Winter Braids Lecture Notes

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Vol. 6 (2019), Course nº I, p. 1-22.

<http://wbln.centre-mersenne.org/item?id=WBLN_2019__6__A1_0>

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Abstract

This is a survey on subgroups of the group of interval exchage transformations. We review definitions and a few properties of the groups of interval exchange transformations. We give examples of subgroups, and obstructions to realise certain subgroups.

Contents

Introduction

Informally speaking, an interval exchange transformation of an interval I consists of breaking I into finitely many pieces, and rearrange them by a permutation, in order to form a new copy of I. More formally, it is a bijection of I that is piecewise translations, with finitely many discontinuity points, and right-continuous. Interval exchange transformation occur as interesting examples of one-dimensional dynamical systems, and play an important role for studying flows on surfaces [1, 27, 38]. The earliest litterature about them seems to be from A. Katok, and A. Stepin, for specific transformations [20], and M. Keane [22] in a more general case.

The dynamics generated by a single interval exchange transformation have been well studied. We only cite a few of the related works here. For instance, Masur and Veech, independently, proved Keane's conjecture that interval exchange transformations associated to irreducible permutations are almost always uniquely ergodic (meaning that, for a measure 1 set of choice of discontinuity points, the Lebesgue measure of the interval is the only measure preserved by the transformation) [27, 35]. Boshernitzan even gave an explicit condition for this unique ergodicity [3]. Avila and Forni proved that a typical interval exchange transformation, whose permutation is irreducible, is either weakly mixing or is an irrational rotation [2].

On the other hand, the group of all of the interval exchange transformations of the interval has been largely mysterious. Its abelianisation and its derived subgroups are known though, throught the so-called Sah-Arnoux-Fathi invariant.

The study of the finer structure of the group was triggered by questions of Franks and Katok.

In comparison with classical linear groups, one frequent question, beside simplicity, is whether a transformation group contains a non-abelian free subgroup. This amounts to ask whether it contains two transformations so independent from each other that they do not satisfy any relation. This cannot happen when the group satisfies a law: recall that a law for a group G is a reduced word in a certain alphabet $S \cup S^{-1}$ that evaluates to 1_G for every choice of value in G of the letters in S. For instance for abelian, or for solvable groups. This obstruction is far from being the only one, yet it is famously virtually the case in linear groups: Tits' alternative states that any finitely generated subgroup of $GL_n(\mathbb{K})$ either contains a non-abelian free subgroup, or a finite index subgroup that is solvable. The group of interval exchange transformations does not satisfy any law, since it contains a copy of every finite group. Thus, one might ask the following question due to A. Katok, whether or not there exists two interval exchange transformations so independent with each other that they generate a non-abelian free group. A related question, due to Franks, asks whether there are subgroups isomorphic to certain Lie groups.

In his thesis C. Novak proved the first results with this point of view [29, 30, 31]. To this date, Katok's question on free subgroups is still not solved. Whereas Katok publicly expressed his early suspicion that there might exist such free subgroups, the general expectation is perhaps now that there are none, after [9].

Recent developments on groups issued from dynamical systems on the Cantor sets, and the modern study of topological full groups, initiated by R. Grigorchuk and K. Medynets [13] (formulating a conjecture on their amenability), and K. Juschenko and N. Monod [17] (proving this conjecture) have given an original insight on some subgroups of interval exchange transformations. This insight seems also to support the suspicion of absence of free subgroups.

In this survey, we review some definitions, and tools to approach interval exchange transformations. We describe the so-called minimal model for each transformation, and the irreducible components of each finitely generated subgroup. Trying to provide examples and counterexamples of subgroups of the group of interval exchange transformations, as in [9, 10], we explain why one may find many solvable groups with torsion, but few non-abelian solvable groups without torsion, and indeed, so far, very few free groups.

The reader interested in the relationship with surfaces is refered to the notes of Yoccoz's course in College de France, [38]. Among extensions and generalisations, we mention the work of I. Liousse and N. Guelman, in which they consider the group of piecewise affine interval exchanges [25]. A fascinating extension of this viewpoint, that aims to replace intervals with domains in $\mathbb{R}^2$ can be found in [12].

I would like to thank Nóra Szoke for her detailed explanations on topological full groups and their applications, and the reviewer for their very valuable comments and suggestions.

1. IET, examples and points of views

1.1. Definitions and settings

1.1.1. Interval exchange transformations of $[a, b)$

We now give a formal definition.

Definition 1.1. *Let a, b be two real numbers, with $a < b$. A map $T : [a, b) \to [a, b)$ is an interval exchange transformation if it is right-continuous, bijective, piecewise translation, with only finitely many discontinuity points.*

The choices of defining T on the half-open interval $[a, b)$ and of requiring the right-continuity are consubstantial, albeit conventional.

Associated to an interval exchange transformation T, one can extract a collection of objects.

- There is an integer $d \geq 1$ such that $d - 1$ is the number of discontinuity points of T on $[a, b)$.
- There is a natural subdivision of $[a, b)$ as $[a, b) = \sqcup_{i=1}^{d} [a_i, a_{i+1})$, in which $a_1 = a$, $a_{d+1} = b$ and each a_i, for $1 < i \leq d$ is a discontinuity points of T.
- There are d numbers ℓ_i, for $i = 1, \ldots, d$, which are $\ell_i = a_{i+1} - a_i$, the lengths of the consecutive intervals on which T is continuous. The tuple $(\ell_1, \ldots, \ell_d)$ is called the length tuple of T.
- There is a permutation $\sigma \in \mathfrak{S}_d$ of $\{1, \ldots, d\}$ defined as follows: $\sigma(i)$ is j if there are exactly $j - 1$ real number of the form $T(a_k)$ that are strictly inferior to $T(a_i)$. In other words, $\sigma(i)$ is the position of $T(a_i)$ among the $T(a_k), (k = 1, \ldots, d)$ for the order of the interval.
- there are $d + 1$ real numbers $t_1, \ldots, t_{d+1}$ such that, in restriction to $[a_i, a_{i+1})$, the map T is equal to the translation by the number t_i. The tuple $(t_1, \ldots, t_d)$ is called the translation tuple of T.

The interpretation of σ should be clearer than its definition: it is the underlying permutation of the *order* between the intervals $[a_i, a_{i+1})$ induced by T: the first such interval, $[a_1, a_2)$ is sent on the $\sigma(1)$-th interval of the subdivision by the $T(a_i)$.

Observe that one has a linear relation between the length tuple, and the translation tuple.

Proposition 1.2. *Consider T an interval exchange transformation of an interval $[a, b)$, and d, σ, $(\ell_1, \ldots, \ell_d)$, $(t_1, \ldots, t_d)$ as above.*

Then, for all $i \leq d$,

$$t_i = \sum_{\sigma(j) < \sigma(i)} \ell_j - \sum_{j < i} \ell_j.$$

This is obtained by expressing the initial position of the i-th subinterval of $[a, b)$ (it starts at the point $a + \sum_{j<i} \ell_j$), and the position of its image by T (which starts at the point $a + \sum_{\sigma(j)<\sigma(i)} \ell_j$).

We also observe that the composition of two interval exchange transformations on $[a, b)$ is still an interval exchange transformation. The set of all of these transformations forms a group for composition, that we denote by $\mathrm{IET}([a, b))$.

1.1.2. The group of interval exchanges over a domain

Let us now turn to minor variations of the model. Instead of an interval, we may as well consider a circle $\mathcal{C}$, with an orientation, and define an interval exchange transformation to be a bijection of $\mathcal{C}$ that is piecewise a rotation, with finitely many discontinuity points, and right-continuous with respect to the orientation. One could call such a transformation an *arc exchange transformation*, however, we will also call it an interval exchange transformation, over the domain $\mathcal{C}$.

As an associated data, we have a cyclically ordered finite family of discontinuity points (let us say there are d such points), the cyclically ordered family of lengths of the maximal arcs on which T is continuous (this is now a family of d numbers), the cyclically ordered rotation angles of the rotations induced by T (again a family of d numbers), and a permutation, up to a cyclic one, that is a double coset of the subgroup of $\mathfrak{S}_d$ generated by the cycle $(i \mapsto i+1\,[mod\, d])$. Observe that, in contrast with Proposition 1.2, this coset together with the lengths or the arcs in the decomposition only does define the transformation up to a rotation.

There is no fundamental difference with an interval exchange transformation. For instance, if the interval is $[0, 2\pi)$, one can also think of it as the unit circle cut at one point. This way, a rotation of the circle gives an interval exchange transformation on $[0, 2\pi)$ with one discontinuity point (see Figure 1.1). It turns out that allowing to cut and past intervals and circles on specific points will allow to better understand a single, or a group of interval exchange transformations. This motivates the following minor variation.

An interval exchange domain $\mathcal{D}$ will be a disjoint union of finitely many oriented circles, and half-open intervals of $\mathbb{R}$ of the form $[a, b)$. Each interval has a natural orientation (from left to right).

A general interval exchange transformation on such a domain $\mathcal{D}$ will then be a locally isometric, orientation preserving transformation with finitely many discontinuity points.

Let us observe that if $T : \mathcal{D} \to \mathcal{D}$ is an interval exchange transformation, then so is $T^{-1} : \mathcal{D} \to \mathcal{D}$. Moreover, if T_1, T_2 are interval exchange transformations of a domain $\mathcal{D}$, then so is their composition $T_1 \circ T_2$. In other words, the set $\mathrm{IET}(\mathcal{D})$ of all interval exchange transformations of the domain $\mathcal{D}$, endowed with the composition, is a group, that we of course call the group of interval exchange transformations of $\mathcal{D}$.

We want to know what kind of group it is.

1.2. Abelianisation

Before going to some examples, let us mention the abelianisation and the derived subgroup of $\mathrm{IET}([0, 1))$. We will not dig into the details of this topic, though.

The Sah-Arnoux-Fathi invariant takes a transformation T of I, and gives

$$\mathcal{SAF}(T) = \int_I 1 \otimes (T(x) - x)dx$$

in the tensor product $\mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R}$.

When T is an interval exchange transformation, let $(\ell_i)_{i=1,\dots,d}$ and $(t_i)_{i=1,\dots,d}$, the lengths and translation tuples of T. Recall that it means that if x is in the i-th interval, the translation of x by T is by t_i. The Sah-Arnoux-Fathi invariant of T is then equal to

$$\mathcal{SAF}(T) = \sum_{i=1}^{m} \ell_i \otimes t_i.$$

Replacing t_i by its value $\sum_{\sigma(j)<\sigma(i)} \ell_j - \sum_{j<i} \ell_j$, given by Proposition 1.2, one can rewrite $\mathcal{SAF}(T)$ as

$$\mathcal{SAF}(T) = \sum_i \left(\sum_{\sigma(j)<\sigma(i)} \ell_i \otimes \ell_j - \sum_{j<i} \ell_i \otimes \ell_j \right).$$

Let us compute the contribution of $\ell_i \otimes \ell_j$ in the expression of $\mathcal{SAF}(T)$. It is

- equal to 0 if both $j < i$ and $\sigma(j) < \sigma(i)$, or if both $j \geq i$ and $\sigma(j) \geq \sigma(i)$ (i.e. if i, j are not inverted by σ)
- or equal to $\ell_i \otimes \ell_j$ if $\sigma(j) < \sigma(i)$ and $j > i$,
- or equal to $-\ell_i \otimes \ell_j$ if $\sigma(j) > \sigma(i)$ and $j < i$.

In all cases, the contribution of $\ell_i \otimes \ell_j$ is the opposite of that of $\ell_j \otimes \ell_i$, and one can write $\mathcal{SAF}(T)$ as a linear combination of the $(\ell_i \otimes \ell_j) - (\ell_j \otimes \ell_i)$, for $i, j \leq d$. It is in the span of elements of the form $(a \otimes b - b \otimes a)$.

Recall that the second symmetric power $\bigwedge^2_{\mathbb{Q}} \mathbb{R}$, is the quotient of the tensor product by the ideal spanned by the elements of the form $(v \otimes v)$.

Classically, any tensor product splits as a combination of an element of this ideal and an element $(b \otimes a - a \otimes b)$. Indeed, for an arbitrary tensor product $v \otimes w$, write $a = \frac{1}{2}(v+w)$ and $b = \frac{1}{2}(v-w)$, so that $v \otimes w = (a+b) \otimes (a-b)$. One thus finds that $v \otimes w = a \otimes a - b \otimes b + (b \otimes a - a \otimes b)$.

Also classically, if some $(a \otimes b - b \otimes a)$ is in the ideal spanned by the elements of the form $(v \otimes v)$, it is fixed by the involution of $\mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R}$ that exchange the factors, and therefore it equals its opposite, hence it is 0. This shows that the image of $\mathcal{SAF}$ embeds injectively in the quotient $\bigwedge^2_{\mathbb{Q}} \mathbb{R}$. We may see the map $\mathcal{SAF}$ as $\mathcal{SAF} : \mathrm{IET}([0,1)) \to \bigwedge^2_{\mathbb{Q}} \mathbb{R}$.

This map $\mathcal{SAF}$, from the group $\mathrm{IET}([0,1))$ to $\bigwedge^2_{\mathbb{Q}} \mathbb{R}$ is actually a surjective group homomorphism, and is the abelianisation map from the group of interval exchange transformations [36, Theorem 1.3], [32]. Its kernel, the derived subgroup, is simple. We refer to [1, 32, 37, 36] for a more complete discussion of this invariant. Further study of this invariant, on variants of the group $\mathrm{IET}([0,1))$, has been undertaken by O. Lacourte in [24].

1.3. Some examples

The earliest example (beside the identity) is the exchange of two intervals: on $[0,1)$ consider θ for which $0 < \theta < 1$, and consider the map R_θ that sends $[0,\theta)$ on $[1-\theta, 1)$ by $(t \mapsto t+1-\theta)$, and that sends $[\theta, 1)$ on $[0, 1-\theta)$ by $(t \mapsto t - \theta)$. This is an interval exchange transformation with only two subintervals, one discontinuity point, the underlying permutation is the non-trivial element in $\mathfrak{S}_2$, the translation numbers are $1-\theta$ and $-\theta$. The figure 1.1 shows the mapping band complex of the transformation, defined in the next section, but probably intuitive enough to provide illustration.

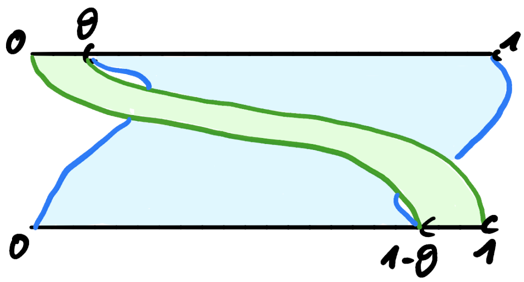

Figure 1.1: The mapping band complex of an interval exchange transformation with only one discontinuity point.

Another example would be the transformation of $[0,1)$ defined as follows: its discontinuity points are θ_1, θ_2 with $0 < \theta_1 < \theta_2 < 1$, and its underlying permutation is the transposition $(1,3) \in \mathfrak{S}_3$. As we noticed in the formula of Proposition 1.2, this completely determines R' as an interval exchange transformation. The picture of the mapping band complex is Figure

1.3. Observe that depending on the values of θ_1, θ_2, the transformation can have order 2 (if $\theta_1 = 1 - \theta_2$) or infinite order, or finite but arbitrarily large order, if both θ_i are rational.

Proposition 1.3. *If $\mathcal{C}$ is a circle,* $\mathrm{IET}(\mathcal{C})$ *contains a subgroup isomorphic to $\mathbb{S}^1$, the multiplicative group of complex numbers of modulus* 1*. Moreover, for all n,* $\mathrm{IET}(\mathcal{C})$ *contains a subgroup isomorphic to $\mathfrak{S}_n$.*

Of course, if $\mathcal{C}$ is a circle, we have, in $\mathrm{IET}(\mathcal{C})$, all the rotations of $\mathcal{C}$. This produces the subgroup isomorphic to $\mathbb{S}^1$. In order to obtain a subgroup isomorphic to $\mathfrak{S}_n$, consider n disjoint arcs of same length in $\mathcal{C}$. Consider transformations in $\mathrm{IET}(\mathcal{C})$ that are the identity outside them, and that are continuous on each of them. It is easy to obtain that these transformations form a group isomorphic to $\mathfrak{S}_n$.

1.4. Topological representations, suspensions, leaves

We introduce here a classical tool for working with interval exchange transformations, from a topological or dynamical point of view.

A suspension of the domain $\mathcal{D} = [0,1)$ by an interval exchange transformation $T : \mathcal{D} \to \mathcal{D}$ is defined as follows.

Start by marking k points $0 < x_1 < \cdots < x_k$ in $\mathcal{D}$, containing the discontinuity points of T. Write also $x_0 = 0$, and $x_{k+1} = 1$.

For each interger i between 1 and $k+1$, consider $B_i = [x_{i-1}, x_i) \times [0,1]$, that we choose to call a band of width $x_i - x_{i-1}$. Each line of the form $\{x\} \times [0,1] \subset B_i$ is called a band leaf, oriented from $\{x\} \times \{0\}$ to $\{x\} \times \{1\}$. The two intervals $[x_{i-1}, x_i) \times \{0\}$ and $[x_{i-1}, x_i) \times \{1\}$ are the widths of the band, the first is the starting width, the second is the arriving width.

We first consider the disjoint union of the B_i and of $\mathcal{D}$: $X = (\sqcup_i B_i) \sqcup \mathcal{D}$. Then, we glue each band B_i on $\mathcal{D}$ as follows : $(x,0) \sim x \in \mathcal{D}$ and $(y,1) \sim T(x) \in \mathcal{D}$.

That is to say that the two widths of the band are glued on $\mathcal{D}$, the first on the i-th interval, in the decomposition of the marked points, the second, on its image by T (see Figure 1.2 for an illustration). The topological space obtained is called the suspension Σ_T of $\mathcal{D}$ by the transformation T.

Observe that, T being a bijection of $\mathcal{D}$, each point of $\mathcal{D}$ is glued to exactly two widths of bands (perhaps of the same band), one starting and one arriving.

A leaf in the suspension Σ_T is a maximal connected union of band leaves.

Observe that any point in $\mathcal{D}$ is the start of one band leaf, and the arrival of one band leaf (possibly the same). It follows that leaves in Σ_T are 1-manifolds, hence homeomorphic to real lines, or to circles. To find the image by T^k of a certain point in $\mathcal{D}$, one just need to follow the leaf passing through x, in the positive direction, for k turns.

Some leaves pass through marked points of T, possibly a discontinuity point (fortunately, not all of them, by uncountability of $\mathcal{D}$). We call those leaves, passing through a marked point of T, a singular leaf. It has at least one subsegment that is a boundary band leaf, that is, of the form $\{x_i\} \times [0,1) \subset B_i$.

A similar construction can be made, and is easier to draw. If $T : \mathcal{D}_1 \to \mathcal{D}_2$ is a piecewise isometric and orientation preserving bijection with finitely many discontinuity points, between two interval exchange domains, one can consider marked points on $\mathcal{D}_1$ containing the discontinuity points, and a band for each arc between consecutive marked points, and glue the starting width of the band on $\mathcal{D}_1$ and the arriving width on $\mathcal{D}_2$ according to the image by T. We thus obtain the mapping band complex of T (see Figure 1.3 for an illustration). In the case where $\mathcal{D}_1 \simeq \mathcal{D}_2$, glueing back $\mathcal{D}_1$ on $\mathcal{D}_2$ gives the suspension. Even when $\mathcal{D}_1 \simeq \mathcal{D}_2$, it might be easier to consider the mapping band complex, instead of the suspension: one can stack n isomorphic band complexes to get a band complex whose leaves realise the monodromy of T^n.

One should be cautious, as there is a temptation to define "the" suspension by taking the marked points as being the discontinuity points of T. Then their images are the discontinuity

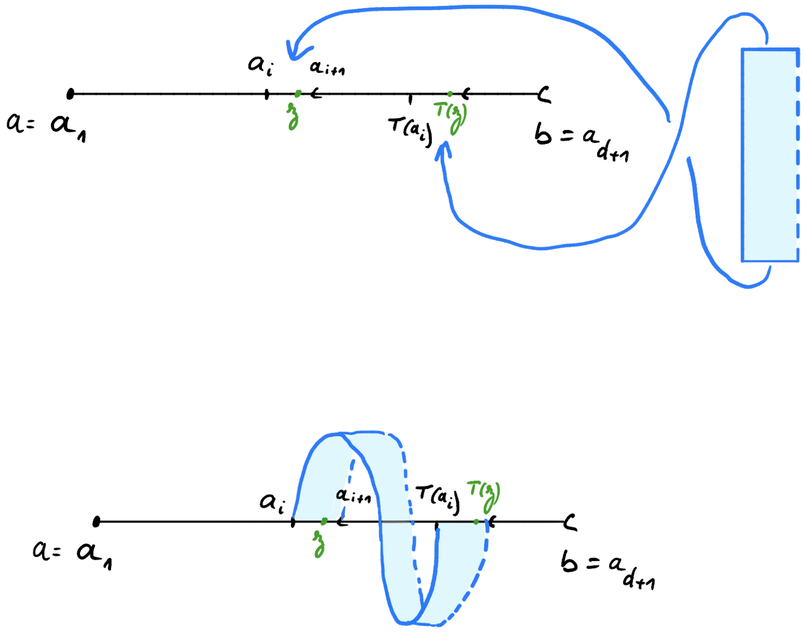

Figure 1.2: A band, glued to an interval. The vertical lines of the bands will link a point x to its image $T(x)$.

points of T^{-1}, and the situation is rather pleasantly described in these terms. However, I think that this is an uncomfortable path, since transformations that we will perform on the suspension will not necessarily preserve this virtue of the marked points of being discontinuity points.

When one considers a more general interval exchange domain $\mathcal{D}$, i.e. a disjoint union of oriented circles and half-open intervals, one can also construct, for each given interval exchange transformation, a suspension of $\mathcal{D}$. First, one describes $\mathcal{D}$ as a disjoint union of circles and arcs, on each of which T is continuous. Then one glues bands of lengths 1 or cylinders of length 1, with isometries on the widths, of the end circles, so that the vertical holonomy is the map T.

For instance, the suspension of a circle by a rotation with no marked point is a torus, endowed with leaves spiralling according to the angle of rotation.

In general, the suspension as we defined it is not a compact space. We can endow it with the metric locally given by the euclidean metric in the interior of the bands, and take the metric completion, $\overline{\Sigma_T}$ of Σ_T. This amounts to add an extra band leaf on the right side of every band. This is the convention taken in [9], and it is convenient for a number of things, however, these extra leaves do not represent the map T, and with this convention, leaves are now more complicated graphs than just lines or circles.

Finally observe that in Σ_T, each point has a very simple neighborhood: each of its points has a neighborhood homeomorphic either to the open unit disc $\mathbb{D}$ of $\mathbb{R}^2$, or to $\mathbb{D} \setminus \{(x, y) : y > 0, -y \leq x < y\}$ or to $\mathbb{D} \setminus \{(x, y) : y \neq 0, -|y| \leq x < |y|\}$ (see Figure 1.4 for illustration). One might also observe that those two later special cases are specific to the marked points, and their images by T, and for the last case, points that are marked and image of a marked point as well. If the last case does not happen, that is if no discontinuity point of T is a discontinuity

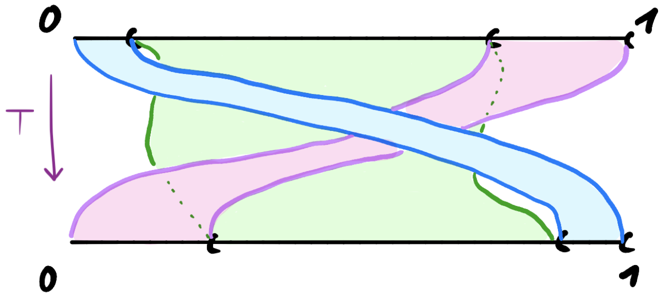

Figure 1.3: Three bands glued to obtain a mapping band complex of a transformation with two discontinuity points (from top to bottom). To get the suspension, one needs to glue back identically the bottom interval on the top interval.

point of T^{-1}, then the metric completion of Σ_T is a topological surface, possibly with boundary and cut points.

Figure 1.4: Local picture of the suspension near a discontinuity point of T: two cases, whether or not the point is also a discpontinuity point of T^{-1}.

1.5. Conjugation between domains

The abundance of the possible domains might be misleading.

Proposition 1.4. *For any non-empty interval exchange domains $\mathcal{D}_1, \mathcal{D}_2$, the groups* $\mathrm{IET}(\mathcal{D}_1)$ *and* $\mathrm{IET}(\mathcal{D}_2)$ *are isomorphic.*

This proposition will justify that we will freely navigate from a domain to another, since, doing this, we merely change the nametag of the group IET, and not the (isomorphism type of) the group itself.

Lemma 1.5. *If the total lengths of $\mathcal{D}_1$ and $\mathcal{D}_2$ are equal, then there exists a bijection that is piecewise isometric, orientation preserving, with finitely many discontinuity points $\mathcal{D}_1 \to \mathcal{D}_2$.*

We first prove the Lemma 1.5. Construct domains $\mathcal{D}'_1$ and $\mathcal{D}'_2$ by cutting open the circles in one point (obtaining, for each, a finite family of intervals), choosing an arbitrary order between the components of the domains, and glueing them (in the fashion of $[a,b) \sqcup [c,d)$ becoming $[a, b-c+d)$) in the chosen order. Thus, there are obvious bijective maps $\mathcal{D}_1 \to \mathcal{D}'_1$ and $\mathcal{D}_2 \to \mathcal{D}'_2$ (piecewise isometric, preserving orientation, with finitely many discontinuity points), and also $\mathcal{D}'_1 \to \mathcal{D}'_2$ since both are intervals of same length.

We now proceed to prove the Proposition 1.4. We first consider the case where $\mathcal{D}_1$ and $\mathcal{D}_2$ have same total length. By the lemma, we obtain an element of $\mathrm{IET}(\mathcal{D}_1 \sqcup \mathcal{D}_2)$ that exchanges the domains $\mathcal{D}_1$ and $\mathcal{D}_2$. It follows that the subgroup of $\mathrm{IET}(\mathcal{D}_1 \sqcup \mathcal{D}_2)$ that induces the identity on $\mathcal{D}_2$ is conjugated to the subgroup that induces the identity on $\mathcal{D}_1$. But these two subgroups are respectively isomorphic to $\mathrm{IET}(\mathcal{D}_1)$ and $\mathrm{IET}(\mathcal{D}_2)$. This shows that if the total lengths of $\mathcal{D}_1$ and $\mathcal{D}_2$ are equal, the groups $\mathrm{IET}(\mathcal{D}_1)$ and $\mathrm{IET}(\mathcal{D}_2)$ are isomorphic. Finally, it is easy to see that the group $\mathrm{IET}(\mathcal{D})$ is isomorphic to $\mathrm{IET}(\lambda\mathcal{D})$ if $\lambda\mathcal{D}$ is a positive rescaling of $\mathcal{D}$. This shows the proposition in all cases.

For instance, the map $[0, 2\pi) \to \mathbb{S}^1$ given by $(t \mapsto e^{it})$ thus conjugates the suitable rescaling of example in Figure 1.1 (so that its domain is $[0, 2\pi)$) to a rotation on the circle $\mathcal{C} = \mathbb{S}^1$. Observe that as a consequence, one can drop the assumption on the domain in Proposition 1.3, and also one can conclude that $\mathrm{IET}(\mathcal{D})$ contains, for each m, a subgroup isomorphic to the torus $(\mathbb{S}^1)^m$. They come from the conjugation between domains, between $\mathcal{D}$ on one hand, and the disjoint union of m circles on the other hand. Such subgroups are called subgroups of multi-rotations in $\mathcal{D}$.

1.6. More on examples

Let us return to examples, with another fun example, the lamplighter group. It is is the group presented by $\langle s, t \,|\, s^2 = 1, \forall k \in \mathbb{Z}, [t^{-k}st^k, s] = 1\rangle$.

It has an enlightening geometric meaning. Consider lamps placed at integer points on an bi-infinite street. An element of the lamplighter group is the data of a configuration of finitely many lit lamps, and of a position of the lighter at one lamp in the street. The generator s correspond to changing the status (lit or unlit) of the lamp at which the lighter is. The generator t correspond to moving the lighter to the next lamp. The commutator relation says that lamps commute: lighting the lamp i, then going to j, lighting j then coming back to i is equivalent to going to j, lighting j then coming back to i and lighting i.

If for all $i \in \mathbb{Z}$, L_i is a copy of $\mathbb{Z}/2\mathbb{Z}$, the lamplighter group is thus the semidirect product $(\oplus_{i\in\mathbb{Z}}(L_i)) \rtimes \mathbb{Z}$ for the shift of indices. The element s is the generator of L_0, and t is a generator of $\mathbb{Z}$.

The notation for the lamplighter group is $L_{\mathbb{Z}/2\mathbb{Z}} = (\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$. The same construction with any group of lamp, and with any group of lighter, is possible and yields the *wreath product* of H by G, denoted by $H \wr G = \left(\oplus_{g\in G} H_g\right) \rtimes G$. We refer to [28, Chap. 8].

Here is the lamplighter group $L_{\mathbb{Z}/2\mathbb{Z}}$ as represented as a subgroup of $\mathrm{IET}(\mathcal{D})$, for $\mathcal{D}$ the disjoint union of two circles, C_1, C_2 of same length. Let $\iota C_1 \to C_2$ be an isometry. One chooses I_1 an arc on C_1, and I_2 its image by ι, in C_2. One defines σ the transformation that exchanges I_1 and I_2, and is the identity elsewhere. One also defines τ the transformation that is, in restriction to each C_i, a rotation of angle θ_0.

Proposition 1.6. *If θ_0 is rationally independent to the length of the circles, then the group generated by τ and σ in $\mathrm{IET}(C_1 \sqcup C_2)$ is isomorphic to the lamplighter group $L_{\mathbb{Z}/2\mathbb{Z}} = (\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$.*

To prove this, one first proves that there is a homomorphism $L_{\mathbb{Z}/2\mathbb{Z}} \to \langle \tau, \sigma \rangle$ that sends t on τ and s on σ. This requires to check that all relations of $L_{\mathbb{Z}/2\mathbb{Z}}$ are satisfied by τ and σ. There are infinitely many such, but most are of the same kind. First, it is clear that $\sigma^2 = Id$. Let us now show that each lamp $\tau^k\sigma\tau^{-k}$ indeed commute with the lamp σ. The transformation $\tau^k\sigma\tau^{-k}$ is the one exchanging $\tau^k I_1$ and $\tau^k I_2$. The commutator $[\tau^k\sigma\tau^{-k}, \sigma]$ simply equals to $(\tau^k\sigma\tau^{-k}\sigma)^2$, and one wants to check that it is the identity on C^1. The reader is then invited to check it on the picture, distinguising whether the considered point is in I_1, $\tau^k I_1$, or both, or neither.

This observation proves that $\langle \sigma, \tau \rangle$ is a quotient of $L_{\mathbb{Z}/2\mathbb{Z}}$. However, $L_{\mathbb{Z}/2\mathbb{Z}}$ doesn't have so many quotients, and it is easy to check that the map is injective. Indeed, consider (ℓ, t^k) in the kernel of this quotient map, for some $\ell \in \oplus_{i\in\mathbb{Z}}(L_i)$ (product of some elements of the form $t^r st^{-r}$), and $k \in \mathbb{Z}$. Let λ be the image of ℓ in $\langle \tau, \sigma \rangle$. Then, one has $\lambda \circ \tau^k = Id_{C^1 \cup C^2}$.

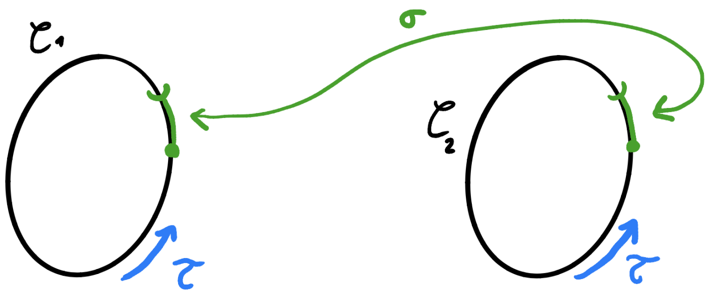

Figure 1.5: The domain is a pair of circles of same radius. The picture shows two transformations: τ which is the simultaneous rotation on those circles, and σ which exchanges two arcs in each circles. For most parameters, these two transformations generate a group isomorphic to the classical lamplighter group.

One can now argue that $k=0$ as follows. Let $\iota' : C_1 \cup C_2 \to C_2$ to be ι on C_1 and the identity on C_2. Then, since ℓ is a product of elements of the form $\tau^r \sigma \tau^{-r}$, in restriction to C_2, the map $\iota' \circ \lambda^{-1} : C_2 \to C_2$ is the identity. Thus $\iota' \circ \tau^k$ is also the identity on C_2, and therefore $k = 0$. single circle C of same length, in which λ, product of elements of the form $\tau^r \sigma \tau^{-r}$, automatically vanishes. Since $\lambda \circ \tau^k$ also vanishes, and τ is an infinite order rotation, one has that $k=0$.

Now to show that ℓ is also null, notice that the arcs $\tau^k I_1$ all have different end points, and given any finite, non-empty collection of them, one can find $x \in C_1$ belonging to an odd number of them. Thus, if ℓ is non-trivial, λ is non trivial at least on this point x. This shows that if (ℓ, t^k) is in the kernel, it is trivial in the lamplighter group.

It is not difficult to adapt this example to show that any group of the form $A \wr \mathbb{Z}$, in which A is finite abelian, embeds in $\mathrm{IET}(\mathcal{D})$, for instance, for $\mathcal{D}$ a disjoint union of $|A|$ circles of same length.

1.7. Some limitations

All this of course triggers the question of knowing which groups one can *not* find in $\mathrm{IET}(\mathcal{D})$.

After having discussed the first examples of lamplighter groups, perhaps the following is an appreciable contrast.

Proposition 1.7. *Let F be a finite group. If $F \wr \mathbb{Z}$ embeds as a subgroup of $\mathrm{IET}(\mathcal{D})$, then F is abelian.*

We refer to [10, Thm. 4.4] for the proof of the Proposition. Nevertheless, we take the opportunity to mention that, for this statement, a crucial observation is that in IET, the growth of orbits is polynomial. This is formalised as follows.

Proposition 1.8. *Let $\mathcal{I} = [0,1)$, and let S be a set of elements of $\mathrm{IET}(\mathcal{I})$, and $x \in \mathcal{I}$. There is an integer $M \geq 0$ such that for all n, the number of points in $S^n.x$ is at most n^M.*

The bound M can be chosen to be the dimension of the $\mathbb{Q}$-vector space generated, in $\mathbb{R}$, by all the translation numbers of elements in S. Indeed, the orbits for the group generated by S are included in the orbits by the group of translations of $\mathbb{R}$ by these values, and these translations have such a polynomial growth rate.

For a different reason, a very popular group, the Heisenberg group over the integers, cannot embed in $\mathrm{IET}(\mathcal{D})$. To see that, we need to stack a definition and two statements.

Let G be a finitely generated group, with a word metric d. If $g \in G$ has infinite order, we say that g is distorted if $\frac{d(1,g^n)}{n}$ goes to 0 as n goes to infinity. Of course, $d(g^k, g^{k+1}) = d(1,g)$, so by triangular inequality, one always has that $\frac{d(1,g^n)}{n} \leq d(1,g)$. Triangular inequality also ensures the subadditivity of $d(1,g^n)$, and therefore by Fekete Lemma, the sequence $\frac{d(1,g^n)}{n}$ always converges.

Distorsion means that there are significant shortcuts in G to reach g^n compared to the path that goes through each g^k, $k = 1, \ldots, n$.

The Heisenberg group $\mathcal{H} = \{\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, a, b, c \in \mathbb{Z}\}$ is iconic with respect to distorsion: consider the elements

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

An easy computation reveals that $A^n B^n A^{-n} B^{-n} = C^{n^2}$. Hence $d(I_3, C^{n^2})/n^2$ tends to 0. This shows that C is distorted in $\mathcal{H}$. Thus, the following statement, due to Novak, forbids to find $\mathcal{H}$ in IET($\mathcal{D}$).

Proposition 1.9. *(C. Novak,* [29]*)*
If $\mathcal{S}$ is a finite set of elements in IET($\mathcal{D}$)*, then the group generated by $\mathcal{S}$ contains no distorted element.*

We will present an argument for this result in Section 2.2.

An early question is whether IET contains non-abelian free groups. This quesion is due to A. Katok, and its answer is is not known at the time of writing. And it is perhaps appropriate not to jump too enthousiatically on first guesses.

Proposition 1.10. [9, Thm. 3.6] *If R is a rotation in* IET(C)*, and if T is any other element of* IET(C)*, then $\langle R, T\rangle$ is not free of rank* 2*.*

Let us sketch the argument of [9, Thm. 3.6] for this Proposition. It is enough to find a non trivial relation between R and T, because, if it is free, then it is free over the basis R, T, by the Hopf property of free groups (see for instance [26, Chap. 6.5]). If R has finite order, this goes without saying. If it has infinite order, then one can take a power R^m so that it is a rotation of a very small angle on C, much smaller than any length of subinterval defining T, or T^{-1}, or any translation length defining T. Then the commutator of R^m and T is the identity away from a few discontinuity points of T or T^{-1}. Its support is located in a small neighborhood of these discontinuity points.

Thus, for some other k, $R^k[R^m, T]R^{-k}$ has support disjoint from the support of $[R^m, T]$. It follows that they commute: $[R^k[R^m, T]R^{-k}, [R^m, T]] = Id$. This is our relation, and the group is not free.

Even worse, one can say that generically a subgroup of IET is not free. Before stating this precisely, let us put that in contrast. Consider for instance an algebraic, or matrix group G (over the field $\mathbb{C}$ for convenience). The mere existence of a single non-abelian free subgroup ensures that free groups are generic, by the following (heuristic) argument. For a pair of generators, seen as indeterminates, the locus of the pairs satisfying any single word relation is an algebraic subvariety of $G \times G$ defined by the word, defined by some polynomial equations. If there is a free subgroup, this locus is not the entire $G \times G$. Therefore, as a proper subvariety, it is a closed subset of empty interior. There are countably many possible reduced word relations, each producing a proper closed subvariety of $G \times G$, with empty interior. Every pair (g_1, g_2) in the complement of the union of these subvarieties satsify no reduced word relation at all, and therefore generates a free group. This complement is a Baire subset of $G \times G$, in other words it is generic.

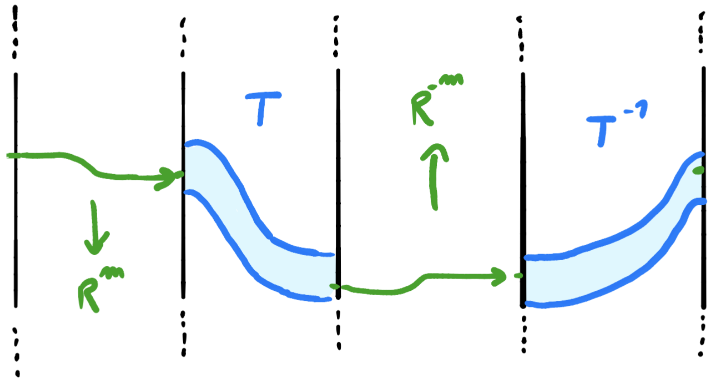

Figure 1.6: In this picture, one sees (parts of) several mapping band complexes stacked from left to right, in order to realise the composition $T^{-1} \circ R^{-m} \circ T \circ R^m$. The blue band in T^{-1} is the same (in reverse orientation) as the one shown for T, and the rotation R^{-m} is small such that the shown point remains in the domain of the blue band. The commutator thus fixes the initial point on the left.

Recall that one can define an interval exchange transformation of $[0,1)$ by choosing $n \geq 1$, an underlying permutation $\sigma \in \mathfrak{S}_n$, and a n-tuple of lengths of subintervals $(\ell_1, \dots, \ell_n)$ all positive, whose sum is 1 (in other words, $(\ell_1, \dots, \ell_n)$ is an element of the standard simplex $\Omega_n = \{v = (v_i)_{i=1\dots,n} \in \mathbb{R}_+^n, \sum_i v_i = 1\}$ of dimension $(n-1)$).

We say that a permutation σ of $\{1, \dots, n\}$ is unsplittable if there is no $k < n$ such that $\sigma(k) = k$ and $\{k+1, \dots, n\}$ is stable by σ. On the contrary, if there is such a k, one could say that σ splits as a permutation of $\{1, \dots, k-1\}$ and another permutation of $\{k+1, \dots, n\}$.

Theorem 1.11. [9, Thm. 5.2] *For all n, m there is an open dense subset of $\Omega_n \times \Omega_m$ such that if σ_n, σ_m are permutations of, respectively $\mathfrak{S}_n$, $\mathfrak{S}_m$, one of which is unsplittable, and T, S are elements of* $\mathrm{IET}([0,1))$ *whose underlying permutations are σ_n, σ_m, and whose pair of length tuples is in $\Omega_n \times \Omega_m$, then $\langle T, S\rangle$ is not free.*

We sketch the argument, which looks like the argument for Proposition 1.10.

We fix the permutations σ_n, σ_m. Assume σ_m is unsplittable. We will argue that, from any rational point in $\Omega_n \times \Omega_m$, there is an open subset accumulating on this point, in which the defined elements do not generate a free subgroup. So we start with a rational point in $\Omega_n \times \Omega_m$ (all coordinates are rational).

By taking a common denominator q, one realises the group $\langle T_0, S_0\rangle$ as a subgroup of the permutation group of $\{[k/q, (k+1)/q), k = 0, \dots, q-1\}$. Therefore $\langle T_0, S_0\rangle$ is finite, and $(T_0)^{q!}$ is trivial. After a slight perturbation of T_0, one reaches some element T for which $T^{q!}$ is almost the identity away from the points of the form k/q. More precisely, for each neighborhood $\mathcal{N}$ of the points k/q, and all $\epsilon > 0$ there exists a neighborhood of T_0 in Ω_n such that any T, restricted to the complement of $\mathcal{N}$, is piecewise a translation of at most ϵ. This plays the role of the "small rotation" of Proposition 1.10. Now, choose a small perturbation S of S_0 and produce the commutator $C = [T, S]$. It is now the identity in restriction to the complement of a neighborhood $\mathcal{N}'$ if the points k/q. If we had a rotation at our disposal, we would conjugate it to produce another element with disjoint support. But of course, we must proceed without rotation at hand. This is a difficulty, and this is why we use the unsplittability assumption. The possible perturbations of S_0 (in Ω_m) are parametrised by a neighborhood of the null vector of the vector space $V_m = \{\vec{v} \in \mathbb{R}^m, \sum v_i = 0\}$. If S_0 has unsplittable permutation, we may find a open cone in V_m so that for each small vector $\vec{v}$ in it, the translations numbers of the thus defined transformation S are all strictly larger than that of S_0. For this, we will use the following observation, that being unsplittable for σ is actually equivalent to the following: for all i, either there is $j > i$ such that $\sigma(j) < \sigma(i)$, or there is $j < i$ such that $\sigma(j) > \sigma(i)$ (assume not: there is k such that both $\{1, \ldots, \ldots k-1\}$ and $\{k+1, \ldots n\}$ are preserved by σ, which forces $\sigma(k) = k$).

Before giving the argument, let us take an example. If $m = 3$ and if σ_m is the transposition $(1,3)$, it is unsplittable. If S_0 is the transformation that sends $[0, 1/10)$ to $[9/10, 1)$, and $[1/10, 7/10)$ to $[3/10, 9/10)$ and $[7/10, 1)$ to $[0, 3/10)$, it has σ_m as underlying permutation, and its translation numbers are $9/10, 1/5$ and $-7/10$. The point defined in $\Omega_m = \Omega_3$ is $(1/10, 3/5, 3/10)$ (those are the lengths of the subintervals), and the vector space of perturbations is $V_3 = \{\vec{v} = (v_1, v_2, v_3), \sum v_i = 0\}$ as follows: a perturbation $\vec{v}$ changes the point $(1/10, 3/5, 3/10) \in \Omega_3$ to the point $(1/10+v_1, 3/5+v_2, 3/10+v_3)$, provided, for instance (and simplicity) that all v_i are smaller than $1/10$. The translation vector is affected as follows: it becomes $(1 - 1/10 - v_1, 3/10 + v_3 - 1/10 - v_1, 3/5 + v_2 + 3/10 + v_3)$, as prescribed by the relations length/translation in 1.2.

Take the intersection in V_3 of the three half spaces $\{v_1 < 0\}, \{v_3 - v_1 > 0\}, \{v_2 + v_3 > 0\}$ is a non-empty open cone, accumulating on $\vec{0}$. All perturbations in this cone increases strictly each translation number.

We now give the general argument. Take $i_0 < m$. By our unsplittability assumption, there exists j_0 such that $i_0 < j_0$, and which gives an inverted pair, thus such that $\sigma(j_0) < \sigma(i_0)$. Name the i_0-th interval $[\alpha, \beta)$, and the j_0-th $[\gamma, \delta)$. If one enlarges the j_0-th interval as $[\gamma - \epsilon, \delta)$ and reduces the length of the i_0-th interval by the same amount, as $[\alpha, \beta - \epsilon)$ (all other intervals keeping same length, those between i_0 and j_0 are just translated by $-\epsilon$), then the i_0-interval is translated strictly further, and the j_0-th interval also sees its (negative !) translation number increase. It turns out that all interval see their translation number either remain constant or increase. More precisely, for the k-th interval, the initial translation number is, by Proposition 1.2, equal to

$$t_k = \sum_{\sigma(j) < \sigma(k)} \ell_j - \sum_{j<k} \ell_j.$$

After our ϵ-perturbation, the first term $\sum_{\sigma(j)<\sigma(k)} \ell_j$ cannot have decreased (it could only decrease if $\sigma(i_0) < \sigma(k)$ and $\sigma(j_0) \geq \sigma(k)$ but this is impossible since $\sigma(j_0) < \sigma(i_0)$). And similarily, the second term $\sum_{j<k} \ell_j$ cannot have increased (only ℓ_{j_0} has increased, and if $j_0 < k$ then $i_0 < k$, and contributions cancel each other). Thus, all translation numbers have increased or remained constant, and those of i_0, j_0 have strictly increased.

Since the vector of translation numbers is a linear image of the vector of lengths of subintervals (Proposition 1.2), by superposing all such perturbations, we obtain a final perturbation that strictly increases all translations numbers.

Now we can play quantitatively. We have this open cone depending only on S_0. Choose a point in it, and a small neighborhood so that the increase of translation is very small compared to $1/q$. It is still larger than some ϵ. We may construct the perturbation of T so small that the support of the commutator C is in the ϵ-neighborhood of the points k/q. Conjugating the commutator by the element S pushes the support outside itself, therefore producing an element SCS^{-1} commuting with C.

2. Irreducibility

2.1. Minimal model

Each interval exchange transformation has prefered domains on which it is represented. Its foremost motivation, in our presentation, is about counting of discontinuity points.

Let $T \in \mathrm{IET}(\mathcal{D})$. Denote by $\delta(T)$ the number of discontinuity points of T in $\mathcal{D}$. First note the following.

Lemma 2.1. *$\delta(T^n) \leq n \times \delta(T)$, for all $n \geq 1$. More precisely, if Δ is the set of discontinuity points of T, the set of discontinuity points of T^n is a subset of $\bigcup_{i=0}^{n-1} T^{-i}(\Delta)$.*

The inequality is clear, but the equality is not automatic. One says that T is discontinuity-wise minimal on $\mathcal{D}$ if the number of discontinuity points $\delta(T^n)$ is $n \times \delta(T)$ for all n.

There are two circumstances that affect the discontinuity-wise minimality.

A *fake boundary* is a boundary component of $\overline{\Sigma_T}$ that consists of one leaf in Σ_T and one leaf in its completion but not in Σ_T, with same initial and terminal points, and same length. If k is this length, this means that T^k is continuous on the initial point of these two leaves. See Figure 2.1

A *boundary connection* is a segment in a leaf in Σ_T, with initial point and terminal point in boundary components of $\overline{\Sigma_T}$, and with no other point in these boundary components. See Figure 2.2.

Proposition 2.2. *Let T be in $\mathrm{IET}(\mathcal{D})$ and Σ_T be its suspension, for which the marked points are exactly the discontinuity points. If Σ_T contains a fake boundary, or a boundary connection, then T is not discontinuity-wise minimal.*

We sketch the proof as follows.

Consider the suspension of T for which the marked points are exactly the discontinuity points of T. The existence of a fake boundary whose leaves have length k, implies that $\delta(T^k) \leq n \times \delta(T) - 1$: the discontinuity point of T that starts the fake boundary has been cancelled, turned into continuity point, after k iteration. Thus, T is not discontinuity-wise minimal.

This is illustrated in Figure 2.1.

If now, there is a boundary connection (and this is illustrated in Figure 2.2) from x to y, then y is a discontinuity point of T, and also, some singular leaf arrives at x, and it correspond to a discontinuity point x' of T (and x is $T^r(x')$ for some r). Moreover, if k is the length of the boundary connection, $T^{r+k}(x') = y$. It follows that $\bigcup_{i=0}^{r+k} T^{-i}(\Delta)$ is not a disjoint union. Therefore, $\delta(T^{r+k+1}) < (r+k+1) \times \delta(T)$. This proves the proposition.

On the other hand, the following is a key proposition.

Proposition 2.3. *If $T \in \mathrm{IET}(\mathcal{D})$, there exists $\mathcal{D}_0$, and $P : \mathcal{D} \to \mathcal{D}_0$ an interval exchange bijection such that $PTP^{-1} (\in \mathrm{IET}(\mathcal{D}_0))$ is discontinuity-wise minimal on $\mathcal{D}_0$.*

The argument is a construction of zipping, and unzipping, to get rid of boundary connections, and fake boundaries.

The procedure is simple: first unzip all boundary connections. There are finitely many such leaves, and unzipping one strictly decreases their numbers. Then, on the new suspension, zip all fake boundaries. Note that zipping a fake boundary diminishes the number of boundary components, thus, this process eventually stops. Also note that if, after zipping a fake boundary, there is a boundary connection, then there existed one such before. Thus, after our process, the new suspension has no boundary connection, and no fake boundary. Of course, the domain has changed, and we actually have a suspension of a conjugate of our initial interval exchange transformation.

Observe that unzipping boundary connections can make new fake boundaries, so the order of the operations cannot be reversed (see Figure 2.3).

Consider the following examples. If $\mathcal{D} = [0;1)$ and if T has only one discontinuity point, as we already seen, the suspension has two bands. There are now two cases.

If the discontinuity point is irrational, there is no boundary connection. However there is a fake boundary, with both leaves of length 2.

Zipping this fake boundary identifies 0 and 1, and turns the domain into a circle. The transformation T is indeed conjugate to a rotation on a circle.

If the discontinuity point is rational, say p/q, then there is a boundary connection (possibly very long). Unzipping it turns the interval $[0, 1)$ into a disjoint union of intervals of the form

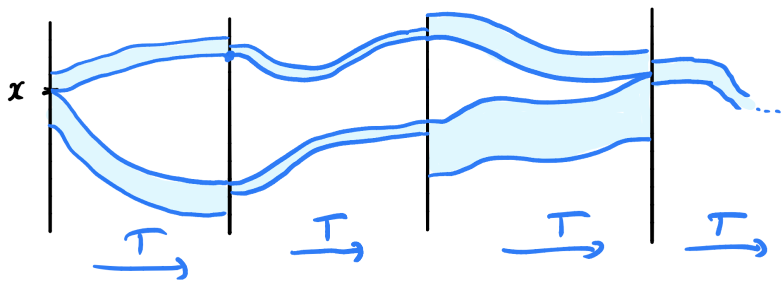

Figure 2.1: A fake boundary: the picture shows parts of four mapping band compexes of a single transformation, stacked from left to right (the domains are the vertical segments, one sees different parts at every step) in order to realise T^4.

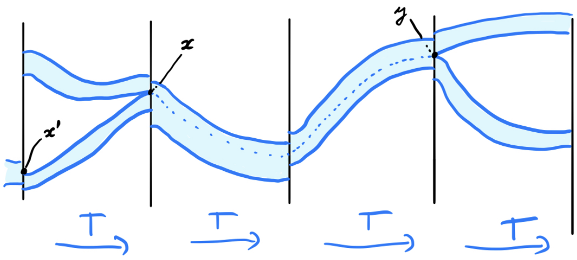

Figure 2.2: A boundary connection on a similar stack of mapping band complexes (see Figure 2.1)

$[k/q, (k+1)/q)$, with exactly one band arriving and one band departing from them. The transformation T is then conjugate to a cyclic permutation of q disjoint intervals. Since there are no more discontinuity points, there is no fake boundary.

2.2. Looking back to the distorsion

Thanks to this minimal model, one can for instance prove Novak's theorem [29][Thm. 1.3] about absence of distorted element. Indeed, the number of discontinuity points is subadditive with respect to the composition in IET. Thus, if T has a minimal model (say on a domain $\mathcal{D}_0$) with positive number of discontinuity points, then T^n has at least n discontinuity points, and in order to reach this with a word with fixed alphabet, one needs a word of length at least ϵn, where ϵ is the inverse of the maximal number of discontinuity points of elements in the alphabet (on the domain $\mathcal{D}_0$).

It remains to check that elements with no discontinuity points in their minimal model are undistorted as well: one recognises that after taking some power, one has a multi-rotation, that is a family of rotations on a family of circles. Let us explain the case of a single rotation. Let R be a rotation of angle $0 < \theta < 2\pi$ on its minimal model $\mathcal{C}$. Assume that $\mathcal{A}$ is a generating set of a subgroup of IET($\mathcal{C}$), containing R. Each element of $\mathcal{A}$ is a piecewise rotation on $\mathcal{C}$. Let $t_1, \dots, t_m$ be the rotation angles appearing in the elements of $\mathcal{A}$, and $t_0 = 2\pi$, the length of $\mathcal{C}$. Since R is in the group they generate, θ is in the $\mathbb{Q}$-vector space spanned by the t_i in $\mathbb{R}$. Take a positive quadratic form q for this vector space, whose kernel is precisely the line $\mathbb{Q}t_0$. The rotation R^n has angle $n\theta$ (modulo 2π), and the norm $q(n\theta) = n^2 q(\theta)$ of this element is thus growing quadratically in n, if $\theta/2\pi$ is irrational. Let A_0 be an upper bound for

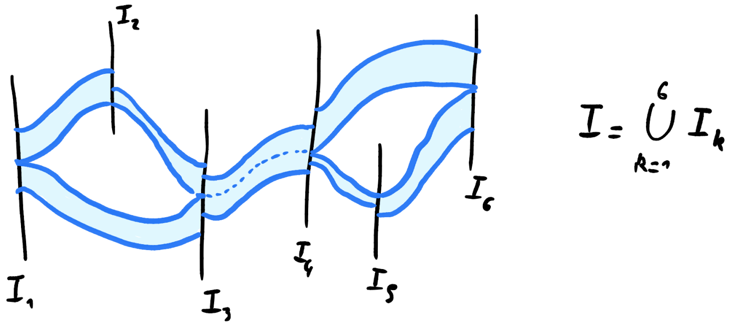

Figure 2.3: A fake boundary appears once one unzips the boundary connection. In this picture, one sees a suspension of T, but the domain has several subintervals.

$q(t_i)$. If R^n is expressed in a word $w = w_1 w_2 \dots w_\ell$ of elements of $\mathcal{A}$, of length ℓ, then fix a point in $\mathcal{C}$, and consider the partial rotations $t_{i_1}, \dots t_{i_\ell}$ that are applied to it by the word. One has $n\theta \equiv \sum_k t_{i_k}$ modulo 2π. By Cauchy-Scwharz, one has $q(n\theta)^{1/2} \leq \sum_k q(t_{i_k})^{1/2}$ frow which follows that $|n| q(\theta)^{1/2} \leq \ell M$ for M bounding all $q(t_i)^{1/2}$, and that ℓ is growing linearly in n. In other words, R is undistorted in the group generated by $\mathcal{A}$.

2.3. Commutation and discontinuity points

The control of discontinuity points is one of our first tools. It allows the construction of the minimal model. Another application is the following result of Novak.

Proposition 2.4. [29]*[Prop. 5.3] Let $T \in \mathrm{IET}(\mathcal{D})$ be irreducible. Assume that, on its minimal model no power of T is a multi-rotation.*

Then its centraliser $Z_{\mathrm{IET}(\mathcal{D})}(T)$ in $\mathrm{IET}(\mathcal{D})$ is virtually cyclic.

We sketch the proof of Novak.

Suppose $TS = ST$. We will use the minimal model of T: its singular points are discontinuity points for T and all T^k, $k \geq 1$, and none of their image by T^{-k} (with $-k < 0$) is a discontinuity point of T.

We want to check that S permutes the orbits of the singular points of T in its minimal model. Take a singular point x in the minimal model, discontinuity of T, and take $r > 0$ such that S is continuous at $T^{-r}x$, and $k > r$ such that S^{-1} is continuous at $T^{k-r}Sx$ (by irreducibility this is always possible). Since we used the minimal model, we have that ST^{-r} is continuous at x. The element $S^{-1}T^{k-r}S$ is equal to T^{k-r} and therefore is discontinuous at x. Decompose it as $S^{-1}T^{k-r}S = S^{-1}T^kST^{-r}$: from our continuity assumptions follows that T^k is discontinuous at $ST^{-r}x$. So S sends the T-orbit of the singular point x of T to another T-orbit of singular point.

This means that S permutes the T-orbits of singular points in the minimal domain for T.

Write $Sx = T^k y$ for x, y two such points (y is among finitely many choices). One deduces that $ST^r x = T^r Sx = T^{r+k}x$. Thus, S is determined uniquely on the T-orbit of x. Since T is irreducible, such an orbit is dense, and S is determined by T. A chase in the choices reveals that S actually lives in a virtually cyclic group, for which the coset representatives of a finite index cyclic subgroup are the given by the different possible choices of permutation of the T-orbits of singular points, that are realised by some S.

Another application will lead us toward the following statement.

Theorem 2.5. [10, Thm 3.1] *If G is a torsion free subgroup of $\mathrm{IET}(\mathcal{D})$ that is finitely generated, solvable, then it is virtually abelian.*

For the time being, we will illustrate it with the following statement.

Proposition 2.6. *If G is a subgroup of* $\mathrm{IET}(\mathcal{D})$ *that contains an irreducible element in its lowest non-trivial derived subgroup, then G is virtually abelian.*

We can sketch the argument, from [10, Prop. 3.6]. First assume that the irreducible element is actually a rotation on its minimal model. Call it R, and assume that $\mathcal{D}$ is its minimal model: a circle. Since R is in the last derived subgroup, if $T \in G$, then TRT^{-1} commutes with R, and so do all its powers TR^kT^{-1}. This forces TR^kT^{-1} to be a rotation as well on $\mathcal{D}$. If T^{-1} had a disconinuity point on $\mathcal{D}$, for some suitable k, TR^kT^{-1} would also have this point as a discontinuity point. But it is a rotation, so it is impossible. So T also is a rotation, and therefore G is contained in the group of rotations on $\mathcal{D}$, which is abelian. A similar argument goes if R is a multi-rotation.

Now assume on the contrary that the irreducible element (now named S) is not a multi-rotation. Then the last derived subgroup D of G is in the centralizer $Z_{\mathrm{IET}(\mathcal{D})}(S)$ of S, which is virtually cyclic, by Novak's proposition. The action of G by conjugation in this subgroup factors throught the finite automorphism group of D. A finite index subgroup of G is therefore in the kernel of this action, hence in the centralizer of D, and in particular of S. So G has a finite index subgroup contained in $Z_{\mathrm{IET}(\mathcal{D})}(S)$, which is virtually cyclic. It follows that G is virtually cyclic.

2.4. Imanishi's theorem

Assuming that certain elements are irreducible can be painfully restrictive. A tool to obtain subdomains with irreducibility properties is provided by the following theorem of Imanishi, also discussed in [11].

Theorem 2.7 (Imanishi's Theorem)**.** [16]

If G is a finitely generated subgroup of $\mathrm{IET}([0,1))$*, there exist $\mathcal{D}$ and $h : [0,1) \to \mathcal{D}$ an interval exchange bijection, and a decomposition $\mathcal{D} = \left(\sqcup_{i=1}^r \mathcal{I}_i\right) \sqcup \left(\sqcup_{i=1}^s \mathcal{J}_i\right)$, such that:*

- *For each $i \leq r, j \leq s$, the domains $\mathcal{I}_i$ and $\mathcal{J}_j$ are G-invariant*
- *For each i, each G-orbit in $\mathcal{I}_i$ is dense in $\mathcal{I}_i$*
- *For each j, the action of G on $\mathcal{J}_j$ factorises through a finite permutation group of the connected components of $\mathcal{J}_j$.*

Moreover, the components $\mathcal{I}_i, i = 1, \dots, r$ are uniquely defined.

Let us sketch the proof, following Gaboriau Levitt and Paulin (see [11, Thm 3.1]).

Consider the simultaneous suspension of $[0,1)$ by the elements of a finite generating set of G. We will perfom surgery on this complex of bands.

First, cut open every finite singular orbit, that is to say, the leaves that are completely bounded by discontinuity points.

Second cut open every purely singular boundary connection, that is every leaf made of regular band leaves, that is finite, cannot be extended by another regular band leaf, and whose end-points are singular points (see Figure 2.4).

At this stage, from any singular point, one can go to infinitely many places, using only leaves on the interior of bands. Also any finite orbit comes with parallel finite orbits that fill up some interval, until this filling meets a singular point.

Thus, there is a finite collection of intervals on which orbits are finite, locally parallel to each other.

Take $\mathcal{D}_c$ a component of the suspension so far, on which there is no finite orbit.

A key fact is that for any orbit closure in D_c, if it is not D_c itself, there is a lower bound on the size of the intervals in the complement. If one accepts this claim, either any orbit in $\mathcal{D}_c$ is

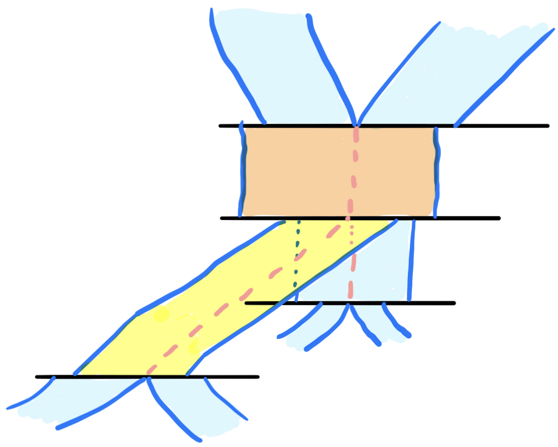

Figure 2.4: A local view of the simultaneous suspension by finitely many interval exchange transformations, and a purely singular boundary connection.

dense, or there is a non-dense orbit, and in that case, G permutes the boundaries of finitely many intervals, that makes for the complement of the closure of this orbit. This makes a finite orbit for G in D_c, which was exluded, and this proves the statement.

Finally, let's argue for the claim. Let $\mathcal{O}$ be an orbit, that we assume not dense in D_c. The candidate lower bound δ is smaller than the distance from O to the singular points of D_c on which it does not accumulate. Consider a component J in the complement of $\bar{\mathcal{O}}$, of length smaller than δ. Take one of its end points x, in the interior of D_c, consider its infinite orbit, and take y, z in this orbit that are very close ($<< \delta$) to each other in D_c. Up to exchanging y and z, there is a path of bands taking a neighborhood of x in J to a neighborhood of y in the arc $[y, z]$. The path of bands cannot be thick enough to reach y, otherwise there is a point very close to x in J in its orbit. Thus, the leaf of x by this particular path of bands passes very close ($<< \delta$) to a singular point of D_c, on which, on the other hand, the orbit does not accumulate by definition of J. This is a contradiction on the definition of δ.

2.5. Fragmentability

Imanishi's theorem allows to define canonical irreducible components for a finitely generated group. Those are the components $\mathcal{I}_i$ of the statement, those on which every orbit is dense.

For instance, when applied to a group generated by a single infinite order element T, we can conclude from Imanishi's theorem that T is irreducible, if and only if every orbit is dense.

However, this may change dramatically if one takes T^2 instead of T. Consider the following example.

$\mathcal{D}$ is the disjoint union of two circles, and T maps isometrically one to the other. In this situation, T^2 preserves both circles and rotate both of them, by a quantity depending on the maps defining T. Hence T is irreducible, and T^2 is not. The Imanishi decomposition is not the same for $\langle T \rangle$ and $\langle T^2 \rangle$.

This phenomenon is called fragmentation. More precisely one says that G is fragmentable if there is a finite index subgroup G_0 of G for which at least one if the irreducible components in the Imanishi's decomposition is a strict subset of a irreducible component of G. Otherwise, we say that G is unfragmentable.

The following statement indicates that fragmentation always terminate.

Theorem 2.8. [10, Thm. 2.11] *For each finitely generated subgroup G of* $\mathrm{IET}(\mathcal{D})$*, there is a finite index subgroup G_0 of G that is unfragmentable.*

This is not obvious even for cyclic subgroups, as the example above perhaps illustrate.

2.6. Applications

We finally can sketch the proof Theorem 2.5, the theorem of absence of torsion free solvable subgroups of IET (beside the virtually abelian ones).

Assume we have a torsion free, finitely generated solvable subgroup G. Consider its last non-trivial subgroup in its derived series. It is a torsion free abelian normal subgroup. Call it $A \triangleleft G$, and choose a non-trivial (hence of infinite order) in it.

Take $b = a^n$ so that its generated subgroup is unfragmentable. It has irreducible components $\mathcal{I}_1, \dots \mathcal{I}_r$.

Now take $g \in G$. The irreducible components of gbg^{-1} are the $g\mathcal{I}_i$. On the other hand, $gbg^{-1} \in A$, hence commutes with b, so that b permutes the $g\mathcal{I}_i$. Some power $b^{r!}$ hence preserves each $g\mathcal{I}_i$.

By unfragmentability of b, the element b itself preserves the $g\mathcal{I}_i$, and by a symmetric argument, gbg^{-1} preserves each $\mathcal{I}_i$.

The intersection $\mathcal{I}_i \cap g\mathcal{I}_k$ is preserved by both b and gbg^{-1}, and therefore is either empty or equal to both, by irreducibility.

Summing up: for all $g \in G$, either $g\mathcal{I}_1$ is disjoint from all $\mathcal{I}_j$, or there exists $k \leq r$ such that $g\mathcal{I}_1 = \mathcal{I}_k$.

Assume for simplicity that G is irreducible (Imanishi's theorem explains how to pass from this case to the general case). The collection $(g\mathcal{I}_1, g \in G)$ is then a finite partition of the domain $\mathcal{D}$. Write the pieces, $g_1\mathcal{I}_1, \dots, g_m\mathcal{I}_1$. So we may pass to a normal finite index subgroup G_0 that preserves each item of this partition.

We thus have representations $\pi_j : G_0 \to \mathrm{IET}(g_j\mathcal{I}_1)$ for each g_j above. It sends a certain power of b on an unfragmentable subgroup of $\mathrm{IET}(g_j\mathcal{I}_1)$. It is either a multi-rotation, or an element with singular minimal model. In each case, we know that the image of G_0, that normalises a subgroup of the centralizer of this element, is virtually abelian.

Since $\oplus\pi_j$ is faithful, G_0 is a subgroup of a product of virtually abelian groups, and so it is virtually abelian.

3. IET and topological full groups

The most notable attempt to establish, without condition of genericity, that there would not be free groups in $\mathrm{IET}(\mathcal{D})$ is through the amenability properties of topological full groups.

Let us recall briefly the concept of amenability for groups (we refer to [15], [5, chap. 5], [19]). Let G be a discrete group. One says that G is amenable if it admits an left invariant mean: a map $\mu : \mathcal{P}(G) \to [0, 1]$ from its set of subsets to $[0, 1]$, that is finitely additive. Many equivalent definitions are known. This property was introduced by von Neumann, in response to the Banach-Tarski paradox. The archetypal counterexample is a non-abelian free group, and indeed, an amenable group cannot contain a non-abelian free group. The group SO_3 contains free groups, which makes it non-amenable (as defined for a discrete group) and this lack of amenability is the fundamental reason of the Banach-Tarski paradox.

Let us now present the setup of topological full groups. Let K be a Cantor set, and G be a group of homeomorphisms of K, that is minimal, meaning that there is no invariant proper open-closed subset in K.

One denotes by $[[G]]$ the group of all homeomorphisms of K that are piecewise equal to the restriction of an element of G (on an open-closed partition). It is indeed a group, and is called the topological full group of G

Notice that by compactness of K, for any element of $[[G]]$ only finitely many elements of G are involved, and we do not need to add the requirement that the decomposition is finite.

A striking result is the following.

Theorem 3.1. • *(K. Juschenko, N. Monod* [17]*) If h is a minimal homeomorphism of the cantor set K, then* $[[\langle h\rangle]]$ *is amenable.*

- *(N. Szoke* [33]*) More generally, if G is a virtually cyclic group of homeomorphisms of K, that is minimal, then* $[[G]]$ *is amenable.*

- *(K. Juschenko, N. Matte-Bon, N. Monod, M. de la Salle* [18] *) If G is a group of homeomorphisms of K, that is minimal, and that is virtually* $\mathbb{Z}$ *or* $\mathbb{Z}^2$*, then the action of* $[[G]]$ *on K is extensively amenable.*

In the last statement, one says that an action of a group G on a set X is extensively amenable if there exists an invariant mean on the set of finite subsets of X that gives, for all $x \in X$, value 1 on the set of finite subsets containing x. For amenable groups, every action is extensively amenable, but the converse fails. We refer to [18], [19, Chap. 5], [34].

With our pretended exclusive interest in IET($\mathcal{D}$), we want to relate topological full groups, and subgroups of IET($\mathcal{D}$). It is indeed possible to prove that some large interesting subgroups of IET($\mathcal{D}$) are actually amenable, as proved in [18]. To explain that, we borrow the presentation proposed in N. Szoke's thesis [34, §3.2].

If $\mathcal{C}$ is a circle, and H is a subgroup of IET($\mathcal{C}$), we define its angle group $\Lambda(H)$ to be the group generated by rotations of $\mathcal{C}$ of angles appearing in the piecewise rotations of elements of H.

Theorem 3.2. *(Juschenko, Matte-Bon, Monod, de la Salle,* [18]*)*
If $\mathcal{C}$ *is a circle, and H is a subgroup of* IET($\mathcal{C}$) *whose angle group* $\Lambda(H)$ *is finite, virtually* $\mathbb{Z}$ *or virtually* $\mathbb{Z}^2$*, then H is amenable.*

We will sketch its proof to some extent.

Consider such a group H and pick $x_0 \in \mathcal{C}$. We blow up every point in the orbit of x_0 under the group of rotations $\Lambda(H)$, into two points, in the fashion of the Devil's staircase.

More precisely, define $K = (\mathcal{C} \setminus \Lambda(H)c_0) \cup_{r\in\Lambda(H)} \{rx_-, rx_+\}$ with the topology of the half-intervals.

K is a Cantor set: it is compact, its connected components are singletons, and it has no isolated point. Moreover each transformation $h \in H$ induces a transformation $\tilde{h} : K \to K$ that is an homeomorphism.

In the following, if X is a set, $\mathfrak{S}_{(0)}(X)$ is its group of finitely supported permutations.

We will use the following, but we will not sketch its proof here.

Theorem 3.3 ([18])**.** *If a group H has an extensively amenable action on X, and embeds in* $\mathfrak{S}_{(0)}(X) \rtimes H$ *by a map* $(h \mapsto (c_h, h))$ *such that* $\{h, c_h = e\}$ *is an amenable subgroup, then H is amenable.*

Such a map c_h is a cocycle, and is characterised by satisfying the relation $c_{gh} = c_g g c_h g^{-1}$.

We apply this theorem to H a group of interval exchange transformations on $\mathcal{C}$ a circle as in the statement of Theorem 3.2.

First we need to check that the action of H on $\mathcal{C}$ is extensively amenable. This is where the assumption on $\Lambda(H)$ is used. By the third point of Theorem 3.1, the action of the topological full group of $\Lambda(H)$ on the blow-up K of $\mathcal{C}$ (over the orbit of H) is extensively amenable. Although H does not embed in $\Lambda(H)$, it does embed as a subgroup in $[[\Lambda(H)]]$, its topological full group. Therefore the action of H on K (through $[[\Lambda(H)]]$) is extensively amenable as well. Consider then the collapse map from K to $\mathcal{C}$: the group H acts on K and $\mathcal{C}$ such that the stabilizer of any preimage (in K) of a point $x \in \mathcal{C}$ is again the stabilizer of x. One easily deduces our first goal: that the action of H on $\mathcal{C}$ is extensively amenable.

Now we need to check that we have an embedding in $\mathfrak{S}_{(0)}(\mathcal{C}) \rtimes H$ whose cocycle has amenable kernel, as in Theorem 3.3. For this, consider, for all $h \in H$, the transformation $\tilde{h}$ that is equal to h almost everywhere, but is left-continuous instead of right-continuous. Define then $c_h = h\tilde{h}^{-1}$. Clearly it has finite support, and is an element of $\mathfrak{S}_{(0)}(S^1)$. It is an easy computation to check that it is a cocycle. One needs to check that the kernel of this cocycle is an amenable subgroup of H. But h is in the kernel of c if and only if it has no discontinuity point on $\mathcal{C}$. This property of absence of discontinuity point is characteristic of rotations. The group of all rotations being abelian, it is amenable, and we are done.

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