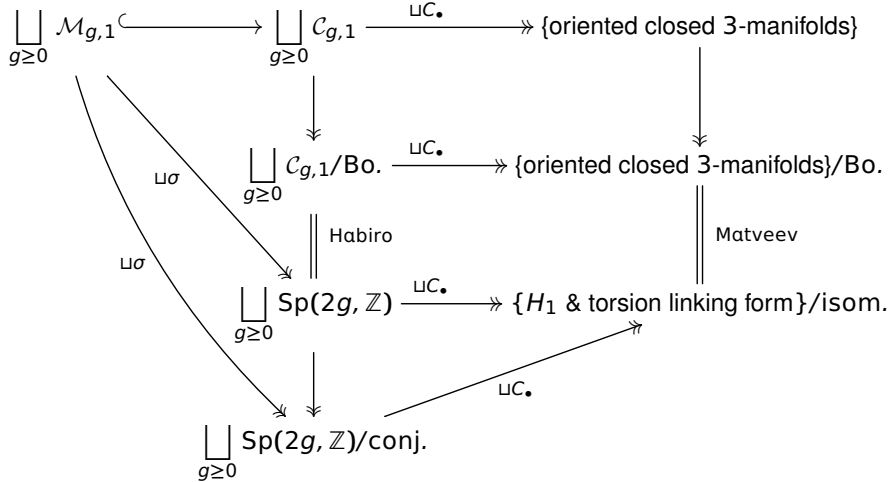
**Theorem 6.8** (Habiro [37], Massuyeau-Meilhan [63]). *Two homology cobordisms  $(M, i_+, i_-), (N, j_+, j_-) \in \mathcal{C}_{g,1}$  are Borromean surgery equivalent if and only if  $\sigma(M, i_+, i_-) = \sigma(N, j_+, j_-)$ . In particular, the monoid  $\mathcal{IC}_{g,1}$  of homology cylinders over  $\Sigma_{g,1}$  is the same as the set of all homology cobordisms which are Borromean surgery equivalent to the trivial cylinder  $1_{g,1}$ .*

Let us observe a more close relationship between  $\mathcal{C}_{g,1}$  and oriented closed 3-manifolds.

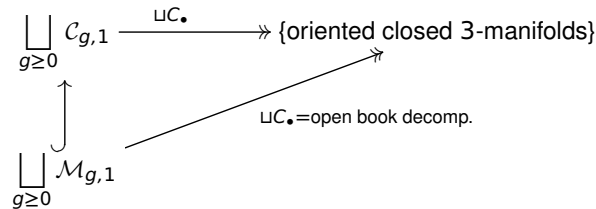
**Definition 6.9.** For a homology cobordism  $(M, i_+, i_-)$  over  $\Sigma_{g,1}$ , the *closure*  $C_M$  of  $(M, i_+, i_-)$  is defined by

$$C_M := M / (i_+(x) \sim i_-(x), x \in \Sigma_{g,1}).$$

It is easily checked that this construction is the same as gluing  $\Sigma_{g,1} \times [0, 1]$  to  $M$  along their boundaries and that if  $(M, i_+, i_-) = (\Sigma_{g,1} \times [0, 1], \text{id} \times 1, f \times 0)$  for  $[f] \in \mathcal{M}_{g,1}$ , we have  $C_M \cong C_f$ . Therefore the closure construction is an extension of open book decompositions. Note that this construction is compatible with the Borromean surgery equivalence. Taking Theorem 6.8 into account, we have the following commutative diagram:



In the diagram, the abbreviation “Bo.” stands for the Borromean surgery equivalence. The surjectivity of the map  $\iota C_\bullet$  follows from the fact proved by Alexander [1], Myers [80], González-Acunã (unpublished) that any oriented closed 3-manifold has a fibered knot and a Seifert surface of some genus. That is, we have a commutative diagram



**Remark 6.10.** Using the above mentioned fact, we can assign to an oriented closed 3-manifold  $X$  two integers

$$\begin{aligned} \text{op}(X) &:= \min\{g \mid X \cong C_f \text{ for some } [f] \in \mathcal{M}_{g,1}\}, \\ \text{hc}(X) &:= \min\{g \mid X \cong C_M \text{ for some } (M, i_+, i_-) \in \mathcal{C}_{g,1}\}. \end{aligned}$$

Studying these numbers might be interesting. By definition, we have  $\text{hc}(X) \leq \text{op}(X)$  for any  $X$ . The Mayer-Vietoris exact sequence for  $X \cong C_M = M \cup (\Sigma_{g,1} \times [0, 1])$  gives the inequality

$$\min\{n \mid H_1(X) \text{ is generated by } n \text{ elements}\} \leq 2\text{hc}(X).$$

It is easily checked that  $\text{op}(X) = 0$  if and only if  $X \cong S^3$ , while  $\text{hc}(X) = 0$  if and only if  $X$  is an integral homology 3-sphere.

Note that  $\text{hc}(X)$  depends only on  $H_1(X)$  and its torsion linking form, namely the Borromean surgery equivalence class. Indeed, suppose oriented closed 3-manifolds  $X, Y$  are given and  $Y$  is obtained from  $X$  by a Borromean surgery along an embedding  $C : H_{(3)} \hookrightarrow X$ . We have  $X \cong C_M = M \cup (\Sigma_{g,1} \times [0, 1])$  for some  $(M, i_+, i_-) \in \mathcal{C}_{g,1}$  with  $g = \text{hc}(X)$ . Then we may move the embedded handlebody  $\Sigma_{g,1} \times [0, 1] \cong H_{(2g)}$  in  $X$  by an isotopy so that it does not intersect  $C(H_{(3)})$ , which is possible because both of  $C(H_{(3)})$  and  $\Sigma_{g,1} \times [0, 1]$  are closed regular neighborhoods of graphs. We have  $Y \cong C_{M_C}$  and hence  $\text{hc}(Y) \leq \text{hc}(X)$ . By the symmetry, we also have  $\text{hc}(X) \leq \text{hc}(Y)$ . Therefore  $\text{hc}(X) = \text{hc}(Y)$  holds.

The author computed  $\text{hc}$  for a number of 3-manifolds (unpublished). Since  $C_{\text{id}} = \#_{2g}(S^1 \times S^2)$  and  $C_{T_\gamma} = \#_{2g-1}(S^1 \times S^2)$  for any Dehn twist  $T_\gamma \in \mathcal{M}_{g,1}$  along a non-separating simple closed curve  $\gamma$ , we see that  $\text{hc}(X) = g$  holds if  $H_1(X)$  is isomorphic to  $\mathbb{Z}^{2g}$  or  $\mathbb{Z}^{2g-1}$ . The situation becomes complicated when  $H_1(X)$  is not free abelian. For example, we can directly check that for connected sums of lens spaces  $L(p, q)$ , we have  $\text{hc}(L(p, \pm 1) \# L(p', \pm 1)) = 1$  for any  $p, p'$ , although  $\text{hc}(L(5, 1) \# L(5, 2)) = 2$ . Recently, Nozaki [83] showed that  $\text{hc}(L(p, q)) = 1$  holds for all lens spaces by a number theoretical method. Also, he generalized some of the facts mentioned here.

On the other hand, the integer  $\text{op}(X)$  has more topological nature and is difficult to compute. Using Baker's results [5, 6], we can determine whether  $\text{op}(X) = 1$  holds when  $X$  is  $L(p, q)$  or  $L(p, q) \# L(p', q')$ .

#### 6.4. Refinement of higher Johnson homomorphisms for homology cobordisms

Let us extend the refinement  $\tilde{\tau}_k = \theta_k$  of the  $k$ -th Johnson homomorphisms in Sections 4.2 and 4.3 to homology cobordisms. Here we extend Heap's construction as follows.

For a given homology cobordism  $(M, i_+, i_-) \in \mathcal{C}_{g,1}[k]$ , we construct the closure  $C_M$  as mentioned above. The fundamental group of  $C_M$  is given by

$$\pi_1(C_M) \cong \pi_1(M) / \langle\langle (i_+)_*(\gamma)(i_-)_*(\gamma^{-1}) \mid \gamma \in \pi \rangle\rangle.$$

Since  $(M, i_+, i_-) \in \mathcal{C}_{g,1}[k]$  implies that  $(i_+)_* = (i_-)_* : N_{k+1} \xrightarrow{\cong} N_{k+1}(\pi_1(M))$ , we have a commutative diagram

$$\begin{array}{ccc} \pi_1(\Sigma_{g,1}) & \xrightarrow{(i_+)_* = (i_-)_*} & \pi_1(C_M) \\ \downarrow & & \downarrow \\ N_{k+1} & \xrightarrow[\cong]{(i_+)_* = (i_-)_*} & N_{k+1}(\pi_1(C_M)) \end{array}$$

whose bottom horizontal map is an isomorphism. Let  $B_M$  be the composition

$$B_M : C_M \xrightarrow{\eta_{C_M}} K(\pi_1(C_M), 1) \longrightarrow K(N_{k+1}(\pi_1(C_M)), 1) \xrightarrow{((i_\pm)_*^{-1})_+} K(N_{k+1}, 1)$$

of the natural continuous maps as before. Define a map

$$\theta_k : \mathcal{C}_{g,1}[k] \longrightarrow H_3(K(N_{k+1}, 1)) = H_3(N_{k+1})$$

by assigning to  $(M, i_+, i_-) \in \mathcal{C}_{g,1}[k]$  the image  $(B_M)_*([C_M])$  of the fundamental class  $[C_M] \in H_3(C_M)$  by  $(B_M)_*$ .

**Theorem 6.11** ([90]). *We have a surjective homomorphism*

$$\theta_k : \mathcal{C}_{g,1}[k] \longrightarrow H_3(N_{k+1})$$

which extends  $\theta_k : \mathcal{M}_{g,1}[k] \rightarrow H_3(N_{k+1})$ . The kernel of  $\theta_k$  is  $\mathcal{C}_{g,1}[2k]$ .

The additivity of  $\theta_k$  follows from the topological consideration given below. For  $(M, i_+, i_-), (N, j_+, j_-) \in \mathcal{C}_{g,1}[k]$ , we construct a 4-manifold

$$W = (M \times [0, 1]) \cup (N \times [0, 1]) \cup ((\Sigma_{g,1} \times [0, 1]) \times [0, 3])$$

by the following gluing rule: We glue  $(\partial(\Sigma_{g,1} \times [0, 1])) \times [0, 1]$  to  $(\partial M) \times [0, 1]$ . We also glue  $(\partial 1_{g,1}) \times [2, 3]$  to  $\partial N \times [0, 1]$  with opposite direction of unit intervals and opposite markings of homology cobordism  $N$  and  $\Sigma_{g,1} \times [0, 1]$ . Taking the definition of the closure into account, we see that  $\partial W$  is the union of  $C_M, C_N$  and

$$(-M \times \{1\}) \cup (-N \times \{1\}) \cup ((\partial(\Sigma_{g,1} \times [0, 1])) \times [1, 2]),$$

which is the closure  $-C_{M \cdot N}$  of the product  $(M, i_+, i_-) \cdot (N, j_+, j_-)$  with opposite orientation. Since  $(M, i_+, i_-)$ ,  $(N, j_+, j_-)$  and their product are all in  $\mathcal{C}_{g,1}[k]$ , the  $(k+1)$ -st nilpotent quotients of their fundamental groups have natural identifications with  $N_{k+1}$  through markings. The continuous map  $B_M \cup B_N \cup B_{M \cdot N} : \partial W \rightarrow K(N_{k+1}, 1)$  naturally extends to  $B_W : W \rightarrow K(N_{k+1}, 1)$ . Then the equality  $\partial([W, \partial W]) = [C_M] + [C_N] - [C_{M \cdot N}]$  of cycles holds, implying the additivity of  $\theta_k$ .

The surjectivity of the extended  $\theta_k$  follows from an argument using Corollary 6.7 and the direct sum decomposition (4.3).

## 6.5. Equivalence relations among homology cobordisms of surfaces

Originally, Goussarov and Habiro independently introduced monoids of homology cobordisms over a surface to apply their surgery theory generalizing Matveev's Borromean surgeries. Their surgery techniques named clover or clasper surgery also preserve homology of 3-manifolds and define the  $Y_k$ -equivalence relation among 3-manifolds for  $k \geq 1$ . Here the  $Y_1$ -equivalence coincides with the Borromean surgery equivalence. In general, two oriented compact 3-manifolds  $M$  and  $M'$  are said to be  $Y_k$ -equivalent if  $M'$  is obtained from  $M$  by cutting  $M$  along an embedded surface  $\Sigma \cong \Sigma_{g,1}$  (of any genus  $g$ ) and reglue it by an element of  $\Gamma_k \mathcal{I}_{g,1}$ . Here, we adopt the definition used in Massuyeau-Meilhan [64] in a number of equivalent definitions of  $Y_k$ -equivalences (see also Massuyeau [61, Appendix]).

Let us now consider  $Y_k$ -equivalences for homology cobordisms over  $\Sigma_{g,1}$ . For details, see a survey paper by Habiro and Massuyeau [39]. Define  $Y_k \mathcal{C}_{g,1}$  to be the subset of  $\mathcal{C}_{g,1}$  consisting of homology cylinders that are  $Y_k$ -equivalent to the trivial cylinder  $1_{g,1}$ . As seen in Theorem 6.8, we have  $Y_1 \mathcal{C}_{g,1} = \mathcal{I} \mathcal{C}_{g,1}$ . It is easily checked from the definition that  $Y_k \mathcal{C}_{g,1}$  is a submonoid of  $\mathcal{I} \mathcal{C}_{g,1}$  and that the  $k$ -th Johnson homomorphism  $\tau_k$  is invariant under  $Y_{k+1}$ -equivalence of homology cobordisms. Goussarov and Habiro showed that  $Y_1 \mathcal{C}_{g,1} / Y_{k+1}$  is a finitely generated nilpotent group and in fact,  $Y_k \mathcal{C}_{g,1} / Y_{k+1}$  is a finitely generated abelian group. In [63] and [64], Massuyeau and Meilhan gave the following explicit descriptions of  $Y_k \mathcal{C}_{g,1} / Y_{k+1}$  for  $k = 1, 2$  by using invariants of homology cobordisms. The map  $Y_1 \mathcal{C}_{g,1} \rightarrow Y_1 \mathcal{C}_{g,1} / Y_2$  is given by the first Johnson homomorphism and the (extension of) Birman-Craggs-Johnson homomorphism. As a result, we have  $Y_1 \mathcal{C}_{g,1} / Y_2 \cong H_1(\mathcal{I} \mathcal{C}_{g,1})$ . The map  $Y_2 \mathcal{C}_{g,1} \rightarrow Y_2 \mathcal{C}_{g,1} / Y_3$  is given by the second Johnson homomorphism, (the extension of) Morita's homomorphism  $d$  and one more invariant coming from a Reidemeister torsion invariant of the pair  $(M, i_{\pm}(M))$ .

In Massuyeau-Meilhan [64], another equivalence relation named the  $J_k$ -equivalence was introduced. The definition is given by replacing  $\Gamma_k \mathcal{I}_{g,1}$  by  $\mathcal{M}_{g,1}[k]$  in the definition of the  $Y_k$ -equivalence. The  $Y_1$ -equivalence is the same as the  $J_1$ -equivalence. Massuyeau and Meilhan gave explicit descriptions of  $J_2$ - and  $J_3$ -equivalences among homology cylinders. By using these results, they recovered and generalized (to the settings of homology cylinders) the results of Morita and Pitsch mentioned in Section 5.2.

The direct sum  $\mathcal{G}^Y \mathcal{C}_{g,1} := \bigoplus_{k \geq 1} Y_k \mathcal{C}_{g,1} / Y_{k+1}$  has a natural structure of a Lie algebra. Habiro-Massuyeau [38] determined the structure of  $\mathcal{G}^Y \mathcal{C}_{g,1} \otimes \mathbb{Q}$  by using the LMO functor defined in Cheptea-Habiro-Massuyeau [15]. See also a paper by Andersen, Bene, Meilhan and Penner [2] for a related work.

**Remark 6.12.** Garoufalidis and Levine [28] defined another equivalence relation among homology cobordisms by considering the 4-dimensional homology cobordism relation. The quotient  $\mathcal{H}_{g,1}$  with respect to this equivalence relation becomes a group. It is easily checked that  $\mathcal{M}_{g,1}$  is embedded in the homology cobordism group  $\mathcal{H}_{g,1}$  and the actions  $\sigma_k : \mathcal{C}_{g,1} \rightarrow \text{Aut}_0(N_k)$  factor through  $\mathcal{H}_{g,1}$ .

Structures of the group  $\mathcal{H}_{g,1}$  and its Johnson filtration were studied in Levine [57], Conant-Schneiderman-Teichner [18], Kitayama [55], Song [95] and Cochran-Harvey-Horn [16] (see also a survey paper [92]). One of the remarkable results was given by Cha, Friedl and Kim in [13], where it is shown that the abelianization  $H_1(\mathcal{H}_{g,1})$  has  $(\mathbb{Z}/2\mathbb{Z})^{\infty}$  as a direct summand for  $g \geq 1$ . Recently, Massuyeau and the author [65] showed that  $H_1(\mathcal{H}_{g,1}) \otimes \mathbb{Q}$  is non-trivial.

**Remark 6.13.** As is discussed in Garoufalidis and Levine [28], Habegger [35] and Meilhan [67] for instance, the relationship among mapping class groups, monoids of homology cobordisms and the homology cobordism group  $\mathcal{H}_{g,1}$  is similar to that among braid groups, monoids of string links and the concordance group of string links.

## 7. Applications to homologically fibered knots

For a knot  $K$  in  $S^3$  with a Seifert surface  $R$ , we have a sutured manifold  $(M_R, K)$  constructed as follows. We take an open regular neighborhood  $N(K)$  of  $K$  and cut  $S^3 - N(K)$  open along  $R$  to obtain a 3-manifold  $M_R$ . The knot  $K$  (more precisely,  $\partial N(K)$ ) is regarded as a suture in the boundary  $\partial M_R$ . When there exists a Seifert surface  $R$  such that  $(M_R, K)$  is a product sutured manifold, the knot  $K$  is said to be *fibered*. In our context, we consider the following class of knots defined by picking up a property satisfied by fibered knots.

**Definition 7.1** (Goda-S. [29]). A knot  $K$  in  $S^3$  is called a *homologically fibered knot of genus  $g$*  if it has the following properties which are equivalent to each other:

- (a) The Alexander polynomial  $\Delta_K(t)$  of  $K$  is monic and its degree is equal to twice the genus  $g = g(K)$  of  $K$ , namely  $\Delta_K(t)$  up to multiplication by  $\pm t^k$  is of the form

$$\Delta_K(t) = t^{2g} + a_{2g-1}t^{2g-1} + \cdots + a_1t + 1.$$

- (b) For any minimal genus Seifert surface  $R$  of  $K$ , its Seifert matrix  $S$  is invertible over  $\mathbb{Z}$ .
- (c) For any minimal genus Seifert surface  $R$  of  $K$ , the sutured manifold  $(M_R, K)$  is a homology cobordism over  $R$ .

Aside from the name, the equivalence of the conditions (a), (b), (c) in the definition was mentioned in Crowell-Trotter [19]. To see the conditions (b) and (c), it suffices to check for *one* minimal genus Seifert surface.

**Example 7.2.** (1) All fibered knots are of course homologically fibered. It follows from Murasugi's result [79] that all homologically fibered knots which are alternating are fibered.

(2) Up to 11 crossings, all homologically fibered knots are known to be fibered. On the other hand, there are totally 13 non-fibered homologically fibered knots of 12 crossings:

genus	number
2	0057, 0258, 0279, 0382, 0394, 0464, 0483, 0535, 0650, 0801, 0815
3	0210, 0214

Here each of the numbers in the right column indicates  $P$  with the non-alternating prime knot of 12-crossings indexed by  $12n_P$  in KnotInfo [14]. Their non-fiberedness was first shown by Friedl and Kim in [25] using twisted Alexander invariants. In [30], Goda and the author gave another proof of the non-fiberedness by using a Reidemeister torsion invariant discussed in [91].

(3) Pretzel knots  $P_n = P(-2n+1, 2n+1, 2n^2+1)$  are homologically fibered knots of genus 1. Using these knots and sutured Floer homology theory, we showed in [31] that the submonoid

$$C_{g,1}^{\text{irr}} := \{(M, i_+, i_-) \in C_{g,1} \mid M \text{ is an irreducible 3-manifold}\}$$

has a non-finitely generated abelian quotient for any  $g \geq 1$ . Note that  $C_{g,1}$  itself has the monoid  $\theta_{\mathbb{Z}}^3$  as a big abelian quotient (recall Example 6.2 (2)).

Constructing a homology cobordism from a given homologically fibered knot has an ambiguity arising from taking a minimal genus Seifert surface and fixing a pair of markings. However, the following holds:

**Proposition 7.3** (Goda-S. [30, Proposition 2.5]). *Let  $R_1$  and  $R_2$  be (maybe parallel) minimal genus Seifert surfaces of a homologically fibered knot of genus  $g$  and let  $M_{R_1}$  and  $M_{R_2}$  be their sutured manifolds. For any markings  $i_{\pm}$  and  $j_{\pm}$  of  $\partial M_{R_1}$  and  $\partial M_{R_2}$ , there exists another homology cobordism  $N \in \mathcal{C}_{g,1}$  such that*

$$(M_{R_1}, i_+, i_-) \cdot N = N \cdot (M_{R_2}, j_+, j_-)$$

*holds as elements of  $\mathcal{C}_{g,1}$ .*

The claim for the case where  $R_1$  and  $R_2$  are disjoint in  $E(K)$  is easily proved. The general case where  $R_1$  and  $R_2$  are not disjoint uses a theorem of Scharlemann and Thompson [96] saying that there exists a sequence of minimal genus Seifert surfaces  $R_1 = S_1 \rightarrow S_2 \rightarrow \cdots \rightarrow S_n = R_2$  such that  $S_i$  and  $S_{i+1}$  are disjoint in  $E(K)$  for  $i = 1, 2, \dots, n-1$ . Using the argument in the first case repeatedly, we have the conclusion.

Proposition 7.3 can be seen as a generalization of the fact that a fibered knot determines an element of the mapping class group of a surface uniquely up to conjugation. It also provides a way to get invariants of homologically fibered knots. For example, abelian quotients of  $\mathcal{C}_{g,1}$  mentioned in Remark 6.12 give invariants. We also have invariants by considering class functions on the group  $\mathrm{Sp}(H) \cong \mathrm{Sp}(2g, \mathbb{Z})$  through the representation  $\sigma$ . For example, we may use the Meyer functions [69] for  $g = 1, 2$ .

Note that, differently from fibered knots, a homologically fibered knot does not necessarily have a unique minimal genus Seifert surface. By using a theorem of Eisner [22], we see that the connected sum of two non-fibered homologically fibered knots, which is again a homologically fibered knot, has infinitely many non-isotopic minimal genus Seifert surfaces.

Finally, we present two results on homologically fibered knots related to Johnson homomorphisms. The first one is that Johnson homomorphisms can be used as fibering obstructions of homologically fibered knots.

**Theorem 7.4** (Goda-S. [32]). *The non-fiberedness of the 13 non-fibered homologically fibered knots of 12 crossings in Example 7.2 (2) is detected by  $\sigma_4 : \mathcal{C}_{g,1} \rightarrow \mathrm{Aut}_0(N_4)$ .*

The precise meaning of this theorem is as follows. We showed that for each of the 13 homologically fibered knots, there exists a minimal genus Seifert surface  $R$  such that the homology cobordism  $(M_R, i_+, i_-)$  obtained by fixing  $i_+$  and  $i_-$  satisfies  $\sigma_4(M_R, i_+, i_-) \notin \sigma_4(\mathcal{M}_{g,1})$ . If the knot were fibered, it had a unique minimal genus Seifert surface and the corresponding homology cobordism (in fact a product sutured manifold) was mapped in  $\sigma_4(\mathcal{M}_{g,1})$  by  $\sigma_4$ . For the proof that  $\sigma_4(M_R, i_+, i_-) \notin \sigma_4(\mathcal{M}_{g,1})$ , we used Yokomizo's result on the image  $\tau_2(\mathcal{K}_{g,1})$  (Example 4.9 (2)).

The second one is that homologically fibered knots which cannot be distinguished by Johnson homomorphisms are ubiquitous.

**Theorem 7.5** (Goda-S. [30]). *Let  $K$  be a homologically fibered knot of genus  $g \geq 1$  with a minimal genus Seifert surface  $R$ . For any positive number  $V$ , there is a homologically fibered knot  $L$  of the same genus and a minimal genus Seifert surface  $R'$  of  $L$  satisfying that*

1. *The homology cobordisms  $(M_R, i_+, i_-)$  and  $(M_{R'}, j_+, j_-)$  cannot be distinguished by  $\sigma_k$  for any  $k \geq 2$ .*
2.  *$L$  is a hyperbolic knot and the hyperbolic volume of the exterior  $S^3 - L$  is bigger than  $V$ .*

The proof relies on Myer's theorem [82] on concordances of Seifert surfaces generalizing his previous result [81].

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Course n° IV— *Johnson-Morita theory*

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