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Abstract

To a branched cover $f : \tilde{\Sigma} \to \Sigma$ between closed surfaces one can associate a combinatorial datum given by the topological types of $\tilde{\Sigma}$ and Σ, the degree d of f, the number n of branching points of f, and the n partitions of d given by the local degrees of f at the preimages of the branching points. This datum must satisfy the Riemann-Hurwitz condition plus some extra ones if either Σ or both Σ and $\tilde{\Sigma}$ are non-orientable. A very old question posed by Hurwitz [14] in 1891 asks whether a combinatorial datum satisfying these necessary conditions is actually *realizable* (namely, associated to some existing f) or not (in which case it is called *exceptional*). Or, more generally, to count the number of realizations of the datum up to a natural equivalence relation. Many partial answers have been given to the Hurwitz problem over the time, but a complete solution is still missing. In this short course we will report on ancient and recent results and techniques employed to attack the question.

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1. Statement of the problem, monodromy, dessins d'enfant, reduction to the sphere

In the first lecture we will state the problem and illustrate its solution in a vast number of cases.

1.1. Surface branched covers and the problem

We will denote by

$$S \qquad T \qquad g\cdot T \qquad \mathbb{P} \qquad k\cdot\mathbb{P}$$

respectively the sphere, the torus, the connected sum of g copies of T (*i.e.*, the orientable surface of genus $g\geqslant 1$), the projective plane, and the connected sum of k copies of $\mathbb{P}$ (*i.e.*, the non-orientable surface of crosscap number $k\geqslant 1$).

A surface branched cover is a continuous function $f:\widetilde{\Sigma}\to\Sigma$ where $\widetilde{\Sigma}$ and Σ are closed and connected surfaces and f is locally modeled on maps of the form

$$(\mathbb{C},0)\ni z\mapsto z^m\in(\mathbb{C},0).$$

If $m>1$ the point corresponding to 0 in the target $\mathbb{C}$ is called a *branching point*, and m is called the local degree at the point corresponding to 0 in the source $\mathbb{C}$. There is a finite number n of branching points, and the points themselves are denoted by $\{x_i\}_{i=1}^n$. Setting

$$\Sigma^\bullet=\Sigma\setminus\{x_i\}_{i=1}^n \qquad \widetilde{\Sigma}^\bullet=f^{-1}(\Sigma^\bullet) \qquad f^\bullet=f\big|_{\widetilde{\Sigma}^\bullet}$$

one gets a genuine cover

$$f^\bullet:\widetilde{\Sigma}^\bullet\to\Sigma^\bullet$$

of some degree d. The local degrees of f at the points of $f^{-1}(x_i)$ form a partition $\pi_i=[d_{ij}]_{j=1}^{\ell_i}$ of d (here square brackets are used to denote an unordered set with repetitions). Setting $\widetilde{n}=\ell_1+\ldots+\ell_n$ we have the following necessary conditions:

1. The Riemann-Hurwitz relation holds:

$$\chi(\widetilde{\Sigma})-\widetilde{n}=d(\chi(\Sigma)-n);$$

2. If Σ is orientable then $\widetilde{\Sigma}$ also is;

3. If Σ is non-orientable but $\widetilde{\Sigma}$ is then d is even and every partition π_i of d splits as $\pi_i'\bigsqcup\pi_i''$ where π_i' and π_i'' are partitions of $d/2$;

4. $n\cdot d\equiv\widetilde{n}\pmod 2$.

The Riemann-Hurwitz relation translates the multiplicativity relation

$$\chi(\widetilde{\Sigma}^\bullet)=d\cdot\chi(\Sigma^\bullet)$$

of the Euler characteristic χ under the ordinary cover $f^\bullet$. The second condition is obvious because a non-orientable $\widetilde{\Sigma}^\bullet$ cannot cover an orientable $\Sigma^\bullet$. The third condition follows from the fact that a cover $f^\bullet:\widetilde{\Sigma}^\bullet\to\Sigma^\bullet$ with orientable $\widetilde{\Sigma}^\bullet$ and non-orientable $\Sigma^\bullet$ factors as $f^\bullet=p^\bullet\circ g^\bullet$ where $p:\overline{\Sigma}\to\Sigma$ is the orientation double cover, $\overline{\Sigma}^\bullet=p^{-1}(\Sigma^\bullet)$, $p^\bullet=p\big|_{\overline{\Sigma}^\bullet}$ and $g^\bullet:\widetilde{\Sigma}^\bullet\to\overline{\Sigma}^\bullet$ is a genuine cover of degree $d/2$. This induces a branched cover $g:\widetilde{\Sigma}\to\overline{\Sigma}$ such that $f=p\circ g$, and π_i' and π_i'' are the local degrees of g at the preimages of the two points in $p^{-1}(x_i)$. The condition that $n{\cdot}d$ and $\widetilde{n}$ have the same parity follows from the previous ones if $\widetilde{\Sigma}$ is orientable (two distinct arguments apply to an orientable and a non-orientable Σ), and it will be proved below for a non-orientable $\widetilde{\Sigma}$.

Let us now call *candidate branch datum* an array of the form

$$\left(\widetilde{\Sigma},\Sigma,d,n;\pi_1,\ldots,\pi_n\right)$$

with $\widetilde{\Sigma}$ and Σ closed and connected surfaces, d and n positive integers, and π_j a partition of d for $j=1,\ldots,n$, such that all the 4 necessary conditions listed above are satisfied. We make the convention that the partitions $\pi_1,\ldots,\pi_n$ after the semicolon are viewed up to reordering (we might write $[\pi_1,\ldots,\pi_n]$ but we will refrain from doing so).

The *Hurwitz problem* asks which candidate branch data are *realizable* (namely, associated to some existing surface branched cover) and which are *exceptional* (non-realizable).

Example 1.1. We describe here some arrays $\left(\widetilde{\Sigma},\Sigma,d,n;\pi_1,\ldots,\pi_n\right)$ with non-orientable Σ that satisfy the Riemann-Hurwitz condition but violate some other necessary condition.

- $(\mathbb{P},\mathbb{P},6,2;[2,2,1,1],[3,2,1])$ satisfies the Riemann-Hurwitz condition because $\chi(\widetilde{\Sigma})-\widetilde{n}=1-7$ equals $d\cdot(\chi(\Sigma)-n)=6\cdot(1-2)$ but $n\cdot d=2\cdot 6$ is even and $\widetilde{n}=7$ is odd, so the last condition is violated;

- $(2\mathbb{P},\mathbb{P},7,3;[3,2,1,1],[3,1,1,1,1],[2,2,1,1,1])$ is an array that satisfies the Riemann-Hurwitz condition because $\chi(\widetilde{\Sigma})-\widetilde{n}=0-14$ equals $d\cdot(\chi(\Sigma)-n)=7\cdot(1-3)$ but $n\cdot d=3\cdot 7$ is odd and $\widetilde{n}=14$ is even, so the last condition is violated;

- $(2T,2\mathbb{P},3,2;[2,1],[2,1])$ satisfies the Riemann-Hurwitz condition because $\chi(\widetilde{\Sigma})-\widetilde{n}=-2-4$ equals $d\cdot(\chi(\Sigma)-n)=3\cdot(0-2)$ and the last condition because $n\cdot d=2\cdot 3$ and $\widetilde{n}=4$ are even, but $d=3$ is odd so the first part of the third condition does not hold;

- $(S,\mathbb{P},6,2;[4,1,1],[2,1,1,1,1])$ satisfies the Riemann-Hurwitz condition because $\chi(\widetilde{\Sigma})-\widetilde{n}=2-8$ equals $d\cdot(\chi(\Sigma)-n)=6\cdot(1-2)$ and the last condition because $n\cdot d=2\cdot 6$ and $\widetilde{n}=8$ are even; morever d is even, so the first part of the last condition holds, but the second part does not, because $[4,1,1]$ does not split as the disjoint union of two partitions of $d/2=3$.

By our convention, these arrays are *not* candidate branch data.

Remark 1.2. For orientable Σ and $\widetilde{\Sigma}$, the Hurwitz realizability problem could be rephrased in the category of Riemann surfaces and in that of algebraic curves. Namely one could ask whether there exist complex structures on Σ and $\widetilde{\Sigma}$ and a holomorphic $f:\widetilde{\Sigma}\to\Sigma$ realizing a given candidate branch datum. Or whether there exist structures of algebraic curve on Σ and $\widetilde{\Sigma}$ and a rational $f:\widetilde{\Sigma}\to\Sigma$ realizing the datum. And it is a deep fact that the answer is always the same whatever category one chooses, either the topological one discussed above, or the complex one, or the algebraic one.

Remark 1.3. Hurwitz's original question was not quite whether a given candidate branch datum is realizable. Instead, he asked *how many* realizations exist, up to a natural equivalence relation. In explaining this, we slightly extend his viewpoint and we confine ourselves to the case where $\widetilde{\Sigma}$ and Σ are oriented (and not merely orientable). The equivalence of $f_0:\widetilde{\Sigma}\to\Sigma$ and $f_1:\widetilde{\Sigma}\to\Sigma$ is always defined in terms of the existence of homeomorphisms $\widetilde{g}:\widetilde{\Sigma}\to\widetilde{\Sigma}$ and $g:\Sigma\to\Sigma$ such that $f_1\circ\widetilde{g}=g\circ f_0$, and one says that f_0 and f_1 are:

- *strongly* equivalent if g can be chosen to be the identity of Σ (which requires fixing the branch set $\{x_i\}_{i=1}^n$ in advance);

- *weakly* equivalent if $\widetilde{g},g$ can be chosen to be orientation-preserving (which requires assuming f_0,f_1 are also orientation-preserving);

- *very weakly* equivalent if no restriction whatsoever is imposed.

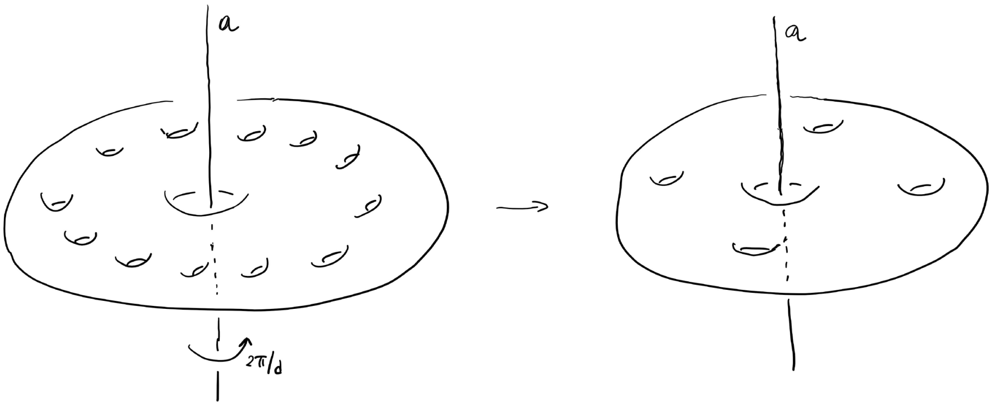

Figure 1.1: An ordinary cover between surfaces of positive genus.

Denoting by $\nu^{(S)}$, $\nu^{(W)}$ and $\nu^{(V)}$ the number of realizations of a given candidate branch datum up to strong, weak and very weak equivalence, we have of course that $\nu^{(S)} \geqslant \nu^{(W)} \geqslant \nu^{(V)}$ and that the three numbers can only vanish symultaneously. Moreover $\nu^{(S)} = \nu^{(W)}$ if the array of partitions $\pi_1, \ldots, \pi_n$ in the candidate branch datum contains no repetitions, but all the possibilities

$$\nu^{(S)} = \nu^{(W)} = \nu^{(V)} \qquad \nu^{(S)} > \nu^{(W)} = \nu^{(V)}$$
$$\nu^{(S)} = \nu^{(W)} > \nu^{(V)} \qquad \nu^{(S)} > \nu^{(W)} > \nu^{(V)}$$

occur (see [34] for the easy proof of this fact and related remarks). The number ν_S was computed in an exact but very implicit fashion in [22, 23]. See also the more recent [24, 12, 19, 21, 20, 31, 32, 33], to the results of some of which we will quickly allude below.

1.2. Easy instances of the problem

We remind that the problem we are facing is to determine whether there exists a map $f : \widetilde{\Sigma} \to \Sigma$ matching a given branch datum $\left(\widetilde{\Sigma}, \Sigma, d, n; \pi_1, \ldots, \pi_n\right)$. Confining ourselves to the case of orientable $\widetilde{\Sigma}$ and Σ, we describe here some cases where the solution of the problem is easy (and always in the affirmative).

For $\Sigma = S$ and $n \leqslant 1$ the Riemann-Hurwitz condition only allows $(S, S, 1, 0; \emptyset)$ which is realized by the identity. For $\Sigma = S$ and $n = 2$ we must have $\chi\left(\widetilde{\Sigma}\right) - \widetilde{n} = 0$, whence $\widetilde{\Sigma} = S$ and $\widetilde{n} = 2$, so the branch datum is $(S, S, d, 2; [d], [d])$ which is realized by the map $(z, t) \mapsto \left(z^d, t\right)$ upon viewing S as the boundary of the cylinder $\Delta \times [0, 1] \subset \mathbb{C} \times \mathbb{R}$, where Δ is the unit disc in $\mathbb{C}$.

For $\Sigma = S$ and $n = 3$ the problem is already hard (and, as a matter of fact, it is this very case to which most attention is currently devoted, as we will see below).

For Σ of positive genus g and $n = 0$, if $\widetilde{\Sigma}$ has genus $\widetilde{g}$ the Riemann-Hurwitz condition reads $2\left(1-\widetilde{g}\right) = d \cdot 2(1-g)$, namely $\widetilde{g}-1 = d(g-1)$. We can then realize the datum as follows. We embed $\widetilde{\Sigma}$ in $\mathbb{R}^3$ with a line a going through one of its holes, and $\widetilde{\Sigma}$ invariant under the rotation of angle $2\pi/(\widetilde{g}-1)$ with axis a. Similarly, we embed Σ (in a different copy of $\mathbb{R}^3$) around a and invariand under the rotation of angle $2\pi/(g-1)$ with axis a. We then get f as the projection in the quotient of the action on $\widetilde{\Sigma}$ of the group of order $d = \frac{\widetilde{g}-1}{g-1}$ generated by the rotation of angle $2\pi/d$ with axis a, as in Fig. 1.1.

For $g > 0$ and $n > 1$ the problem is non-trivial, but we will see later in this lecture that it always has an affirmative solution.

1.3. Dessins d'enfant

A technique that has been employed with success to face the Hurwitz existence problem is based on the notion of *dessin d'enfant*, popularized by Grothendieck [5, 13] but actually known before, see [17] and the references therein. It applies to candidate branch data of the form

$$\left(\widetilde{\Sigma}, S, d, 3; \pi_1, \pi_2, \pi_3\right),$$

namely with the sphere as base surface and 3 branching points.

Definition 1.4. We call *graph* in the surface $\widetilde{\Sigma}$ a subset Γ of $\widetilde{\Sigma}$ consisting of a finite number of points (the *vertices*) and of a finite number of (possibly closed) simple arcs (the *edges*) such that each arc has its ends at two vertices or its only end at a vertex, and two distinct arcs have disjoint interiors. We call *valence* of a vertex v the number of edges of which it is one end plus twice the number of edges of which it is both ends. Namely, the valence of v is the number of *germs* of edges incident to v. We say that Γ is bipartite if a black/white colouring of its vertices is given so that each edge has ends of distinct colours. We call *complementary region* of Γ a component R of $\widetilde{\Sigma} \setminus U$, where U is the interior of a regular neighbourhood N of Γ in $\widetilde{\Sigma}$. If Γ is bipartite we place on ∂R coloured vertices by pulling back to $\partial R \subset \partial N$ the vertices of Γ under the restriction to ∂R of the natural retraction $N \to \Gamma$. Note that on each component of ∂R the black and white vertices alternate, so the number of black vertices on ∂R is the same as the number of white vertices, and we call it the *length* of R. Note that these definitions are independent of N up to coloured homeomorphism, and that one can actually see R as the closure of a component of $\widetilde{\Sigma} \setminus \Gamma$ but keeping in mind that this picture can fail to be an embedding on the boundary, so some vertices can contribute in a multiple fashion to the length. We call *dessin d'enfant* on $\widetilde{\Sigma}$ a bipartite graph on $\widetilde{\Sigma}$ whose complementary regions are discs.

Proposition 1.5. *A branch datum $\left(\widetilde{\Sigma}, S, d, 3; \pi_1, \pi_2, \pi_3\right)$ is realizable if and only if there exists in $\widetilde{\Sigma}$ a dessin d'enfant Γ such that the valences of its black vertices are the entries of the partition π_1 of d, the valences of its white vertices are the entries of π_2, and the lengths of its complementary regions are the entries of π_3.*

Proof. Suppose that there exists $f : \widetilde{\Sigma} \to S$ realizing the datum, with branching points $x_1, x_2, x_3 \in S$. In S take the dessin d'enfant Γ_0 given by one edge with black end at x_1 and white end at x_2. Note that its complementary region is a disc R_0 of length 1 centered at x_3. Now set $\Gamma = f^{-1}(\Gamma_0)$ and note that Γ is a graph by declaring its vertices to be the pull-backs of those of Γ_0. Pulling back the colours as well we see that Γ is bipartite, and of course the valences of its vertices are as described in the statement. If R is a complementary region of Γ then the restriction of f to R is a branched cover of R_0 with only one branching point at x_3. This implies that R is itself a disc and contains only one preimage of x_3. Moreover the restriction $R \to R_0$ of f is modelled on $z \mapsto z^k$ where k is an entry of π_3, and the conclusion on the lengths of the complementary regions of Γ easily follows.

The opposite implication is proved along the same lines. If Γ exists, we first define f on Γ by mapping each edge to Γ_0 so to respect the colours. Then we extend f to each complementary region R (in doing which it is convenient to view R as an abstract disc with embedded interior and boundary immersed onto a subset of Γ). $\square$

With reference to the previous proof, if a graph Γ has black vertices of valences π_1 and white vertices of valences π_2, we will say that Γ *matches* π_1, π_2. Note that this notion applies to any abstract graph with coloured vertices, Γ need not be embedded in a surface nor bipartite (we will need this later on).

Example 1.6. Let us consider candidate branch data of the form

$$\left(\widetilde{\Sigma}, S, 7, 3; [4, 2, 1], [5, 2], \pi_3\right)$$

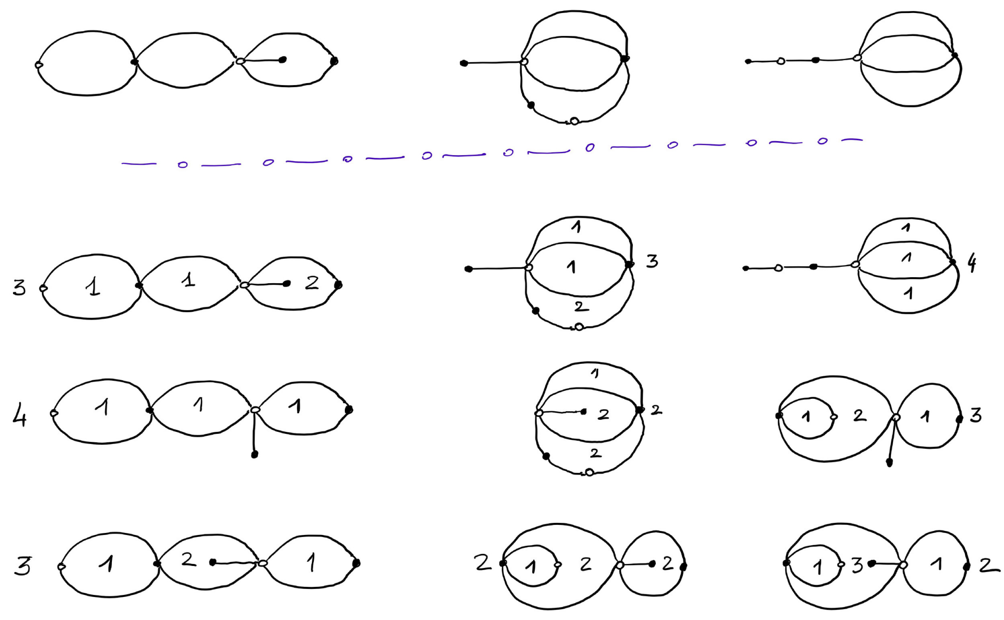

Figure 1.2: The abstract bipartite graphs matching $[4,2,1],[5,2]$ and their embeddings in S with lenghts of the complementary regions.

noting that, if $\widetilde{\Sigma}$ has genus $\widetilde{g}$ and π_3 has length ℓ, the Riemann-Hurwitz condition reads

$$2(1-\widetilde{g})-(3+2+\ell)=7\cdot(2-3) \quad \Rightarrow \quad \widetilde{g}=2-\ell/2$$

therefore we can only have $\widetilde{\Sigma}=S$ and $\ell=4$ or $\widetilde{\Sigma}=T$ and $\ell=2$. The abstract graphs matching $[4,2,1],[5,2]$ are the three shown in the top part of Fig. 1.2. With a little attention one can enumerate all their possible embeddings in S up to homeomorphism of S and compute the lenghts of the complementary regions, as in the bottom part of Fig. 1.2, concluding that all the possibilities

$$[4,1,1,1] \qquad [3,2,1,1] \qquad [2,2,2,1]$$

are realized for π_3 (each by several different dessins d'enfant). Turning to the torus, we refrain from enumerating *all* the embeddings in T of the relevant abstract graphs, and merely show for each of the possibile π_3's

$$[6,1] \qquad [5,2] \qquad [4,3]$$

one dessin d'enfant in T realizing it. In Fig. 1.3 this is illustrated using a convention that we will employ throughout: to describe the embedding of a graph Γ in a surface $\widetilde{\Sigma}$ different from S we will present an immersion of Γ in S, with double points represented as crossings in a knot diagram. The whole of $\widetilde{\Sigma}$ is then recovered by individuating the circles that bound a regular neighbourhood of the immersion of Γ, keeping in mind that crossings do not give double points of Γ in $\widetilde{\Sigma}$.

1.4. Exceptionality from dessins d'enfant

We now prove the first result showing that the solution to the Hurwitz problem is not always in the affirmative:

Proposition 1.7. *The candidate branch datum* $(S,S,4,3;[2,2],[2,2],[3,1])$ *is exceptional.*

Proof. A dessin d'enfant matching $[2,2],[2,2]$ is actually a circle, and its only embedding in S has complementary regions of lengths $[2,2]$, not $[3,1]$. This is illustrated in Fig. 1.4. □

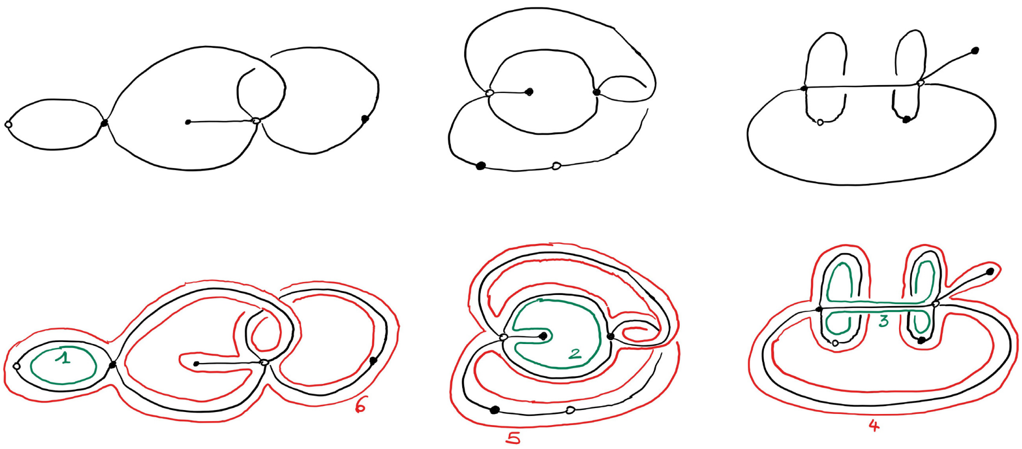

Figure 1.3: Top: three dessins d'enfant matching $[4, 2, 1], [5, 2]$ embedded in T. Bottom: computation of the lenghts of the complementary regions.

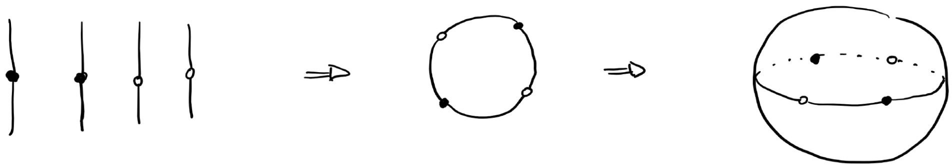

Figure 1.4: The only dessin d'enfant in S matching $[2, 2], [2, 2]$.

1.5. The monodromy approach

The following was already known to Hurwitz:

Theorem 1.8. *A candidate branch datum $\left(\widetilde{\Sigma}, g\cdot T, d, n; \pi_1, \ldots, \pi_n\right)$ is realizable if and only if there exist $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \theta_1, \ldots, \theta_n \in \mathfrak{S}_d$ such that:*

- $\prod_{p=1}^{g} [\alpha_p, \beta_p] \cdot \prod_{i=1}^{n} \theta_i = id$;
- *The cycles of θ_i have lengths π_i;*
- *The subgroup of $\mathfrak{S}_d$ generated by $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \theta_1, \ldots, \theta_n$ acts transitively on $\{1, \ldots, d\}$.*

Proof. Suppose that f is realizable and let $f^\bullet : \widetilde{\Sigma}^\bullet \to (g\cdot T)^\bullet$ be the associated genuine cover between punctured surfaces. Fix a basepoint x_0 in $(g\cdot T)^\bullet$ and note that

$$\pi_1\left((g\cdot T)^\bullet, x_0\right) = \left\langle a_1, b_1, \ldots, a_g, b_g, t_1, \ldots, t_n : \prod_{p=1}^{g} [a_p, b_p] \cdot \prod_{i=1}^{n} t_i \right\rangle$$

with a_p, b_p, t_i the homotopy classes of the loops in Fig. 1.5 (denoted by the same letters for simplicity). Let us label the points in $f^{-1}(x_0)$ as $y_1, \ldots, y_d$. Then we have an associated representation $\rho : \pi_1\left((g\cdot T)^\bullet, x_0\right) \to \mathfrak{S}_d$ where $\rho(c)(k) = h$ if the lift of (a loop representing) c

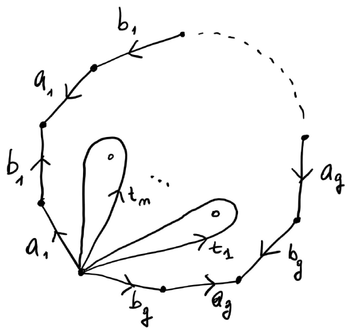

Figure 1.5: Generators of the fundamental group of an orientable punctured surface.

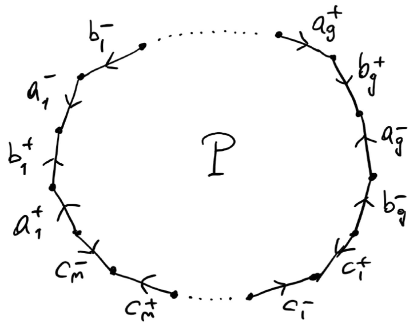

Figure 1.6: The surface of genus g with n marked points realized as a polygon P with edges glued in pairs.

that starts at y_h ends at y_k. Setting $\alpha_p = \rho(a_p), \beta_p = \rho(b_p), \theta_i = \rho(t_i)$ we of course have the first condition in the statement.

Now let us examine the lifts under f of the disc centered at x_i and bounded by t_i. We know it consists of ℓ_i discs and that f restricted to the j-th one is modelled on $z \mapsto z^{d_{ij}}$, where $\pi_i = [d_{ij}]_{j=1}^{\ell_i}$. This implies that on the boundary of the j-th disc there will be d_{ij} elements of $f^{-1}(x_0) = \{y_1, \ldots, y_d\}$, which gives a cycle of length d_{ij} in θ_i, and the second condition is established.

Finally, if we take in $\widetilde{\Sigma}$ a path $\widetilde{c}$ from y_k to y_h we have that $c = f \circ \widetilde{c}$ is a loop at x_0 and $\rho(c)(k) = h$, whence the last condition.

Now suppose $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \theta_1, \ldots, \theta_n \in \mathfrak{S}_d$ are given and satisfy the conditions. We cut $g \cdot T$ open as suggested in Fig. 1.6, so $g \cdot T$ is obtained from the $(4g + 2n)$-polygon P with boundary

$$a_1^+ b_1^+ a_1^- b_1^- \cdots a_g^+ b_g^+ a_g^- b_g^- c_1^+ c_1^- \cdots c_n^+ c_n^-$$

by identifying each a_p^+ to a_p^-, each b_p^+ to b_p^-, and each c_i^+ to c_i^-. Now we take d copies $P_1, \ldots, P_d$ of P, with P_q having boundary

$$a_{1,q}^+ b_{1,q}^+ a_{1,q}^- b_{1,q}^- \cdots a_{g,q}^+ b_{g,q}^+ a_{g,q}^- b_{g,q}^- c_{1,q}^+ c_{1,q}^- \cdots c_{n,q}^+ c_{n,q}^-$$

and define X as the surface obtained from the disjoint union of $P_1, \ldots, P_d$ by gluing each $a_{p,q}^+$ to $a_{p,\alpha_p(q)}^-$, each $b_{p,q}^+$ to $b_{p,\beta_p(q)}^-$ and each $c_{i,q}^+$ to $c_{i,\theta_i(q)}^-$. A map $h : X \to g \cdot T$ is defined as the quotient of the disjoint union of the identities $P_q \to P$, and h is a branched cover realizing a

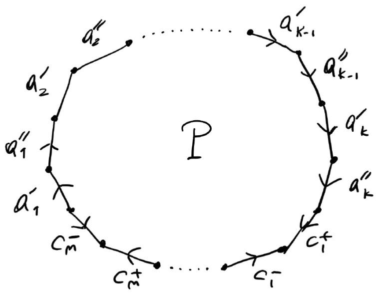

Figure 1.7: The non-orientable surface with crosscap number k and n marked points realized as a polygon P with edges glued in pairs.

branch datum

$$(X, g \cdot T, d, n; \pi_1, \ldots, \pi_n),$$

but $\chi(X) = \chi\big(\widetilde{\Sigma}\big)$ because $(X, g \cdot T, d, n; \pi_1, \ldots, \pi_n)$ satisfies the Riemann-Hurwitz condition and so does $\big(\widetilde{\Sigma}, g \cdot T, d, n; \pi_1, \ldots, \pi_n\big)$. Moreover X is connected thanks to the last condition, and it is orientable, so $X = \widetilde{\Sigma}$ and the proof is complete. □

The representation ρ in the previous proposition is called the *monodromy* of $f^\bullet$. Here comes the non-orientable version of the previous result:

Theorem 1.9. *A candidate branch datum $\big(\widetilde{\Sigma}, k \cdot \mathbb{P}, d, n; \pi_1, \ldots, \pi_n\big)$ is realizable if and only if there exist $\alpha_1, \ldots, \alpha_k, \theta_1, \ldots, \theta_n \in \mathfrak{S}_d$ such that:*

- $\prod_{p=1}^{k} \alpha_p^2 \cdot \prod_{i=1}^{n} \theta_i = id$;
- *The cycles of θ_i have lengths π_i;*
- *The subgroup $\mathfrak{H}$ of $\mathfrak{S}_d$ generated by $\alpha_1, \ldots, \alpha_k, \theta_1, \ldots, \theta_n$ acts transitively on $\{1, \ldots, d\}$;*
- *$\widetilde{\Sigma}$ is non-orientable if and only there exists $h \in \mathfrak{H}$ such that h is the product of an odd number of α_p's plus some θ_i's and h has some fixed point.*

Proof. The proof is similar to the previous one, we only spell out the sufficiency part. We cut $k \cdot \mathbb{P}$ open as suggested in Fig. 1.7, so $k \cdot \mathbb{P}$ is obtained from the $(2k+2n)$-polygon P with boundary $a_1' a_1'' \cdots a_k' a_k'' c_1^+ c_1^- \cdots c_n^+ c_n^-$ by identifying each a_p' to a_p'' and each c_i^+ to c_i^-. Now we take d copies $P_1, \ldots, P_d$ of P, with P_q having boundary $a_{1,q}' a_{1,q}'' \cdots a_{k,q}' a_{k,q}'' c_{1,q}^+ c_{1,q}^- \cdots c_{n,q}^+ c_{n,q}^-$ and define X as the surface obtained from the disjoint union of $P_1, \ldots, P_d$ by gluing each $a_{p,q}'$ to $a_{p,\alpha_p(q)}''$ and each $c_{i,q}^+$ to $c_{i,\theta_i(q)}^-$. A map $f : X \to k \cdot \mathbb{P}$ is defined as the quotient of the disjoint union of the identities $P_q \to P$, and f is a branched cover realizing a branch datum $(X, k \cdot P, d, n; \pi_1, \ldots, \pi_n)$, but $\chi(X) = \chi\big(\widetilde{\Sigma}\big)$ because $(X, g \cdot T, d, n; \pi_1, \ldots, \pi_n)$ satisfies the Riemann-Hurwitz condition and so does $\big(\widetilde{\Sigma}, g \cdot T, d, n; \pi_1, \ldots, \pi_n\big)$. Moreover X is connected by the transitivity condition, so to conclude that $X = \widetilde{\Sigma}$ we must show that X is orientable if and only if $\widetilde{\Sigma}$ is. Now one easily sees that X is non-orientable if and only if there exists an orientation-reversing loop in $k \cdot \mathbb{P}$ that lifts to a loop in X. But a loop in $k \cdot \mathbb{P}$ is orientation-reversing if and only if it is the product of an odd number of a_p's plus some c_i's, and it has a lift that is a loop if and only if its monodromy has a fixed point, whence the conclusion. □

Theorems 1.8 and 1.9 imply that the realizability of a given candidate branch datum $\big(\widetilde{\Sigma}, \Sigma, d, n; \pi_1, \ldots, \pi_n\big)$ could be analyzed by computer, by enumerating all the permutations with prescribed cycle lengths and checking whether the relevant conditions are met. Note that two permutations have the same cycle lengths if and only if they are conjugate to each other. However, conjugacy classes of permutations in $\mathfrak{S}_d$ rapidly become very vast as d grows, so this approach is hardly feasible.

1.6. Proof of the last necessary condition

We stated above that for an array $\left(\tilde{\Sigma}, \Sigma, d, n; \pi_1, \ldots, \pi_n\right)$ associated to a branched cover $f : \tilde{\Sigma} \to \Sigma$ with non-orientable $\tilde{\Sigma}$ and Σ one must have $n \cdot d \equiv \tilde{n} \pmod 2$, where $\tilde{n}$ is the sum of the lengths of $\pi_1, \ldots, \pi_n$, and now we can prove this fact.

Let f have associated permutations $\alpha_1, \ldots, \alpha_k, \theta_1, \ldots, \theta_n$ as in Theorem 1.9. Since the signature of a cycle of length m is $(-1)^{m-1}$, the signature of θ_i is $(-1)^{d-\ell_i}$, and the signature of $\theta_1 \cdots \theta_n$ is $n \cdot d - \tilde{n}$. But $\theta_1 \cdots \theta_n$ is a product of squares, so it is even, and the conclusion follows.

1.7. Exceptionality from monodromy

We can now prove using the monodromy approach an extension of what already seen using dessins d'enfant:

Proposition 1.10. *The following candidate branch datum is exceptional:*

$$((n-3) \cdot T, S, 4, n; [2,2], \ldots, [2,2], [3,1]).$$

Proof. Note that the Riemann-Hurwitz condition reads

$$2(1-(n-3)) - 2n = 4(2-n)$$

so it is satisfied. To realize the candidate we should find $\theta_1, \theta_2, \ldots, \theta_{n-1} \in \mathfrak{S}_4$ with cyclic structures $[2,2]$ such that $\theta_1 \cdot \theta_2 \cdots \theta_{n-1}$ has cyclic structure $[3,1]$. However for the product of two permutations of cyclic structure $[2,2]$ we can assume that the the first one is $(1,2)(3,4)$ and the second one is either $(1,2)(3,4)$ or $(1,3)(2,4)$. But then the product is respectively the identity or $(1,4)(2,3)$, and the conclusion easily follows. □

1.8. Realizability in non-positive Euler characteristic

The following result was proved in [8] after partial achievements in the same direction in [15, 9].

Theorem 1.11. *A candidate branch datum $\left(\tilde{\Sigma}, \Sigma, d, n; \pi_1, \ldots, \pi_n\right)$ is realizable if $\chi(\Sigma) \leqslant 0$ and $\tilde{\Sigma}$ and Σ are either both orientable or both non-orientable.*

We slightly postpone the proof to record the following easy consequence:

Corollary 1.12. *A candidate branch datum $\left(\tilde{\Sigma}, \Sigma, d, n; \pi_1, \ldots, \pi_n\right)$ is realizable if $\chi(\Sigma) \leqslant 0$ and $\tilde{\Sigma}$ is orientable while Σ is non-orientable.*

Proof. Let $p : \overline{\Sigma} \to \Sigma$ be the orientation double cover. By assumption d is even and $\pi_i = \pi_i' \bigsqcup \pi_i''$ where π_i' and π_i'' are partitions of $d/2$. Then we have a candidate branch cover

$$\left(\tilde{\Sigma}, \overline{\Sigma}, d/2, 2n; \pi_1', \pi_1'', \ldots, \pi_n', \pi_n''\right)$$

which is realizable by Theorem 1.11 because $\tilde{\Sigma}$ and $\overline{\Sigma}$ are orientable and $\chi(\overline{\Sigma}) \leqslant 0$. Let $h : \tilde{\Sigma} \to \overline{\Sigma}$ be a branched cover realizing it, and suppose π_i' and π_i'' are respectively the local degrees of h at the preimages of the points x_i' and x_i'' of $\overline{\Sigma}$. Suppose that $p^{-1}(p(x_i')) = \{x_i', x_i'''\}$. Since $\overline{\Sigma}$ is connected, one easily sees that for $F \subset \overline{\Sigma}$ finite and $y, z \notin F$ there exists a self-homeomorphism of $\overline{\Sigma}$ fixed on F and mapping y to z. Then there exists a self-homeomorphism u of $\overline{\Sigma}$ such that $u(x_i') = x_i'$ and $u(x_i'') = x_i'''$ for all i. Therefore we have $(p \circ u)(x_i') = (p \circ u)(x_i'')$ and we conclude by setting $f = p \circ u \circ h$. □

We now report on the (elementary) algebraic machinery developed in [8] to prove Theorem 1.11. For $\eta \in \mathfrak{S}_d$ we make the following conventions:

- The *number of cycles* of η includes those of length 1;
- η is a q-cycle if its cycle decomposition consists of one cycle of length q and $d-q$ cycles of length 1.

Lemma 1.13. *If $\theta \in \mathfrak{S}_d$ has ℓ cycles and $t \in \mathbb{N}$ is such $\ell + 2t \leqslant d$ then we can write $\theta = \sigma \cdot \tau$ with σ a d-cycle and τ is an $(\ell+2t)$-cycle.*

Proof. Suppose

$$\theta = (1,\ldots,a_1)(a_1+1,\ldots,a_2)\cdots(a_{\ell-1}+1,\ldots,a_\ell)$$

with $a_\ell = d$. Choose $b_1,\ldots,b_{2t} \in \{1,\ldots,d\} \setminus \{a_1,\ldots,a_\ell\}$ arranged in increasing order, and define

$$\delta = (a_1,\ldots,a_\ell,b_1,\ldots,b_{2t}).$$

Now note that $\delta = \delta_0 \cdot \delta_1$ where

$$\delta_0 = (a_1,\ldots,a_\ell) \qquad \delta_1 = (b_1,\ldots,b_{2t},a_\ell).$$

Moreover

$$\begin{aligned}
\theta \cdot \delta_0 &= (1,\ldots,a_1)(a_1+1,\ldots,a_2)\cdots(a_{\ell-1}+1,\ldots,a_l)\cdot(a_1,\ldots,a_\ell)\\
&= (1,\ldots,a_1,a_1+1,\ldots,a_2,a_2+1,\ldots)\\
&= (1,\ldots,d).
\end{aligned}$$

Now we recall again that $a_\ell = d$ and we calculate

$$\begin{aligned}
\theta \cdot \delta &= \theta \cdot \delta_0 \cdot \delta_1\\
&= (1,\ldots,d)\cdot(b_1,\ldots,b_{2t},d)\\
&= (1,2,\ldots,b_1,b_2+1,b_2+2,\ldots,b_3,b_4+1,b_4+2,\ldots,b_5,\ldots,\\
&\quad\ b_{2t-2}+1,b_{2t-2}+2,\ldots,b_{2t-1},b_{2t}+1,b_{2t}+2,\ldots d,\\
&\quad\ b_1+1,b_1+2\ldots,b_2,b_3+1,b_3+2,\ldots,b_4,\ldots,\\
&\quad\ b_{2t-1}+1,b_{2t-1}+2,\ldots,b_{2t})
\end{aligned}$$

finding that $\theta \cdot \delta$ is a d-cycle, so we conclude by setting $\sigma = \theta \cdot \delta$ and $\tau = \delta^{-1}$. □

Proposition 1.14. *Every even $\theta \in \mathfrak{S}_d$ can be written in both the following forms:*

- *$\theta = [\alpha_1, \beta_1]$ with α_1 a d-cycle;*
- *$\theta = \alpha_1^2 \cdot \alpha_2^2$ with $\alpha_1 \cdot \alpha_2$ a d-cycle.*

Proof. If θ has ℓ cycles, since θ is even we have that $d-\ell$ is even, so we can set $t = (d-\ell)/2$ and use the previous lemma to write $\theta = \sigma \cdot \tau$ with σ, τ d-cycles.

- Since τ and σ^{-1} are d-cycles, τ is conjugate to σ^{-1}, so there exists $\beta_1 \in \mathfrak{S}_d$ such that $\tau = \beta_1 \cdot \sigma^{-1} \cdot \beta_1^{-1}$. Therefore $\theta = [\sigma, \beta_1]$ and we only need to set $\alpha_1 = \sigma$;
- Since σ and τ are d-cycles, σ is conjugate to τ, so there exists $\alpha_1 \in \mathfrak{S}_d$ such that $\sigma = \alpha_1 \cdot \tau \cdot \alpha_1^{-1}$. Therefore $\theta = \alpha_1 \cdot \tau \cdot \alpha_1^{-1} \cdot \tau = \alpha_1^2 \cdot (\alpha_1^{-1} \cdot \tau)^2$ and we only need to set $\alpha_2 = \alpha_1^{-1} \cdot \tau$: since $\alpha_1 \cdot \alpha_2 = \tau$ we have the conclusion.

□

Proof of 1.11. Choose arbitrarily $\theta_i \in \mathfrak{S}_d$ with cycle lengths π_i, and set $\theta = \theta_1 \cdots \theta_n$. The condition $n \cdot d \equiv \tilde{n}$ (mod 2) implies that θ is even.

For orientable Σ we have $\Sigma = g \cdot T$ with $g \geqslant 1$. We apply the first item of the previous proposition to θ^{-1} to write $\theta^{-1} = [\alpha_1, \beta_1]$ with α_1 a d-cycle. Then we conclude by setting $\alpha_2 = \beta_2 = \ldots = \alpha_g = \beta_g = \mathrm{id}$ and invoking Theorem 1.8.

For non-orientable Σ we have $\Sigma = k \cdot \mathbb{P}$ with $k \geqslant 2$. We apply the second item of the previous proposition to θ^{-1} to write $\theta^{-1} = \alpha_1^2 \cdot \alpha_2^2$ with $\alpha_1 \cdot \alpha_2$ a d-cycle. Then we conclude by setting $\alpha_3 = \ldots = \alpha_k = \mathrm{id}$. Invoking Theorem 1.9, we are only left to note that $\alpha_1 \cdot \alpha_2$ is a d-cycle, so there exists m such that $\alpha_1 \cdot (\alpha_1 \cdot \alpha_2)^m$ has a fixed point, and the conclusion follows. □

1.9. Reduction to the sphere

The next result is another major achievement of [8]:

Theorem 1.15. *A candidate branch datum*

$$\left(\tilde{\Sigma}, \mathbb{P}, d, m; \pi_1, \ldots, \pi_n\right)$$

is realizable if $\tilde{\Sigma}$ is not orientable.

The proof uses a technology similar to that employed to establish Theorem 1.11. In this case one has to find $\alpha_1, \theta_1, \ldots, \theta_n \in \mathfrak{S}_d$ such that the θ_i's have cycles of lengths π_i, and $\alpha_1^2 \cdot \theta_1 \cdots \theta_n = \mathrm{id}$. As opposed to the case already discussed, however, one cannot choose $\theta_1, \ldots, \theta_n$ randomly in their conjugacy class and then α_1. Instead, one has to choose representatives in a careful manner using more work on $\mathfrak{S}_d$, for which we refer to [8].

Combining Theorems 1.11 and 1.15 we conclude that a candidate branch datum is guaranteed to be realizable unless $\Sigma = S$ or $\Sigma = \mathbb{P}$ and $\tilde{\Sigma}$ is orientable. But the same argument showing Corollary 1.12 proves the following:

Proposition 1.16. *A candidate branch datum*

$$\left(\tilde{\Sigma}, \mathbb{P}, d, n; \pi_1, \ldots, \pi_n\right)$$

with orientable $\tilde{\Sigma}$ is realizable if and only if it is possible to split π_i as $\pi_i' \bigsqcup \pi_i''$ so that

$$\left(\tilde{\Sigma}, S, d/2, 2n; \pi_1', \pi_1'', \ldots, \pi_n', \pi_n''\right)$$

is realizable.

This implies that solving the Hurwitz existence problem for $\Sigma = S$ would give the full solution.

Example 1.17. The array $(S, \mathbb{P}, 20, 2; [2, \ldots, 2], [6, 2, 2, 2, 1, \ldots, 1])$, with ten 2's in the first partition π_1 and eight 1's in the second one π_2, is a candidate branch cover because $2-(10+4+8) = 20(1-2)$ and π_i does split as $\pi_i' \bigsqcup \pi_i''$ with π_i', π_i'' partitions of 10 for $i = 1, 2$. However any such splitting lead to a candidate branch datum

$$\left(S, S, 10, 4; [2, \ldots, 2], [2, \ldots, 2], \pi_2', \pi_2''\right)$$

with π_2' containing an entry 6, and an application of Theorem 2.6 below implies that any such datum is exceptional. Therefore the original datum with $\Sigma = \mathbb{P}$ is exceptional as well.

1.10. The prime-degree conjecture

We now prove using dessins d'enfant a result also established in [8] using permutations:

Proposition 1.18. *If d is a composite integer then there exist exceptional candidate branch data of the form $(S, S, d, 3; \pi_1, \pi_2, \pi_3)$.*

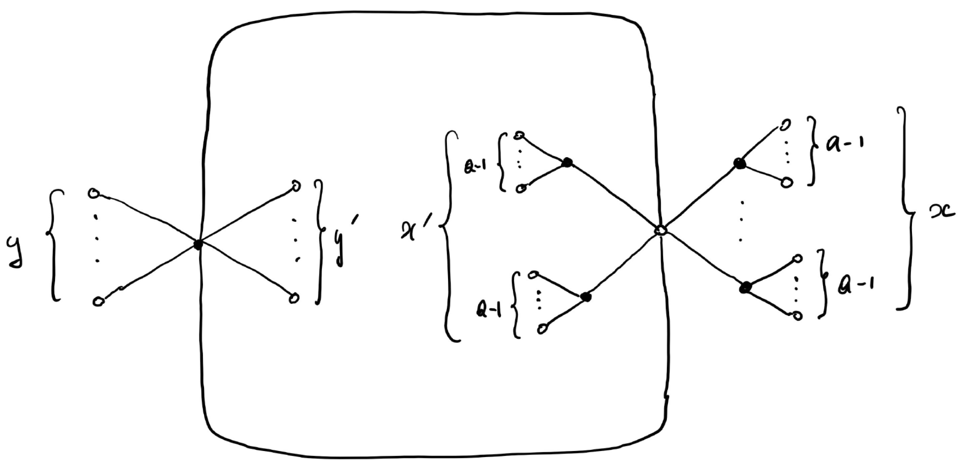

Figure 1.8: A dessin d'enfant matching $[1,\ldots,a],[b+1,1,\ldots,1]$.

Proof. Suppose that $d = ab$ with $a, b > 1$, choose q with $0 < q < b$ and set

$$\pi_1 = [a,\ldots,a] \qquad \pi_2 = [b+1,1,\ldots,1] \qquad \pi_3 = [qa,(b-q)a].$$

Since $\widetilde{n} = b + (1 + (ab - (b+1))) + 2 = ab + 2$ the Riemann-Hurwitz condition holds. Suppose there exists a dessin d'enfant Γ in S matching π_1, π_2. Since Γ must be connected, b of the $b+1$ edges leaving the white vertex v of valence $b+1$ join v to the b black vertices of valence a. The last edge leaving v therefore creates a double connection between v and one black vertex w. For each black vertex u except w there is one edge joining u to v, while the other $a-1$ edges join u to 1-valent white vertices. For w, instead, there are two edges joining w to v and the other $a-2$ joining w to 1-valent white vertices. This implies that Γ appears as in Fig. 1.8, with

$$0 \leqslant x \leqslant b-1 \qquad 0 \leqslant y \leqslant a-2 \qquad x' = b-1-x \qquad y' = a-2-y.$$

Therefore Γ realizes a branch datum with $\pi_3 = [z, ab-z]$ with

$$z = y + 1 + x + x(a-1) = 1 + y + ax$$

(we have counted left to right the white vertices adjacent to the external region). Since $1 + y + ax$ is never a multiple of a, the proof is complete. □

This fact (and many more that we will state in the next lectures) motivate the following:

Prime-degree conjecture If d is a prime number then every candidate branch datum $\left(\widetilde{\Sigma}, S, d, n; \pi_1, \ldots, \pi_n\right)$ is realizable.

The next result, whose proof is sketched again in [8], actually implies that proving the conjecture for $n = 3$ would imply its validity for all n. This is why in the sequel we will be mostly concerned with the case $n = 3$.

Proposition 1.19. *Given d, if all the candidate branch data*

$$\left(\widetilde{\Sigma}, S, d, n; \pi_1, \ldots, \pi_n\right)$$

are realizable for $n = 3$ then they are for all n.

We only provide a vague idea of the argument underlying this result, proceeding by induction on $n \geqslant 3$ and using the monodromy approach. The base of the induction is the very assumption. For $n \geqslant 4$ we take some $\left(\widetilde{\Sigma}, S, d, n; \pi_1, \ldots, \pi_n\right)$ and randomly choose $\theta_{n-1}, \theta_n \in \mathfrak{S}_d$ with cycle lengths π_{n-1}, π_n, and set $\theta'_{n-1} = \theta_{n-1} \cdot \theta_n$. Suppose that θ'_{n-1} has cycle lengths π'_{n-1}. Now *if* there exists a candidate branch datum of the form

$$\left(\widetilde{\Sigma}', S, d, n-1; \pi_1, \ldots, \pi_{n-2}, \pi'_{n-1}\right)$$

we can apply the inductive assumption and find $\theta_1, \ldots, \theta_{n-2}$ such that $\theta_1, \ldots, \theta_{n-2}, \theta'_{n-1}$ realize the datum (note that one permutation can always be chosen at will within its conjugacy class). It readily follows that $\theta_1, \ldots, \theta_{n-2}, \theta_{n-1}, \theta_n$ realizes $\left(\widetilde{\Sigma}, S, d, n; \pi_1, \ldots, \pi_n\right)$. However, *it is not true in general* that for randomly chosen θ_{n-1}, θ_n there exists a candidate branch datum as stated. As a matter of fact, to carry out induction one has to first suitably reorder $\pi_1, \ldots, \pi_n$ and then suitably choose θ_{n-1}, θ_n. See [8], and note that in the statement one could suppose the data are realizable for $n = k$, with $k \geqslant 3$ a given integer, and conclude that they are for all $n \geqslant k$.

2. Short partitions, enumeration, decomposability, very even candidates, checkerboard graphs

We recall that we have reduced to the question whether a candidate branch datum of the form $\left(\widetilde{\Sigma}, S, d, n; \pi_1, \ldots, \pi_n\right)$ is realizable, and that, in view of the prime-degree conjecture, we mostly care about the case $n = 3$.

We begin by reviewing some results where one of the partitions π_i is "short". The next was proved in [36] for $\widetilde{\Sigma} = S$ and in [8] in general. See also [18] where uniqueness issues are discussed with the aid of the action on the braid group on the monodromy (a topic we will not face in these lectures):

Theorem 2.1. *If $\pi_1 = [d]$ then any candidate branch datum*

$$\left(\widetilde{\Sigma}, S, d, n; \pi_1, \ldots, \pi_n\right)$$

is realizable.

We will give an almost complete proof of this result using a more modern technology in Lecture 3.

We then turn to branch data with $\Sigma = S$ and $\pi_1 = [k, d-k]$ to say that their realizability was completely classified in the following cases:

- For all $\widetilde{\Sigma}$, all n and $k = 1$ in [8];
- For $n = 3$, all k and $\pi_2 = \pi_3 = [2, \ldots, 2]$, whence $\widetilde{\Sigma} = S$, in [8];
- For all $\widetilde{\Sigma}$, $n = 3$ and $k = 2$ in [29];
- For $\widetilde{\Sigma} = S$, all n and all k in [25].

We will discuss some of these results, and particularly those of [25], later in this course.

Remark 2.2.

- A degree-d polynomial $f(z) = a_0 + a_1 \cdot z + \ldots + a_d \cdot z^d$ can be viewed as a map from the complex projective line $\mathbb{P}^1(\mathbb{C})$ to itself mapping ∞ to ∞. Moreover $f^{-1}(\infty) = \{\infty\}$ and f is a local homeomorphism away from the finitely many points z at which $f'(z) = 0$. This easily implies that f gives a branched cover matching some $(S, S, d, n; [d], \pi_2, \ldots, \pi_n)$. Conversely, Theorem 2.1 and the facts announced in Remark 1.2 imply that every candidate branch datum of the form $(S, S, d, n; [d], \pi_2, \ldots, \pi_n)$ is realized by a polynomial.

- Similarly, a Laurent polynomial
$$f(z)=a_{-h}\cdot z^{-h}+a_{-h+1}\cdot z^{-h+1}+\ldots+a_{k-1}\cdot z^{k-1}+a_k\cdot z^k$$
with $a_{-h},a_k\neq 0$ gives $f:\mathbb{P}^1(\mathbb{C})\to\mathbb{P}^1(\mathbb{C})$ with $f^{-1}(\infty)=\{0,\infty\}$ and f is a branched cover matching some $(S,S,k+h,n;[k,h],\pi_2,\ldots,\pi_n)$. And, conversely, by the facts announced in Remark 1.2, any realizable candidate branch datum of the form
$$(S,S,d,n;[k,d-k],\pi_2,\ldots,\pi_n)$$
is realized by a Laurent polynomial.

2.1. Special cases

We will mention here without proof (and sometimes somewhat vaguely) some realizability results that are known for a candidate branch datum $\left(\widetilde{\Sigma},S,d,n;\pi_1,\ldots,\pi_n\right)$, and related issues faced in the literature. Some proofs or sketches of will be provided below.

- If $n\cdot d-\widetilde{n}\geqslant 3(d-1)$ then the candidate is realizable [8] except if it is
$$((n-3)\cdot T,S,4,n;[2,2],\ldots,[2,2],[3,1]).$$

- If $\widetilde{\Sigma}=S$ and there exists r such that $\ell_1+\ldots+\ell_r=(r-1)d+1$ then the candidate is realizable [1]. Note that for $r=1$ this reduces to Theorem 2.1.

- If $\widetilde{\Sigma}=S$ and $n\geqslant d$ then the candidate is realizable [1].

- If $\widetilde{\Sigma}=S$, all d_{ij} are at most 2 and $\ell_i\geqslant d-\sqrt{d/2}$ for all i then the candidate is realizable [1].

- Some cases with $\widetilde{\Sigma}=S$, $n=3$ and all π_i of the form $[a_i,\ldots,a_i,1]$ where analyzed in [11] and later in [27].

- It was shown in [3] and later rediscovered in [37] and [26] that for $\widetilde{\Sigma}=S$, $n=3$ and $\pi_3=[k,1,\ldots,1]$, setting
$$x=\mathrm{GCD}\{d_{ij}:\ i=1,2,\ j=1,\ldots,\ell_i\},$$
the datum is realizable if and only if $k\leqslant d/x$.

- It was shown in [3] that for $\widetilde{\Sigma}\neq S$, $n=3$ and $\pi_3=[k,1,\ldots,1]$ the datum is always realizable.

- The results of [3] were extended in [35] to show that for $\widetilde{\Sigma}=S$, any $n\geqslant 3$ and $\pi_i=[k_i,1,\ldots,1]$ for all $i\geqslant 3$, setting
$$x=\mathrm{GCD}\{d_{ij}:\ i=1,2,\ j=1,\ldots,\ell_i\},$$
the datum is realizable if and only if $k_i\leqslant d/x$ for all $i\geqslant 3$.

- Certain very special data with $\widetilde{\Sigma}=T$, namely
$$\begin{array}{l}(T,S,d,3;[3,5,4,\ldots,4],[4,\ldots,4],[2,\ldots,2])\\(T,S,d,3;[2,4,3,\ldots,3],[3,\ldots,3],[3,\ldots,3])\end{array}$$
were shown to be exceptional in [16] and then again using totally different techniques in [7, 10].

- A series of paper by Gonçalves, Zieschang and their collaborators and followers, initiated by [4], deals with the problem of realizing candidate branch data by *indecomposable* maps f, namely such that no expression $f=g\circ h$ exists with g,h non-trivial branched covers.

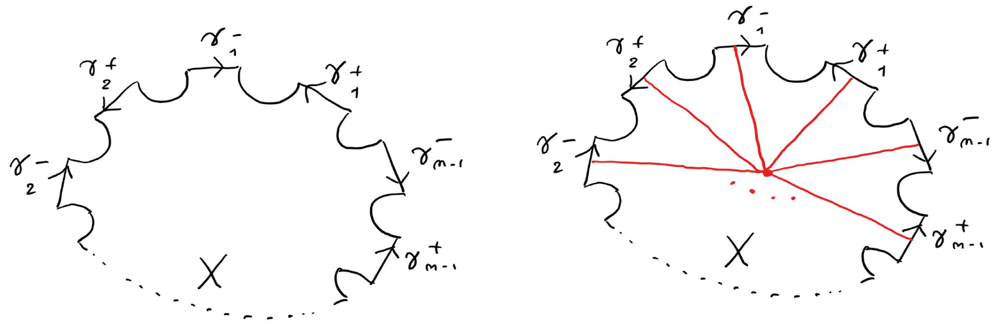

Figure 2.1: A polygon X with $4(n-1)$ edges, half of which are paired to give an n-punctured sphere, and a spine of X.

- Some attention, starting from [2] has been devoted to the issue of uniqueness for realizations of *simple* candidate branch data, namely data such that all π_i have the form $[2,1,\ldots,1]$.

2.2. Zheng's computational approach

We mention here the very interesting results from [38] that have led to a complete computer-assisted classification of all the realizable and exceptional candidate branch data

$$\left(\widetilde{\Sigma}, S, d, n; \pi_1, \ldots, \pi_n\right)$$

with $d \leqslant 20$.

The idea is that a degree-d branched cover of the sphere with n branch points is obtained by suitably gluing d copies $X_1,\ldots,X_d$ of the $4(n-1)$-polygon X shown in Fig. 2.1-left, with each γ_i^- of each X_q glued to some γ_i^+ of another (or the same) $X_{q'}$. Using the graph contained in X shown in Fig. 2.1-right one then shows that the cover is determined by a connected graph with d vertices of valence $2(n-1)$ together with some structure that allows to reconstruct the gluing (we do not spell out this structure here, called *fat graph* in [38]). The theoretical facts proved in [38] are then, roughly speaking, as follows:

- The numbers of fat graphs matching specific candidate branch data can be arranged in a generating function;
- The analysis of this generating function can be reduced to that of one for which the connectivity assumption is dropped;
- The coefficients of the latter generating function can be expressed in terms of characters of certain representations of $\mathfrak{S}_d$;
- These characters can be computed in terms of a certain polynomial $p_\sigma(z)$ and rational function $q_\sigma(z)$ associated to each $\sigma \in \mathfrak{S}_d$;
- These $p_\sigma(z)$ and $q_\sigma(z)$ can be determined by computer for $d \leqslant 20$.

We mention here that, based on his experimental results, Zheng conjectured certain infinite families of candidate branch data to be exceptional, all of which were later proved to be so in [29, 30].

2.3. A sample of counting results

In [31, 32, 33] the number $\nu^{(V)}$ of realizations up to very weak equivalence was computed for the following candidate branch data

$$(g\cdot T, S, 2k, 3; [2,\ldots,2], [2h+1,1,2,\ldots,2], \pi_3)$$
$$(g\cdot T, S, 2k, 3; [2,\ldots,2], [2h+1,3,2,\ldots,2], \pi_3)$$
$$(g\cdot T, S, 2k+1, 3; [1,2,\ldots,2], [2h+1,2,\ldots,2], \pi_3)$$

for arbitrary k and small values of g and h. For given g, h the value of $\nu^{(V)}$ in terms of k is sometimes easy and sometimes rather intricate. We provide here as examples two cases belonging to the last of the three families listed above (the most interesting one, since the degree is odd so it can be a prime).

- For $(S, S, 2k+1, 3; [1,2,\ldots,2], [5,2,\ldots,2], [p,q,r])$ one has
$$\nu^{(V)} = \begin{cases} 0 & \text{if } p=q=r \\ 1 & \text{if two of } p,q,r \text{ but not all three are equal to each other} \\ 2 & \text{if } p,q,r \text{ are different from each other and one is } > k \\ 3 & \text{if } p,q,r \text{ are different from each other and all are } \leqslant k; \end{cases}$$
- For $(2T, S, 2k+1, 3; [1,2,\ldots,2], [9,2,\ldots,2], [2k+1])$ one has $\nu^{(V)} = 10$ for $k=4$ and
$$\nu^{(V)} = \frac{k}{16}(7k^3 - 42k^2 + 72k - 37) + \frac{5}{8}(2k-3)\left[\frac{k}{2}\right]$$
otherwise.

2.4. Special families with many 2's

To give a flavour of how the arguments based on dessins d'enfant go, we will prove now the following from [29]:

Proposition 2.3. *The candidate branch datum*

$$(T, S, d, 3; [2,\ldots,2], [5,3,2,\ldots,2], [p,q])$$

is realizable if and only if $p \neq q$.

Proof. Let us first check that the Riemann-Hurwitz condition is satisfied. If $d = 2k$ we have $\ell_1 = k$, $\ell_2 = k-2$ and $\ell_3 = 2$, so $\tilde{n} = 2k$, and the condition reads $0 - 2k = 2k(2-3)$, so it holds.

We now analyze the dessins d'enfant Γ matching $[2,\ldots,2], [5,3,2,\ldots,2]$. Ignoring the vertices of valence 2, that do not contribute to the topology of Γ, we only have the abstract graphs in Fig. 2.2-top. These graphs embed in T as in Fig. 2.2-center. Now, restoring the valence-2 vertices, we have the possibilities shown in Fig. 2.2-bottom, where an edge marked by $a \geqslant 0$ contains a white and $a+1$ black valence-2 vertices, and $x+y+z+w = k-4$. Counting the lengths of the complementary regions is a routine matter, showing that these graphs realize respectively the following π_3's:

$$\begin{array}{ll} [x+1, 2k-(x+1)] & [x+1, 2k-(x+1)] \\ [y+z+2, 2k-(y+z+2)] & [x+y+z+3, 2k-(x+y+z+3)]. \end{array}$$

This readily implies that $[k,k]$ is not realized while all $[p,q]$ with $p<q$ are. $\square$

Similar arguments lead to the following:

Proposition 2.4. *The candidate branch datum*

$$(S, S, d, 3; [2,\ldots,2], [5,3,2,\ldots,2], \pi_3)$$

(with π_3 of length 4) is realizable if and only π_3 does not have the form $[p,p,q,q]$ or $[p,p,p,3p]$.

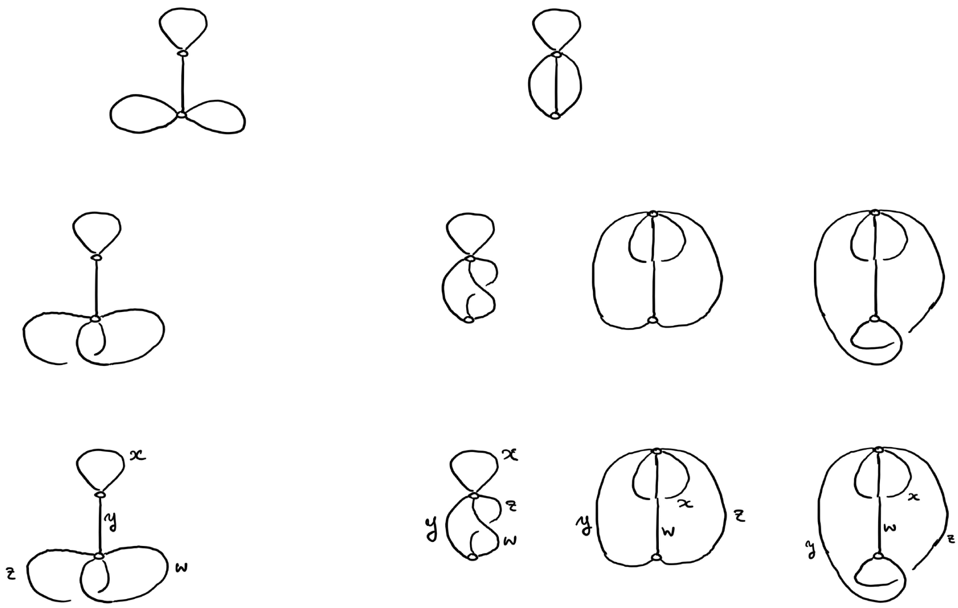

Figure 2.2: Abstract graphs with two vertices of valences $[5,3]$, their embeddings in T and their decorations with valence-2 vertices.

Proposition 2.5. *The candidate branch datum*

$$(S,S,d,3;[2,\dots,2],[3,3,2,\dots,2],\pi_3)$$

(with π_3 of length 3) is realizable if and only if π_3 does not contain $d/2$.

2.5. Exceptionality and realizability from decomposability

To confirm the exceptionality of some families of candidate branch data conjectured to be so in [38], the following was shown in [29]:

Theorem 2.6. *If a candidate branch datum $(S,S,d,n;\pi_1,\dots,\pi_n)$ is realizable and all the entries of π_1 and π_2 are even, then π_i is the union of two partitions of $d/2$ for all $i \geqslant 3$.*

The proof is based on the idea that certain conditions on a candidate branch datum force a map realizing it, if any, to be the composition of two non-trivial maps. More exactly, Theorem 2.6 readily follows from the following fact fully proved below:

Proposition 2.7. *If $f : S \to S$ is a branched cover realizing*

$$(S,S,d,n;\pi_1,\dots,\pi_n)$$

with all the entries of π_1 and π_2 even, then $f = g \circ h$ where $g : S \to S$ is the map $g(z) = z^2$ realizing $(S,S,2,2;[2],[2])$ and $h : S \to S$ is a branched cover realizing some

$$\left(S,S,d/2,2n-2;[d_{1j}/2]_{j=1}^{\ell_1},[d_{2j}/2]_{j=1}^{\ell_2},\pi_3',\pi_3'',\dots,\pi_n',\pi_n''\right)$$

where $\pi_i = \pi_i' \bigsqcup \pi_i''$ for $i \geqslant 3$.

Before proving Theorem 2.6 we illustrate a consequence that demonstrated quite powerful in practice. This uses the next result also established in [29]:

Proposition 2.8. *If a candidate branch datum $(S,S,d,n;\pi_1,\dots,\pi_n)$ is realizable and there exists $k \geqslant 2$ such that each entry of π_1 and π_2 is a multiple of k, then each entry of π_i is at most d/k for all $i \geqslant 3$.*

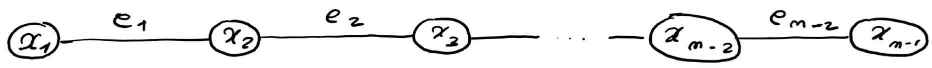

Figure 2.3: A simple path through $n-1$ of the branching points.

We do not reproduce here the argument leading to this proposition, we only mention that it uses the exceptionality of certain candidate branch data $(S, S, d, n; \pi_1, \ldots, \pi_n)$ with $\pi_1 = [k, 1, \ldots, 1]$ and $\pi_i = [2, 1, \ldots, 1]$ for $i \geqslant 4$, together with a direct argument exploiting the monodromy approach. But we show how Theorem 2.6 and Proposition 2.8 combine to give the following:

Corollary 2.9. *If a candidate branch datum $(S, S, d, n; \pi_1, \ldots, \pi_n)$ is realizable and there exists $k \geqslant 2$ such that d is multiple of $2k$, each entry of π_1 is a multiple of k and each entry of π_2 and π_3 is even, then each entry of π_2 and π_3 is at most d/k and each entry of π_i is at most $d/2k$ for all $i \geqslant 4$.*

Proof. We apply Proposition 2.7 to see that any f realizing the datum is $f = g \circ h$ with $g(z) = z^2$ and h realizing

$$\left(S, S, d/2, 2n-2; \pi_1', \pi_1'', [d_{2j}/2]_{j=1}^{\ell_2}, [d_{3j}/2]_{j=1}^{\ell_3}, \pi_4', \pi_4'', \ldots, \pi_n', \pi_n''\right)$$

with $\pi_i = \pi_i' \bigsqcup \pi_i''$ for $i \neq 2, 3$. Now k divides all the entries of π_1' and π_1'' so by Proposition 2.8 we have $d_{ij}/2 \leqslant (d/2)/k$ for $i = 2, 3$ and $d_{ij} \leq (d/2)/k$ for $i \geqslant 4$. $\square$

Before establishing Proposition 2.7 we state without proof the following result from [29] also based on the idea of taking compositions of branched covers:

Theorem 2.10. *Consider a candidate branch datum $\left(\widetilde{\Sigma}, S, d, 3; \pi_1, \pi_2, \pi_3\right)$. If there exists $k \geqslant 3$ such that all the entries of all the π_i's are divisible by k then the candidate is realizable.*

Note that this result only applies for high genus of $\widetilde{\Sigma}$, because $\widetilde{n}$ must be small if all the entries of all π_i are divisible by k. For instance, it shows the realizability of the following candidate branch datum:

$$(3T, S, 12, 3; [6, 3, 3], [6, 3, 3], [6, 6]).$$

2.6. First generalization of dessins d'enfant

The proof of Proposition 2.7 depends on a geometric and an algebraic argument. Here we illustrate the former. Assuming some $f : \widetilde{\Sigma} \to S$ realizes some candidate branch datum $\left(\widetilde{\Sigma}, S, d, n; \pi_1, \ldots, \pi_n\right)$ with $n \geqslant 4$, we consider in S a simple path $e = e_1 \cdots e_{n-2}$ going through the branching points $x_1, \ldots, x_{n-1}$ in this order, see Fig. 2.3. Setting $\Gamma = f^{-1}(e)$ we see that Γ is a graph with vertices and edges labeled x_i and e_p, such that:

- The valences of the vertices labeled x_i are the entries of π_i for $i = 1$ and $i = n-1$, and those of $2\pi_i$ for $i = 2, \ldots, n-2$;
- Each edge labeled e_p has ends at vertices labeled x_p and x_{p+1};
- Around each vertex labeled x_i for $i = 2, \ldots, n-2$ the edges labeled e_{i-1} and e_i alternate;

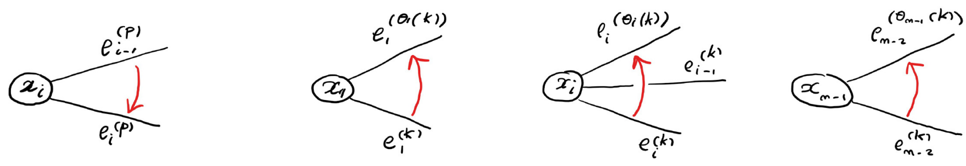

Figure 2.4: How to enumerate the edges e_i and how to compute the action of θ_i.

- The complementary regions of Γ are discs and on the boundary of each of them the cycle $x_1, x_2, \dots, x_{n-2}, x_{n-1}, x_{n-2}, \dots, x_2$ repeats a certain number of times, called the *length* of the region;

- The lengths of the complementary regions of Γ are the entries of π_n.

Conversely, any such Γ gives a realization f of the candidate branch datum.

Remark 2.11. If Γ comes from a given f of course we see that it contains precisely d edges for each of the labels $e_1, \dots, e_{n-2}$. Conversely, if Γ satisfies the above combinatorial conditions, one sees that the same conclusion holds, either by a direct counting argument or invoking the fact that Γ defines a map f.

We will need below an explicit descprition of the monodromy elements $\theta_1, \dots, \theta_{n-1}$ associated as in Theorem 1.8 to the map f realizing a branch datum associated to Γ as above. The method is as follows (see Fig. 2.4):

- Enumerate the edges with label e_1 as $e_1^{(1)}, \dots, e_1^{(d)}$ in some arbitrary fashion;

- Recursively enumerate the edges with label e_i with $i \geqslant 2$ as $e_i^{(1)}, \dots, e_i^{(d)}$ in such a way that around each vertex labeled x_i the edge $e_i^{(p)}$ follows $e_{i-1}^{(p)}$ in a *clockwise* order;

- Now for $i = 1, \dots, n-2$ and $k \in \{1, \dots, d\}$ the value of $\theta_i(k)$ is computed by locating $e_i^{(k)}$ and looking at the next $e_i^{(*)}$ in a *counterclockwise* order around the vertex labeled x_i of $e_i^{(k)}$; if that $e_i^{(*)}$ is $e_i^{(h)}$ then $\theta_i(k) = h$; note that for $i = 1$ we have that $e_i^{(h)}$ is simply the next edge after $e_i^{(k)}$, while for $i = 2, \dots, n-2$ there is one $e_{i-1}^{(*)}$ in between;

- The value of $\theta_{n-1}(k)$ is found similarly, by locating $e_{n-2}^{(k)}$ and finding the next edge $e_{n-2}^{(*)}$ around the end of $e_{n-2}^{(k)}$ labeled x_{n-1}, which is just the next edge regardless of the label;

- Finally, $\theta_n = \theta_{n-1}^{-1} \cdots \theta_1^{-1}$.

The justification of this rule is simply given, remembering that the action of θ_i is obtained by lifting a circle going aroung x_i. We leave the details to the reader.

2.7. Block decompositions

For the algebraic machinery we develop here, see the references in [29].

Definition 2.12. If k is a divisor of d, we call *k-block decomposition* of $\gamma \in \mathfrak{S}_d$ a partition

$$\{1, \dots, d\} = \bigsqcup_{m=1}^{d/k} B_m$$

such that each B_m has cardinality k and $\gamma(B_m) = B_{\widehat{\gamma}(m)}$ with $\widehat{\gamma} \in \mathfrak{S}_{d/k}$.

Proposition 2.13. *A degree-d branched cover $f : \widetilde{\Sigma} \to \Sigma$ with monodromy $\rho : \pi_1(\Sigma^\bullet) \to \mathfrak{S}_d$ can be expressed as $g \circ h$ with $h : \widetilde{\Sigma} \to \Omega$ a degree-k and $g : \Omega \to \Sigma$ a degree-(d/k) branched cover if and only if there exists a common k-block decomposition for all γ in $\mathrm{Im}(\rho)$.*

Proof. Note first that ρ is only well-defined up to conjugation, because it depends on the choice of an ordinary point x_0 of f as a basepoint for $\pi_1(\Sigma^\bullet)$, and on the numbering $y_1, \ldots, y_d$ for $f^{-1}(x_0)$. However, if $B_1, \ldots, B_{d/k}$ is a common k-block decomposition for the elements of $\mathrm{Im}(\rho)$ and $\rho' = \alpha^{-1} \cdot \rho \cdot \alpha$ with $\alpha \in \mathfrak{S}_d$, then setting $B'_m = \alpha^{-1}(B_m)$ we see that $B'_1, \ldots, B'_{d/k}$ is a common k-block decomposition for the elements of $\mathrm{Im}(\rho')$. So the existence of such a decomposition is well-defined.

Suppose now that an expression $f = g \circ h$ exists, let $g^{-1}(x_0)$ consist of the points $z_1, \ldots, z_{d/k}$ and set $B_m = \{i : \ y_i \in h^{-1}(z_m)\}$. Of course each B_m has cardinality k. Moreover, if $\gamma \mapsto \widehat{\gamma}$ is the monodromy of g, we readily see that $\rho(\gamma)(B_m) = B_{\widehat{\gamma}(m)}$ and we are done.

Conversely, if a common block decomposition exists for all the elements of $\mathrm{Im}(\rho)$, we can subdivide into blocks of k points not only the fibre $f^{-1}(x_0)$ of the basepoint, but any fibre $f^{-1}(x)$. The labels of the blocks are only defined up to conjugation, but not the blocks themselves, so it makes sense to define $h : \widetilde{\Sigma} \to \Omega$ as the map that collapses each block to a point, and then $g : \Omega \to \Sigma$ as the map such that $f = g \circ h$, and we are done. □

2.8. Checkerboard graphs

We introduce here a notion we will employ to prove Proposition 2.7 and also later in this course.

Definition 2.14. A graph Γ in a surface Σ is called a *checkerboard graph* if its complementary regions are discs that can be coloured black and white so that each edge separates black from white.

Proposition 2.15. *A graph Γ in the sphere S is a checkerboard graph if and only if its complementary regions are discs and all its vertices have even valence.*

Proof. We proceed by induction on the number $m \geqslant 1$ of vertices of Γ. For $m = 1$ we have that Γ is a bouquet of $k \geqslant 1$ circles and we proceed by induction on k. For $k = 1$ the conclusion is obvious, while for $k > 1$ we select and innermost such circle γ, we delete it to get a bouquet Γ', we apply the induction assumption to Γ' and we colour the empty disc bounded by γ of the opposite colour of the complementary disc of Γ' in which γ lies.

For $m > 1$ we note that Γ must be connected because its complement consists of discs, so there is an edge e of Γ with distinct ends. Then we define Γ' as the graph obtained from Γ by collapsing e to a point, we apply the inductive assumption to Γ' and, for the first time, we use the assumption that the ends of e have even valence to conclude as suggested in Fig. 2.5 (on the right the picture is not a general one, we make an example). □

Remark 2.16. The previous result fails to be true for surfaces other than the sphere. For instance, in the torus T there is a bouquet of two circles whose complement is a single disc.

We are eventually ready to conclude the argument giving Theorem 2.6.

Proof of 2.7. To use the above notation, we rearrange the branch datum associated to f assuming that the even entries are those of π_1 and π_{n-1}. Now we associate to f a graph Γ with vertices labeled $x_1, \ldots, x_{n-1}$ and edges labeled $e_1, \ldots, e_{n-2}$, and we enumerate the edges labeled e_i as $e_i^{(1)}, \ldots, e_i^{(d)}$, as explained above. By our choice, all the vertices of Γ have even valence, so Γ is a checkerboard graph.

By Proposition 2.13 (and its proof, to be completely honest) it is now enough to prove that $\{1, \ldots, d\}$ splits into two blocks of $d/2$ elements such that θ_1 and θ_{n-1} switch them and $\theta_2, \ldots, \theta_{n-2}, \theta_n$ leave them invariant. To define the blocks we orient all e_i's from x_i to x_{i+1} and we declare an integer $p \in \{1, \ldots, d\}$ to be black (respectively, white) if the orientation

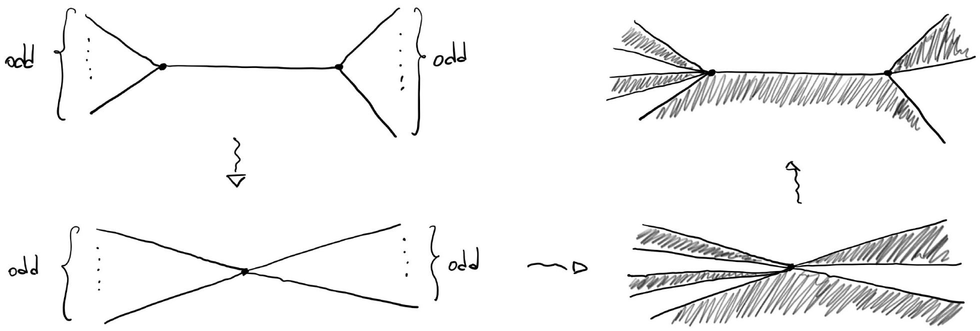

Figure 2.5: A graph in the sphere with vertices of even valence is a checkerboard graph.

Figure 2.6: Edge colours are well defined.

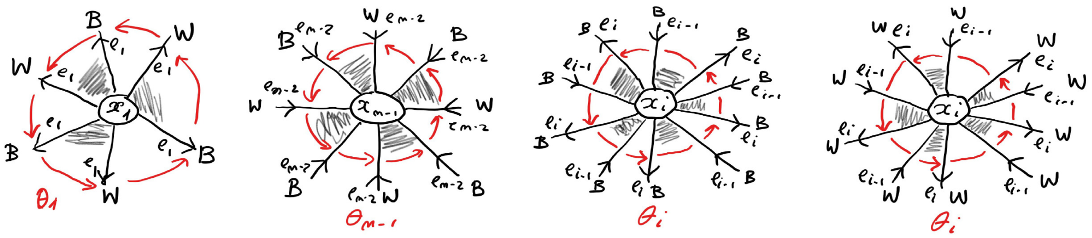

Figure 2.7: Action of the θ_i's on the black and white edges.

of $e_i^{(p)}$ is induced by the incident black (respectively, white) complementary region. We are left to show that:

- This definition is independent of i, which is shown in Fig. 2.6;
- The permutations θ_1 and θ_{n-1} switch the black and the white blocks while $\theta_2, \ldots, \theta_{n-2}$ leave them invariant, which is shown in Fig. 2.7,
- The permutation θ_n leaves the blocks invariant, which follows from above because $\theta_n = \theta_{n-1}^{-1} \cdots \theta_1^{-1}$.

□

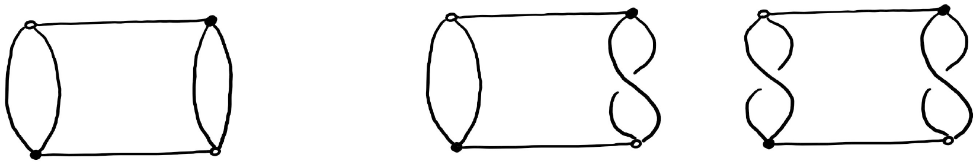

Figure 2.8: The dessin d'enfant matching $[3,3],[3,3]$ and its embeddings in T.

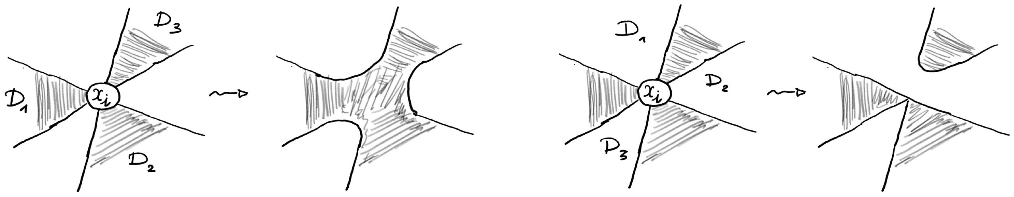

Figure 2.9: A merging as on the left is performed if D_1, D_2, D_3 are distinct discs. One as on the right if D_1 and D_2 are distinct but perhaps D_3 coincides with one of them.

2.9. More checkerboard graphs

An interesting technique was introduced in [1] and applied to show some realizability and exceptionality results for candidate branch data with both the base and the candidate covering surface equal to the sphere S. The method was generalized to any covering surface in [30], proving for instance the following:

Theorem 2.17. *Any candidate branch datum of the form*

$$\left(\widetilde{\Sigma}, S, d, 3; \pi_1, \pi_2, [d-2, 2]\right)$$

is realizable, except

$$(T, S, 6, 3; [3,3], [3,3], [4,2]).$$

The fact that $(T, S, 6, 3; [3,3], [3,3], [4,2])$ is exceptional is easily seen using dessins d'enfant: if Γ matches $[3,3],[3,3]$ then abstractly it is as in Fig. 2.8, where also its only embeddings in T are shown, and one readily sees that as π_3 they realize $[5,1]$ and $[3,3]$.

The constructive part of the previous result relies on the following construction, that we describe omitting some details. Let a map f realize a candidate branch datum

$$\left(\widetilde{\Sigma}, S, d, n; \pi_1, \ldots, \pi_n\right),$$

and arrange $x_1, \ldots, x_n$ as the n-th roots of unity on the boundary of unit disc $D \subset \mathbb{C} \subset \mathbb{P}^1(\mathbb{C}) = S$. Moreover paint D black and its complementaty disc white. Pull back ∂D through f to get a checkerboard graph $\Gamma \subset \widetilde{\Sigma}$ with vertices marked x_i having valences equal to the entries of $2\pi_i$, and complementary discs each with a cycle of vertices $x_1, \ldots, x_n$ on its boundary, positively arranged for the black discs and negatively for the white ones. Now we start merging together distinct black discs and distinct white discs, as shown in Fig. 2.9. Note that a merging of discs is performed at some vertex, and that there is no unique merging strategy: as long as two or more distinct discs of the same colour are incident to the same vertex, they can be merged together. While performing this merging, however, we keep track of what we have done by inserting a tree with the same label as the vertex through which we have done the merging, as shown in Fig. 2.10. Note that collapsing the tree to a point results in undoing the merging.

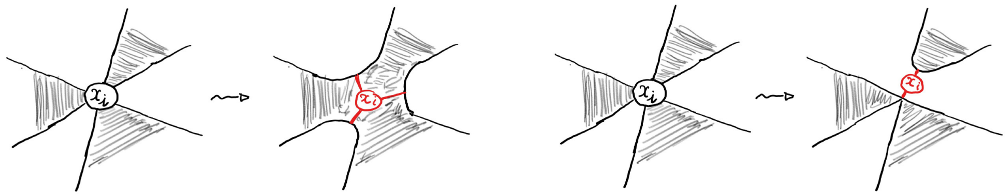

Figure 2.10: Collapsing these trees results in undoing the mergings of Fig. 2.9.

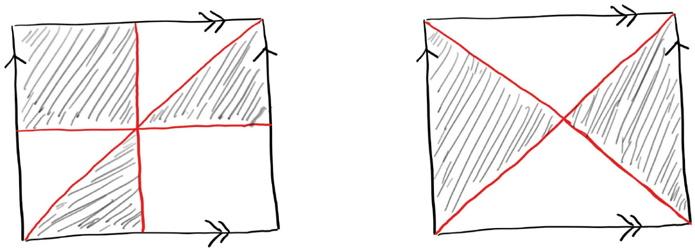

Figure 2.11: Minimal checkerboard graphs on the torus. In both cases the boundary of the square is *not* part of the graph.

Doing this as long as possible, we end up with a *minimal checkerboard graph*, namely one with just one black and one white disc. Of course for $\widetilde{\Sigma} = S$ this is just a circle, but for other $\widetilde{\Sigma}$ there are other possibilities. For instance for $\widetilde{\Sigma} = T$ the minimal checkerboard graphs are those shown in Fig. 2.11.

Example 2.18. Let us consider the candidate branch datum

$$(S, S, 7, 4; [2, 2, 2, 1], [4, 2, 1], [3, 3, 1], [2, 1, 1, 1, 1, 1]).$$

A checkerboard graph associated to a map realizing this datum is shown in Fig. 2.12, where for simplicity we write i instead of x_i (and we use light green rather than black). In Fig. 2.13 we show the result of a maximal disc merging for this graph, and in Fig. 2.14, we show the same picture again in a tidier fashion.

The construction of a checkerboard graph with decorated trees attached to it can actually be reversed, which gives a result of the following type: *a candidate branch datum*

$$\left(\widetilde{\Sigma}, S, d, n; \pi_1, \ldots, \pi_n\right)$$

is realizable if and only if there exists on $\widetilde{\Sigma}$ *a minimal checkerboard graph* Γ *and a finite family of trees with labels* $1, \ldots, n$ *having their valence-0 and valence-1 vertices on* Γ*, but otherwise disjoing from* Γ *and from each other, satisfying. . .* a long list of combinatorial conditions depending on the π_i's. The details of the statement are too complicated to be reproduced here (see [1] for the case $\widetilde{\Sigma} = S$ and [30] for the general case). But it is using this result and an algorithmic machinery to construct minimal checkerboard graphs in $\widetilde{\Sigma}$ that Theorem 2.17 was established in [30].

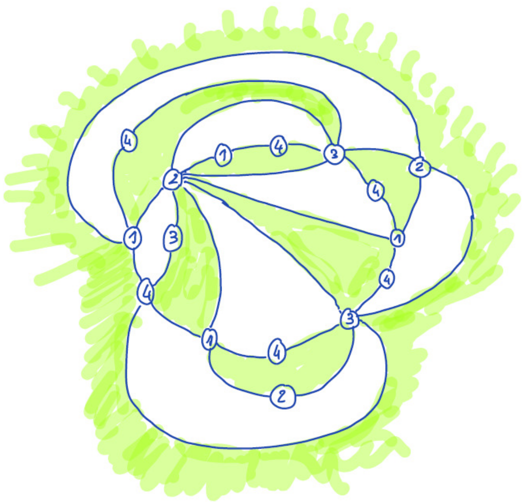

Figure 2.12: A checkerboard graph.

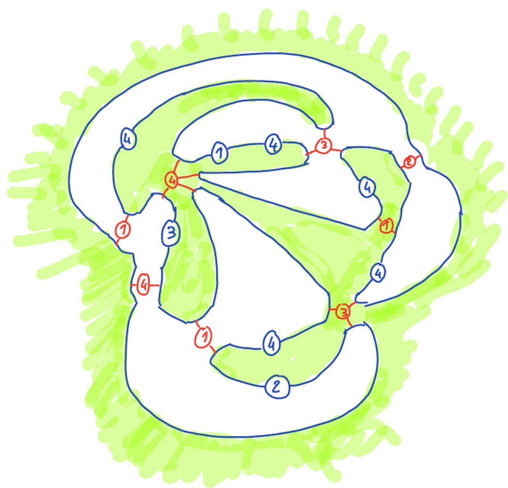

Figure 2.13: A minimal checkerboard graph.

3. Constellations and geometric orbifolds

In this lecture we describe two approaches that have led to major advancements towards the solution of the Hurwitz existence problem.

3.1. Data with one partition of length 2

We now provide a full statement of the following result from [25] already announced above:

Theorem 3.1. *A candidate branch datum* $(S, S, d, n; \pi_1, \ldots, \pi_n)$ *with* $\ell_n = 2$ *is always realizable for* $n \geqslant 4$*, while for* $n = 3$ *it is if and only if it does not belong to the following list:*

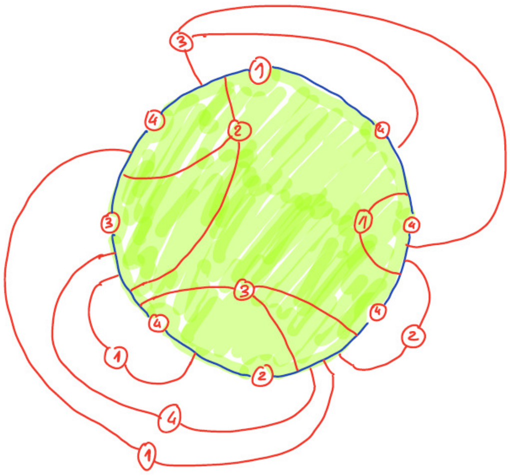

Figure 2.14: The same minimal checkerboard graph as in Fig. 2.13.

- $(S, S, 12, 3; [2, \ldots, 2], [3, 3, 3, 1, 1, 1], [6, 6])$
- $(S, S, 2a, 3; [2, \ldots, 2], [2, \ldots, 2], [b, 2a - b])$ *with* $b \neq a$
- $(S, S, ab, 3; [a, \ldots, a], [b + 1, 1, \ldots, 1], [qa, (b - q)a])$
- $(S, S, 2a, 3; [2, \ldots, 2], [k, k - 1, 1, \ldots, 1], [b, 2a - b])$ *with* $k \geqslant 3$
- $(S, S, 2a, 3; [2, \ldots, 2], [k, k, 1 \ldots, 1], [a - 1, a + 1])$ *with* $k \geqslant 2$
- $(S, S, 2a, 3; [2, \ldots, 2], [3, 1, 2, \ldots, 2], [a, a])$.

Exceptionality of the listed items is very easily established using dessins d'enfant (note that the third one was already discussed in Lecture 1). Realizability of the other candidate branch data relies on a topological argument that we now explain with some detail.

Remark 3.2. The statement in [25] contains 7 exceptional families, rather than 6, but two of them actually coincide. More precisely, using the notation of [25] (employed only within the present remark because incompatible with our current one), the families (4) and (5)

$$\Pi_1 = \{2, \ldots, 2\}, \Pi_2 = \{1, \ldots, 1, d, d\}, \Pi_3 = \{2d - 3, n - 2d + 3\} \quad d \geqslant 3$$
$$\Pi_1 = \{2, \ldots, 2\}, \Pi_2 = \{1, \ldots, 1, d, d\}, \Pi_3 = \{2d - 1, n - 2d + 1\} \quad d \geqslant 3$$

are the same, because the Riemann-Hurwitz condition reads

$$2 - \left(\frac{n}{2} + (2 + n - 2d) + 2\right) = n(2 - 3) \;\Rightarrow\; n = 4d - 4$$

therefore $n - 2d + 3 = 2d - 1$ and $n - 2d + 1 = 2d - 3$, whence

$$\{2d - 3, 2d - 1\} = \left\{\frac{n}{2} - 1, \frac{n}{2} + 1\right\}$$

which leads to the expression we have used in our penultimate item above.

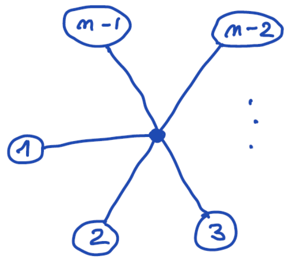

Figure 3.1: An $(n-1)$-star.

3.2. Constellations

Let us call $(n-1)$-*star* a planar graph as shown in Fig. 3.1, namely one consisting of $n-1$ edges with one common unlabeled end (named the *centre* of the star) and distinct other ends labeled $1, \ldots, n-1$, arranged around the centre according to the orientation of the plane. We then term $(n-1)$-*constellation* in $\widetilde{\Sigma}$ a graph $\Gamma \subset \widetilde{\Sigma}$ consisting of some number of stars such that:

- Any two stars share at most some vertices with equal labels;
- The vertices labeled $1, \ldots, n-1$ appear positively arranged around the centre of each star according to the orientation of $\widetilde{\Sigma}$;
- The complementary regions of Γ are discs.

Note that around the boundary of each complementary region R we see the labels $1, \ldots, n-1$ negatively arranged and appearing a certain number p of times, called the *length* of R.

To a map $f : \widetilde{\Sigma} \to S$ realizing some candidate branch datum

$$\left(\widetilde{\Sigma}, S, d, n; \pi_1, \ldots, \pi_n\right)$$

we can associate a constellation Γ defined as the pull-back through f of the star in Fig. 3.1 not containing the branching point x_n and with the vertices labeled $1, \ldots, n-1$ at the branching points $x_1, \ldots, x_{n-1}$. Then Γ contains d stars, the set of valences of its vertices labeled i (denoted henceforth by $\pi_i(\Gamma)$) is π_i, and the set of lengths of the complementary discs of Γ (denoted henceforth by $\pi_n(\Gamma)$) is π_n. Moreover, this construction can be reversed, so the existence of a constellation matching a candidate branch datum implies that the latter is realizable.

Example 3.3. In Fig. 3.2 we show a 4-constellation in the sphere, consisting of 10 stars. Counting the valences of the vertices labeled 1, 2, 3, 4 and the lengths of the complementary regions we see that the constellation realizes

$$\begin{aligned}(S, S, 10, 5; \quad & [2,2,2,1,1,1,1], [3,3,1,1,1,1], [4,3,1,1,1], \\ & [4,1,1,1,1,1,1], [3,2,1,1,1,1,1]).\end{aligned}$$

3.3. The idea underlying Pakovich's argument

The proof of Theorem 3.1 is too long to be reproduced here, but we will explain the idea it is based on. To do so, we introduce the following notation: for an array π of positive integers, we denote by $\overline{\pi}$ the same array π with all 1's removed. The argument of [25] is in three steps:

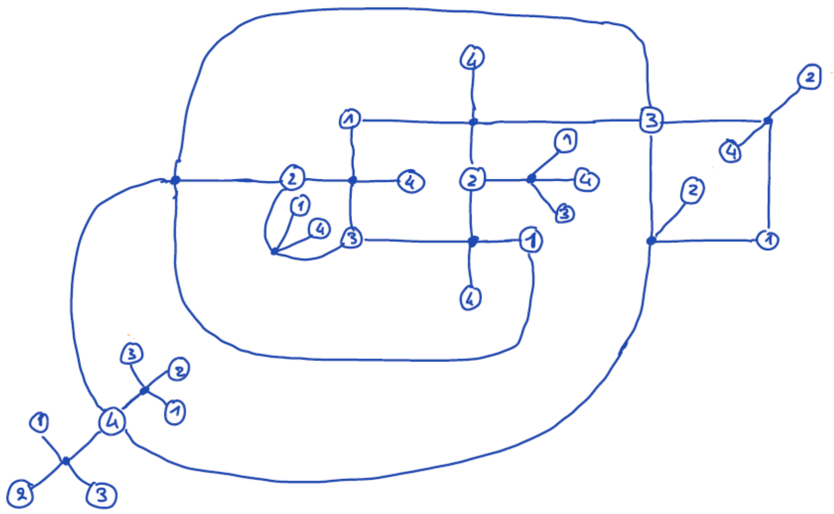

Figure 3.2: A constellation.

Step 1: Show that to realize the given branch datum $(S, S, n, d; \pi_1, \ldots, \pi_n)$ with $\pi_n = [s, d-s]$ it is enough to construct an $(n-1)$-constellation Γ with:

- $\overline{\pi}_i(\Gamma) = \overline{\pi}_i$ for $i = 1, \ldots, n-1$;
- $\pi_n(\Gamma)$ containing s.

This means that in constructing Γ we do not need to count its valence-1 vertices and the number of stars it consists of.

Step 2: Suppose first that s is "small" compared to the total length of $\overline{\pi}_1, \ldots, \overline{\pi}_{n-1}$ and construct Γ as required in two passages: first define $\overline{\pi}_i'$ as an array of 2's of the same length as $\overline{\pi}_i$, and constuct Γ with $\overline{\pi}_i(\Gamma) = \overline{\pi}_i'$ for $i = 1, \ldots, n-1$ and $\pi_n(\Gamma)$ containing s as a circle with interior of length s, no centres of stars in the interior but some branches outside, as suggested in Fig. 3.3. Since all the 2's in $\pi_i(\Gamma)$ are incident to the outside of the circle one can now modify Γ by adding stars to the outside until $\overline{\pi}_i(\Gamma) = \overline{\pi}_i$ for $i = 1, \ldots, n-1$ and $\pi_n(\Gamma)$ still contains s.

Step 3: Given a constellation Γ as above with $\overline{\pi}_i(\Gamma) = \overline{\pi}_i$ for $i = 1, \ldots, n-1$ and $\pi_n(\Gamma)$ containing s, we can increase s without affecting the $\pi_i(\Gamma)$'s by reflecting some of the outer branches with respect to the circle, from the outside to the inside. For instance in Fig. 3.4 we show a reflection that increases s by 3 (but in the same situation we could increase it by 1 or by 4). The conclusion of the proof is then obtained by showing that the flexibility in the choice of the branches to reflect is enough to realize all the values of s. This is based on some numerical estimates on the lengths of the π_i's, and the upshot is that for $n \geqslant 4$ a sufficient flexibility is always guaranteed, while for $n = 3$ some exceptions arise.

We will not provide all the details for Steps 2 and 3, but we will for Step 1, and we will actually consider the general case of a candidate branch datum $\left(\widetilde{\Sigma}, S, d, n; \pi_1, \ldots, \pi_n\right)$ with arbitrary $\widetilde{\Sigma}$. To this end denote by g the genus of $\widetilde{\Sigma}$ and we define m_i as the length of $\overline{\pi}_i$ and u_i as $\ell_i - m_i$. We further assume that entries of π_i are arranged non-increasingly, so π_i is $\overline{\pi}_i$ followed by u_i repetitions of 1.

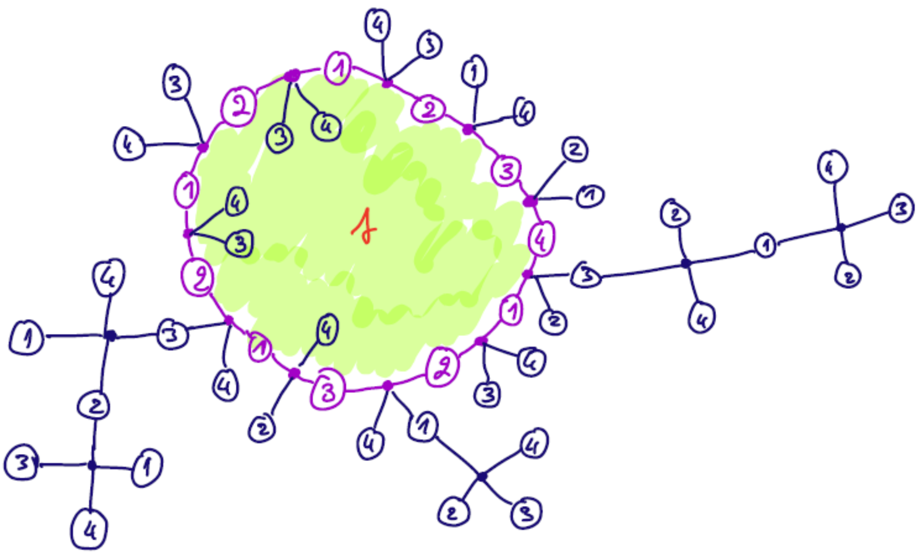

Figure 3.3: A circle with branches outside.

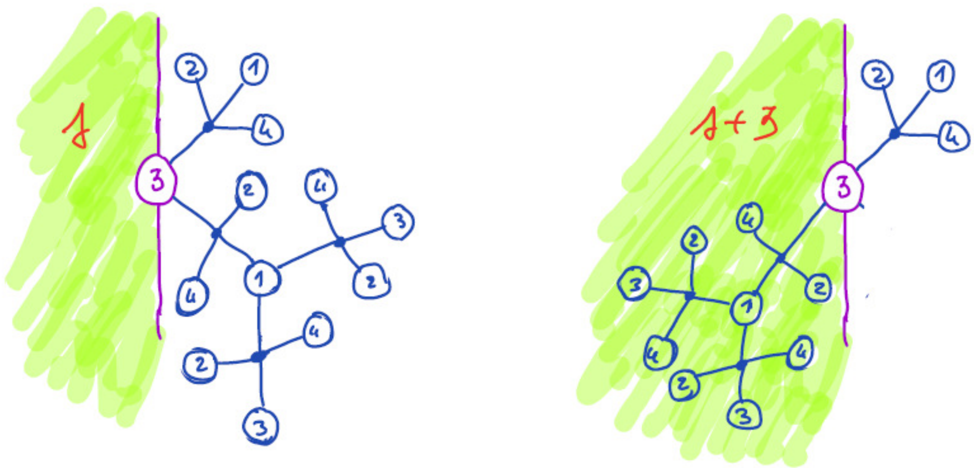

Figure 3.4: Reflection of a branch.

Proposition 3.4. *If $\ell_n = 2$ then*

$$\sum_{i=2}^{n-1}\sum_{j=1}^{m_i}(d_{ij}-2) = u_1 + m_1 - (m_2 + \ldots + m_{n-1}) + 2g.$$

Corollary 3.5. *A candidate branch datum $(\widetilde{\Sigma}, S, d, n; \pi_1, \ldots, \pi_n)$ with $\widetilde{\Sigma}$ of genus g and $\pi_n = [s, d-s]$ is determined by $g, n, \overline{\pi}_1, \ldots, \overline{\pi}_{n-1}, s$.*

Proof. $\overline{\pi}_1, \ldots, \overline{\pi}_{n-1}$ determine $m_1, \ldots, m_{n-1}$, so by the previous result u_1 is determined, and hence also

$$d = \sum_{j=1}^{m_1} d_{1j} + u_1$$

and whence all

$$u_i = d - \sum_{j=1}^{m_i} d_{ij}.$$

□

Proof of 3.4. The Riemann-Hurwitz condition reads

$$2(1-g) - (\ell_1 + \ldots + \ell_{n-1} + 2) = d(2-n),$$

namely $\ell_1 + \ldots + \ell_{n-1} = d(n-2) - 2g$. Now we compute

$$\begin{aligned} \sum_{i=2}^{n-1}\sum_{j=1}^{m_i}(d_{ij}-2) &= \sum_{i=1}^{n-1}\sum_{j=1}^{m_i}(d_{ij}-2) - \sum_{j=1}^{m_1}(d_{1j}-2) \\ &= \sum_{i=1}^{n-1}\sum_{j=1}^{m_i}(d_{ij}-2) + 2m_1 - \sum_{j=1}^{m_1} d_{1j}. \end{aligned}$$

We next evaluate another sum in two different ways:

$$\begin{aligned} \sum_{i=1}^{n-1}\sum_{j=1}^{\ell_i}(d_{ij}-2) &= \sum_{i=1}^{n-1}\sum_{j=1}^{m_i}(d_{ij}-2) + \sum_{i=1}^{n-1}\sum_{j=m_i+1}^{\ell_i}(-1) \\ &= \sum_{i=1}^{n-1}\sum_{j=1}^{m_i}(d_{ij}-2) - \sum_{i=1}^{n-1} u_i \\ \sum_{i=1}^{n-1}\sum_{j=1}^{\ell_i}(d_{ij}-2) &= \sum_{i=1}^{n-1}\sum_{j=1}^{\ell_i} d_{ij} - 2\sum_{i=1}^{n-1}\ell_i \\ &= (n-1)d - 2(d(n-2)-2g) \end{aligned}$$

where we have used the Riemann-Hurwitz condition in the last passage. This shows, after easy computations, that

$$\sum_{i=1}^{n-1}\sum_{j=1}^{m_i}(d_{ij}-2) = d(3-n) + 4g + \sum_{i=1}^{n-1} u_i.$$

Substituting this in the first formula above we get

$$\begin{aligned} &\sum_{i=2}^{n-1}\sum_{j=1}^{m_i}(d_{ij}-2) \\ = \; & d(3-n) + 4g + \sum_{i=1}^{n-1} u_i + 2m_1 - \sum_{j=1}^{m_1} d_{1j} \\ = \; & d(3-n) + 4g + \sum_{i=1}^{n-1}(u_i + m_i) - \sum_{i=1}^{n-1} m_i + 2m_1 - (d - u_1) \\ = \; & d(3-n) + 4g + (d(n-2) - 2g) - \sum_{i=1}^{n-1} m_i + 2m_1 - (d - u_1) \end{aligned}$$

where we have used the Riemann-Hurwitz condition again. And now a direct calculation shows that the last expression is

$$u_1 + m_1 - (m_2 + \ldots + m_{n-1}) + 2g$$

as desired. □

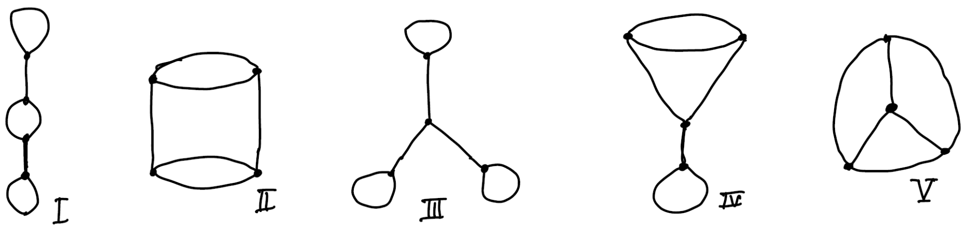

Figure 3.5: Abstract graphs with 4 trivalent vertices.

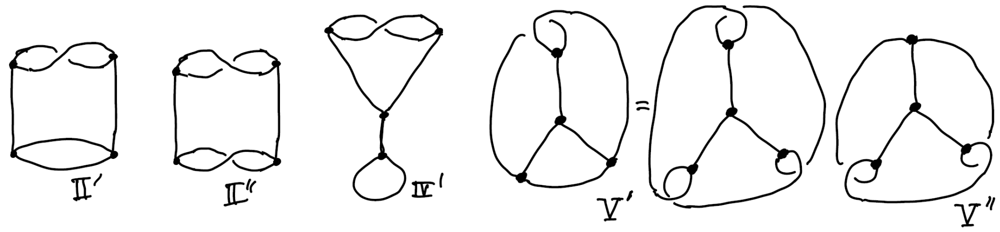

Figure 3.6: Embeddings in the torus.

3.4. Proofs by constellations

As already mentioned, the proof of Theorem 3.1 in [25] is rather complicated and long, so instead of presenting it we will provide other applications of the approach to the Hurwitz existence problem via constellations.

To begin, we give an alternative proof of Proposition 1.10 concerning the exceptionality of $\big((n-3)\cdot T, S, 4, n; [2,2], \ldots, [2,2], [3,1]\big)$. A constellation realizing such a datum would be a graph Γ in the surface of genus $g = n-3$ with only 4 actual vertices of valence $n-1$, edges labeled $1, \ldots, n-1$ (two edges for each label), the labels $1, \ldots, n-1$ positively arranged around each vertex, and two complementary regions incident respectively to $n-1$ and to $3(n-1)$ vertices (with multiplicity).

We prove that no such Γ exists for $n = 4$ (and $g = 1$), leaving the general case to the reader. The abstract graphs with 4 trivalent vertices are those in Fig. 3.5 (*I* and *II* are those in which the only maximal tree is a segment, *III*, *IV* and *V* are those in which there is a maximal tree with a trivalent vertex). Their only relevant embeddings in T are those in Fig. 3.6 (namely, *I* and *III* do not embed in T so that the complement consists of discs, *IV* embeds in one way only, and *II* and *V* embed in two ways —not three, as one might think: the two embeddings *V′* are the same). Now, one sees that the numbers of vertices to which the complementary regions are incident are $[10,2]$ for *II′*, $[6,6]$ for *II″*, $[11,1]$ for *IV′*, $[9,3]$ for *V′* and $[8,4]$ for *V″*. And, finally, Fig. 3.7 proves that for *II″* a choice of the labels as required is possible (but this constellation realizes $\pi_4 = [2,2]$, not $[3,1]$) while it is impossible for *V′*: after an acceptable choice is made for the three bottom vertices, the ordering around the top vertex is not the good one. The argument is complete.

We now turn to Theorem 2.1 according to which any candidate branch datum

$$\big(\widetilde{\Sigma}, S, d, n; \pi_1, \ldots, \pi_n\big)$$

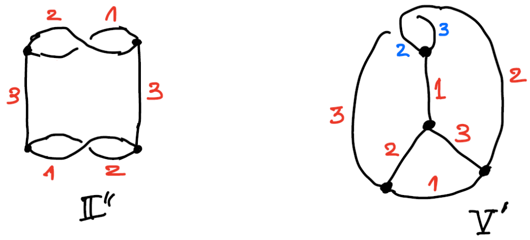

Figure 3.7: Labels for the edges.

is realizable if $\pi_1 = [d]$.

Case 1: $\widetilde{\Sigma} = S,\ n = 3$. In this case we can use dessins d'enfant and proceed by induction on $d \geqslant 3$. For the base step we have the candidate $(S, S, 3, 3; [3], [2, 1], [2, 1])$ which is realized by the only dessin matching $[2, 1], [2, 1]$ (a segment with two internal vertices). For the inductive step, note that the Riemann-Hurwitz condition reads $2 - (1 + \ell_2 + \ell_3) = d(2 - 3)$, namely $\ell_2 + \ell_3 = d + 1$. We claim that up to change of notation π_3 contains a 1. Otherwise we would have $d_{ij} \geqslant 2$ for $i = 2, 3$ and all j, so $\ell_2, \ell_3 \leqslant d/2$, a contradiction. Assuming $d_{21} > 1$ and $d_{31} = 1$ we can inductively realize

$$(S, S, d-1, 3; [d-1], [d_{21}-1, d_{22}, \dots, d_{2\ell_2}], [d_{32}, \dots, d_{3\ell_3}])$$

by a dessin d'enfant matching $[d_{21}-1, d_{22}, \dots, d_{2\ell_2}], [d_{32}, \dots, d_{3\ell_3}]$, and we get the conclusion by attaching to this dessin a segment with a white end to the black vertex corresponding to $d_{21} - 1$.

Case 2: $\widetilde{\Sigma} = S,\ n \geqslant 4$. The argument is very similar using constellations. To simplify the base of the induction we remove here the assumption that each π_i should be different from $[1, \dots, 1]$, so we can start with $d = 1$, all $\pi_i = [1]$ and the identity. For the inductive step, we write the Riemann-Hurwitz condition $\ell_2 + \ldots + \ell_n = (n-2)d + 1$ and claim that up to changing notation $\pi_3, \dots, \pi_n$ contain a 1. Otherwise, π_2, π_3 do not contain a 1, so $\ell_2, \ell_3 \leqslant d/2$, whence

$$(n-2)d + 1 = \ell_2 + \ldots + \ell_n \leqslant d/2 + d/2 + (n-3)d = (n-2)d,$$

a contradiction. So we can assume $d_{21} > 1$ (otherwise we induct on n) and $d_{31} = \ldots = d_{n1} = 1$ and we apply the induction assumption to

$$(S, S, d-1, n; [d-1], [d_{21}-1, d_{22}, \dots, d_{2\ell_2}], [d_{32}, \dots, d_{3\ell_3}], \dots, [d_{n2}, \dots, d_{n\ell_n}])$$

finding a constellation with vertices having labels $2, \dots, n$. We then attach a star to the vertex labeled 2 corresponding to $d_{21} - 1$ (and all other vertices $3, \dots, n$ free) and we are done.

Case 3: $\widetilde{\Sigma} = g \cdot T,\ g \geqslant 1,\ n = 3$. To face this case we define the *extra valence* of the given candidate branch datum as the number

$$e = \sum_{i=2}^{3} \sum_{x \in \overline{\pi}_i} (x - 2)$$

and we claim that $e \geqslant 4g - 2$. In fact we have $2(1-g) - (1 + \ell_2 + \ell_3) = d(2-3)$ by the Riemann-Hurwitz condition, whence $\ell_2 + \ell_3 = d - 2g + 1$. Joining π_2 and π_3 we get a partition η of $2d$ of length $\ell_2 + \ell_3$. Note that $e = \sum_{x \in \overline{\eta}} (x - 2)$. If η contains both a 1 and an entry $x > 2$ we can reduce the value of e without changing the length of η by replacing 1 by 2 and x by $x - 1$. Eventually we get to some η' such that one of the following holds:

- $x \leqslant 2$ for all $x \in \eta'$; then $\ell_2 + \ell_3 \geqslant d$ which contradicts the equality $\ell_2 + \ell_3 = d - 2g + 1$;

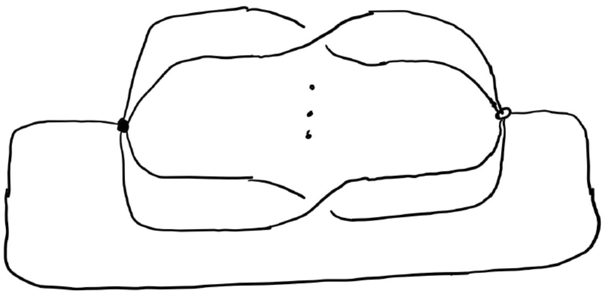

Figure 3.8: A dessin d'enfant with $2g+1$ edges in the surface of genus g. Its complement is a single disc.

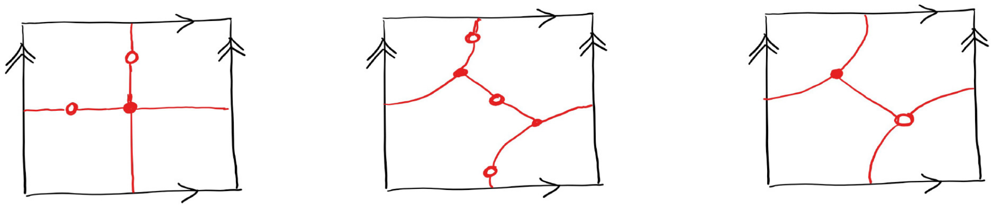

Figure 3.9: Dessins d'enfant on the torus; in each of these graphs k white and k black vertices should be alternately added on any edge with black and white ends.

- $x \geqslant 2$ for all $x \in \eta'$; then

$$e(\eta') = \sum_{x\in\eta}(x-2) = \sum_{x\in\eta} x - 2(\ell_2+\ell_3) = 2d - 2(d-2g+1) = 4g-2.$$

Our claim is proved.

We now note that $d-2g+1 = \ell_2+\ell_3 \geqslant 2$, whence $d \geqslant 2g+1$, and we proceed by induction on d, with fixed $g \geq 1$ and $n=3$, so we use dessins d'enfant. For $d = 2g+1$ the candidate is

$$(g\cdot T, S, 2g+1, 3; [2g+1], [2g+1], [2g+1])$$

and it is realized by the dessin in Fig. 3.8. For $d > 2g+1$ we proceed by induction on $e \geqslant 4g-2$. The base step $e = 4g-2$ is actually here much harder than the inductive step. We first note that the above argument proving that $e \geqslant 4g-2$ implies that for $e = 4g-2$ all the entries of π_2 and π_3 are at least 2. So for instance for $g=1$ we only have for π_1, π_2, π_3 the possibilities

$$\begin{array}{l}
[4+2k], [4,2,\ldots,2], [2,\ldots,2] \\
[6+2k], [3,3,2,\ldots,2], [2,\ldots,2] \\
[3+2k], [3,2,\ldots,2], [3,2,\ldots,2]
\end{array}$$

with $k \geqslant 0$. These partitions are realized by the graphs shown in Fig. 3.9. For general g, we first show that any triple of partitions of the form

$$[d], [4g-p, 2,\ldots,2], [2+p, 2,\ldots,2]$$

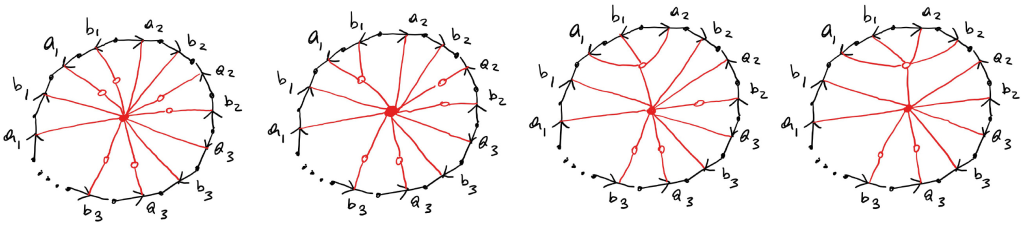

Figure 3.10: Dessins d'enfant on the genus-g surface; again, in each of these graphs k white and k black vertices should be alternately added on any edge with black and white ends.

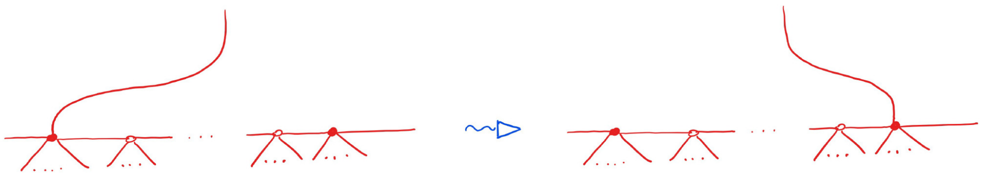

Figure 3.11: Transfer of valence.

can be realized. For $p = 0, 1, 2, 3$ these triples can be described as

$$[4g+2k], [4g, 2, \ldots, 2], [2, \ldots, 2]$$
$$[4g-1+2k], [4g-1, 2, \ldots, 2], [3, 2, \ldots, 2]$$
$$[4g-2+2k], [4g-2, 2, \ldots, 2], [4, 2, \ldots, 2]$$
$$[4g-3+2k], [4g-3, 2, \ldots, 2], [5, 2, \ldots, 2]$$

for $k \geqslant 0$, and Fig. 3.10 shows dessins d'enfant realizing them and suggests how to operate for arbitrary p. To conclude the base of the induction we only need to remark that any triple of partitions with $e = 4g-2$ can be reached from one of the form $[d], [4g-p, 2, \ldots, 2], [2+p, 2, \ldots, 2]$ by the black valence transfer move of Fig. 3.11, and its white analogue. Moving to the inductive step, suppose that $e > 4g-2$. Then $e \geqslant 3$, so the union of π_2 and π_3 contains at least an entry greater than or equal to 3. But the above argument showing the inequality $e \geqslant 4g-2$ implies that the union of π_2 and π_3 also contains at least an entry equal to 1. Now we have two cases: either up to changing notation we have $d_{21} > 2$ and $d_{31} = 1$, or $\pi_2 = [2, \ldots, 2]$. In the first case we realize

$$(g \cdot T, S, d-1, 3; [d-1], [d_{21}-1, d_{22}, \ldots, d_{2\ell_2}], [d_{32}, \ldots, d_{3\ell_3}])$$

by a dessin d'enfant Γ matching the last two partitions and then we attach an edge with a white end to the black vertex of Γ corresponding to $d_{21}-1$, getting a dessin d'enfant realizing the relevant candidate branch datum. In the second case the candidate branch datum to realize is

$$(g \cdot T, S, 2k, 3; [2k], [2, \ldots, 2], \pi_3)$$

for some k, with $\ell_3 = 1 + k - 2g$. Finding a corresponding dessin d'enfant is now an easy exercise: one first deals with the case

$$\pi_3 = [k+2g, 1, \ldots, 1],$$

which is done with an explicit construction similar to that of Fig. 3.8, and then one uses moves similar to that in Fig. 3.11 to do the general case.

To prove Theorem 2.1 in the general case $g \geqslant 1$ and $n \geqslant 4$ one should now combine the above induction argument on the extra valence with the use of constellations. We leave this to the reader.

3.5. Geometric 2-orbifolds

As a last topic, we describe here a geometric approach to the Hurwitz existence problem. To do so we start with some very general notions.

We will call n-orbifold a compact Hausdorff topological space X covered by open charts U with homeomorphisms $\varphi : V/\Gamma \to U$, where V is an open subset of $\mathbb{R}^n$ and Γ is a finite group of self-diffeomorphisms of V. The charts should be compatible in the sense that two of them should intersect in a subchart of both, where a subchart of $\varphi : V/\Gamma \to U$ is $\varphi' : V'/\Gamma' \to U'$ where $V' \subset V$ and $\Gamma' = \{\gamma \in \Gamma : \ \gamma(V') = V'\} = \{\gamma \in \Gamma : \ \gamma(V') \cap V' \neq \emptyset\}$. The formal definition as usual requires the choice of a maximal atlas of compatible charts.

For an orbifold X and $x \in X$ we can define the point group Γ_x as the minimal Γ such that there exists a chart $\varphi : V/\Gamma \to U$ with $x \in U$. From now on we will confine ourselves to *locally orientable* orbifolds, namely such that all the groups Γ as above consist of orientation-preserving diffeomorphisms. And in this case it is not difficult to see that Γ_x can be identified with a subgroup of $\mathrm{SO}(n)$. For $n = 2$ this implies that each Γ_x is some cyclic group C_p generated by the rotation of angle $2\pi/p$ around 0 in $\mathbb{R}^2$, therefore a (locally orientable) 2-orbifold X is topologically a surface Σ, except that at finitely many points $x \in \Sigma$ the point group is C_p with some $p > 1$, and the differentiable structure at x is a singular one. Any such point x will be called a *cone point* of order p, and globally X will be denoted by $\Sigma(p_1, \ldots, p_k)$ if it has cone points of orders $p_1, \ldots, p_k$.

We define the *singular locus* of an orbifold X as the set

$$\mathrm{Sing}(X) = \{x \in X : \ \Gamma_x \neq \{1\}\}.$$

Note that $X \setminus \mathrm{Sing}(X)$ is a manifold. We then define a *Riemannian metric* μ on X as a Riemannian metric on $X \setminus \mathrm{Sing}(X)$ such that for any chart $\varphi : V/\Gamma \to U$ there is a Riemannian metric ν on V where:

- Γ acts isometrically with respect to ν;
- for any $y \in V$ with $x = \varphi(y)$ non-singular, φ is an isometry between $\nu(y)$ and $\mu(x)$.

It is a fact that an orbifold X admits a cell decomposition $\mathcal{C}$ such that for any $c \in \mathcal{C}$ there is a group Γ_c with $\Gamma_x \cong \Gamma_c$ for all $x \in C$. We can now define the *orbifold Euler characteristic* of X as

$$\chi^{\mathrm{orb}}(X) = \sum_{c \in \mathcal{C}} \frac{(-1)^{\dim(c)}}{\#(\Gamma_c)}.$$

For a 2-dimensional $X = \Sigma(p_1, \ldots, p_k)$ we have

$$\begin{aligned}
\chi^{\mathrm{orb}}(X) &= \sum_{c \in \mathcal{C},\ \dim(c)=2} 1 - \sum_{c \in \mathcal{C},\ \dim(c)=1} 1 + \sum_{p \geqslant 1} \sum_{c \in \mathcal{C},\ \dim(c)=0,\ \mathrm{order}(c)=p} \frac{1}{p} \\
&= \left(\sum_{c \in \mathcal{C},\ \dim(c)=2} 1 - \sum_{c \in \mathcal{C},\ \dim(c)=1} 1 + \sum_{c \in \mathcal{C},\ \dim(c)=0} 1 \right) \\
&\quad - \sum_{p \geqslant 2} \sum_{c \in \mathcal{C},\ \dim(c)=0,\ \mathrm{order}(c)=p} \left(1 - \frac{1}{p}\right) \\
&= \chi(\Sigma) - \sum_{j=1}^{k} \left(1 - \frac{1}{p_j}\right).
\end{aligned}$$

We can now prove an orbifold version of the Gauss-Bonnet theorem. To this end note that the curvature κ and the area form $\mathcal{A}$ of a Riemannian metric on a 2-orbifold X are defined outside a finite set, so the integral of κ over X with respect to $\mathcal{A}$ is well-defined:

Theorem 3.6. $\int\limits_X \kappa\, d\mathcal{A} = 2\pi \cdot \chi^{\mathrm{orb}}(X)$.

Proof. The infinitesimal version of the Gauss-Bonnet theorem says that for a geodesic triangle T with inner angles α, β, γ one has

$$\int_T \kappa\, \mathrm{d}\mathcal{A} = \alpha + \beta + \gamma - \pi.$$

We can now take a triangulation $\mathcal{T}$ of X such that the set $\mathcal{T}^{(0)}$ of the vertices of $\mathcal{T}$ contains $\mathrm{Sing}(X)$, and $\mathcal{T}^{(1)}$, the set of the edges of $\mathcal{T}$, consists of geodesic segments. Note that for $v \in \mathcal{T}^{(0)}$ the sum of the angles at v of the triangles in $\mathcal{T}^{(2)}$ containing v is $2\pi/p$ if v has order p (in particular, it is 2π for non-singular v). We then have

$$\begin{aligned}
\int_X \kappa\, \mathrm{d}\mathcal{A} &= \sum_{T \in \mathcal{T}^{(2)}} \int_T \kappa\, \mathrm{d}\mathcal{A} \\
&= -\pi \cdot \sum_{T \in \mathcal{T}^{(2)}} 1 + \sum_{v \in \mathcal{T}^{(0)}} \frac{2\pi}{\mathrm{order}(v)} \\
&= 2\pi \cdot \left(\sum_{T \in \mathcal{T}^{(2)}} 1 - \frac{3}{2} \sum_{T \in \mathcal{T}^{(2)}} 1 + \sum_{v \in \mathcal{T}^{(0)}} \frac{1}{\mathrm{order}(v)} \right) \\
&= 2\pi \cdot \chi^{\mathrm{orb}}(X)
\end{aligned}$$

because $3\#(\mathcal{T}^{(2)}) = 2\#(\mathcal{T}^{(1)})$. □

If $\mathbb{X}^2$ is one of the constant curvature model spaces $\mathbb{S}^2$, $\mathbb{E}^2$ or $\mathbb{H}^2$ we will say that a 2-orbifold X has a *geometric structure* of, respectively, *sperical*, *Euclidean* or *hyperbolic type* if it is endowed with a Riemannian metric locally modeled on the quotient of a disc in $\mathbb{X}^2$ under the action of an isometric rotation of order $2\pi/p$ around its centre. Theorem 3.6 implies that X can be spherical only if $\chi^{\mathrm{orb}}(X) > 0$, it can be elliptic only if $\chi^{\mathrm{orb}}(X) = 0$, and it can be hyperbolic only if $\chi^{\mathrm{orb}}(X) < 0$. Moreover the following is easily established:

Proposition 3.7. *Let X be a 2-orbifold with an underlying surface which is compact, connected, orientable and without boundary.*

- $\chi^{\mathrm{orb}}(X) > 0$ *if and only if X is one of the following:*
$$S \quad S(p) \quad S(p,q) \quad S(2,2,p) \quad S(2,3,3) \quad S(2,3,4) \quad S(2,3,5);$$
- $\chi^{\mathrm{orb}}(X) = 0$ *if and only if X is one of the following:*
$$T \quad S(2,4,4) \quad S(2,3,6) \quad S(3,3,3) \quad S(2,2,2,2).$$

3.6. Orbifold covers

An orbifold cover $f : \widetilde{X} \to X$ is a map such that each $x \in X$ has an open neighbourhood U with $f^{-1}(U)$ a disjoint union of open sets $\widetilde{U}$ for which there exist charts $\widetilde{\varphi} : V/\widetilde{\Gamma} \to \widetilde{U}$ and $\varphi : V/\Gamma \to U$ where $\widetilde{\Gamma} < \Gamma$ and $f \circ \widetilde{\varphi} = \varphi \circ \pi$, for $\pi : V/\widetilde{\Gamma} \to V/\Gamma$ the natural projection. For locally orientable 2-orbifolds, an orbifold cover is simply a map locally modeled on the natural projection $\Delta/C_{\widetilde{p}} \to \Delta/C_p$, where Δ is the unit disc in $\mathbb{C}$ and $\widetilde{p}$ is a divisor of p.

In any dimension the following generalizations of what is known for ordinary covers hold:

- The orbifold Euler characteristic is multiplicative under orbifold covers;

- Every orbifold X has an *orbifold universal cover* $\pi : Y \to X$, namely one such that for any orbifold cover $f : \widetilde{X} \to X$ there exixts an orbifold cover $g : Y \to \widetilde{X}$ with $\pi = f \circ g$.

We will say that an orbifold X is *good* if it is orbifold covered by manifold (or, equivalently, if its orbifold universal cover is a manifold), and *bad* if it is not good. The following is due to Thurston:

Theorem 3.8. *Let X be a 2-orbifold with an underlying surface which is compact, connected, orientable and without boundary. Then X is bad if and only if it is $S(p)$ with $p > 1$ or $S(p,q)$ with $p > q > 1$. If X is good then it is geometric, and more precisely it can be realized globally as a quotient $\mathbb{X}^2/\Gamma$, with Γ a discrete group of isometries of some $\mathbb{X}^2$.*

Note that by Theorem 3.6 the type of geometry of X is dictated by its orbifold Euler characteristic.

3.7. Candidate orbifold covers and the spherical case

We can now spell out the connection of the theory of 2-orbifolds with the Hurwitz existence problem. For a candidate branch datum

$$\left(\widetilde{\Sigma}, \Sigma, d, n; \pi_1, \ldots, \pi_n\right)$$

we define p_i as the least common multiple of the entries of π_i and $q_{ij} = p_i/d_{ij}$. Then one sees quite easily that the candidate is realizable if and only if there exists an orbifold cover $\widetilde{\Sigma}(\{q_{ij}\}) \xrightarrow{d:1} \Sigma(\{p_i\})$ with each cone point of order q_{ij} mapped to the cone point of order p_i. Note that the original candidate branch datum is determined by $\widetilde{\Sigma}(\{q_{ij}\})$, $\Sigma(\{p_i\})$, d and the instructions on which q_{ij} should be mapped to which p_i.

We now introduce the symbol $\widetilde{\Sigma}(\{q_{ij}\}) \overset{d:1}{\dashrightarrow} \Sigma(\{p_i\})$, termed *candidate orbifold cover*, to denote a possibly non-existent orbifold cover as required. Of course we restrict our attention to candidate orbifold covers $\widetilde{X} \overset{d:1}{\dashrightarrow} X$ with X having base surface S and $\widetilde{\Sigma}$ having orientable base surface, and satisfying $\chi^{\rm orb}\left(\widetilde{X}\right) = d \cdot \chi^{\rm orb}(X)$, which coincides with the Riemann-Hurwitz condition for the associated candidate branch datum.

In [27, 28] we have analyzed the realizability of candidate orbifold covers $\widetilde{X} \overset{d:1}{\dashrightarrow} X$ using the geometry of X and $\widetilde{X}$, and we have deduced the realizability or exceptionality of several families of candidate branch data. Note that X and $\widetilde{X}$ have concordant orbifold Euler characteristics, whence almost always the same geometry, except if they both have positive orbifold Euler characteristics and one of them is bad (or both).

For the case of positive orbifold Euler characteristic the following proves crucial:

Proposition 3.9. *A candidate orbifold cover $\widetilde{X} \overset{d:1}{\dashrightarrow} X$ with good X and bad $\widetilde{X}$ is exceptional.*

Proof. Since X is good, the orbifold universal cover of X is a manifold Y. If the candidate is realized by some map $\widetilde{X} \to X$ then Y covers $\widetilde{X}$ as well, but $\widetilde{X}$ is bad. □

The following is shown in [27]:

Theorem 3.10. *A candidate orbifold cover $\widetilde{X} \overset{d:1}{\dashrightarrow} X$ with positive $\chi^{\rm orb}$ is realizable unless X is good and $\widetilde{X}$ is bad.*

The argument underlying this result follows these steps:

- Enumeration of all the possible candidate orbifold covers $\widetilde{X} \overset{d:1}{\dashrightarrow} X$ with $\chi^{\rm orb} > 0$;
- Verification that X is never bad for any of them;
- For good $\widetilde{X}$, description of the spherical structures $\widetilde{X} = \mathbb{S}^2/\widetilde{\Gamma}$ and $X = \mathbb{S}^2/\Gamma$, and

- Verification that $\widetilde{\Gamma}$ can be realised as a subgroup of Γ.

The exceptional candidate branch data corresponding to the exceptional candidate orbifold covers of the previous statement are as follows:

- The infinite series
$$(S, S, 2k, 3; [2, \ldots, 2], [2, \ldots, 2], [a, b])$$
for $a \neq b$, for which the associated candidate orbifold covers is one of the following:
$$S(p) \overset{2k:1}{\dashrightarrow} S(2, 2, r) \quad p > 1 \qquad S(p, q) \overset{2k:1}{\dashrightarrow} S(2, 2, r) \quad p > q > 1;$$
note that the exceptionality of the candidate branch data described is easily proved using dessins d'enfant;

- 11 sporadic cases, among which for instance
$$(S, S, 16, 3; [2, \ldots, 2], [3, \ldots, 3, 1], [5, 5, 5, 1])$$
with associated $S(3, 5) \overset{16:1}{\dashrightarrow} S(2, 3, 5)$, and
$$(S, S, 45, 3; [2, \ldots, 2, 1], [3, \ldots, 3], [5, \ldots, 5])$$
with associated $S(2) \overset{45:1}{\dashrightarrow} S(2, 3, 5)$.

3.8. The Euclidean case

The analysis of the candidate orbifold covers with $\chi^{\mathrm{orb}} = 0$ has led to the most interesting results, and we explain here the ideas it is based on.

Proposition 3.11. *If $f : \widetilde{X} \xrightarrow{d:1} X$ is an orbifold cover with*
$$\chi^{\mathrm{orb}}(X) = \chi^{\mathrm{orb}}(\widetilde{X}) = 0$$
then there exist discrete groups $\Gamma, \widetilde{\Gamma}$ of isometries of $\mathbb{E}^2$ such that:

- *X can be identified to the quotient $\mathbb{E}^2/\Gamma$ with projection $c : \mathbb{E}^2 \to X$;*
- *$\widetilde{X}$ can be identified to the quotient $\mathbb{E}^2/\widetilde{\Gamma}$ with projection $\widetilde{c} : \mathbb{E}^2 \to \widetilde{X}$;*
- *X and $\widetilde{X}$ have the same area with respect to the Euclidean structures thus defined;*
- *There exists an affine map $\widetilde{f}(z) = \lambda \cdot z + \mu$ from $\mathbb{E}^2$ (viewed as $\mathbb{C}$) to itself such that $c \circ \widetilde{f} = f \circ \widetilde{c}$ and $d = |\lambda|^2$.*

Proof. Choose any Euclidean structure on X given by a group Γ of isometries and a projection $c : \mathbb{E}^2 \to X$. Pull-back this structure to $\widetilde{X}$ via f, getting some isometry group $\overline{\Gamma}$ and projection $\overline{c} : \mathbb{E}^2 \to \widetilde{X}$. Then there exists $\overline{f} : \mathbb{E}^2 \to \mathbb{E}^2$ such that $c \circ \overline{f} = f \circ \overline{c}$. Now $\overline{f}$ is an orbifold cover, so it is a genuine cover and actually a homeomorphism. In addition it is a local isometry, whence a global isometry and hence an affine map. Now the area of $\widetilde{X}$ with respect to $\overline{c}$ is d times the area of X with respect to c. Therefore we obtain equal area by rescaling the metric on $\widetilde{X}$ by a factor $1/\sqrt{d}$. If $\widetilde{c}$ is the corresponding projection we still have that there exists an affine $\widetilde{f}(z) = \lambda \cdot z + \mu$ with $c \circ \widetilde{f} = f \circ \widetilde{c}$, but now $\widetilde{f}$ is $\sqrt{d}$ times an isometry, whence $|\lambda|^2 = d$. □

Corollary 3.12. *With notation as in the previous proposition, Γ and $\widetilde{\Gamma}$ have maximal sublattices Λ and $\widetilde{\Lambda}$, and $\lambda \cdot \widetilde{\Lambda} \subset \Lambda$.*

To analyze the realizability of all $\widetilde{X} \overset{d:1}{\dashrightarrow} X$ with $\chi^{\rm orb} = 0$, the first step is to list all the possibilities for the cone points $\{q_{ij}\}$ and $\{p_i\}$ with q_{ij} a divisor of p_i. Excluding the case $\widetilde{X} = T$, which is always realizable, there are 7 cases for $\widetilde{X}$ and X, and for each of them several possibilities for which q_{ij} should be mapped to which p_i. For instance for $\widetilde{X} = X = S(2,4,4)$ we have the following list:

1. $2 \mapsto 2,\ 4' \mapsto 4',\ 4'' \mapsto 4''$ with associated candidate branch datum $(S, S, 4k+1, 3; [2\ldots,2,1], [4,\ldots,4,1], [4,\ldots,4,1])$;
2. $2 \mapsto 2,\ 4', 4'' \mapsto 4'$ with *no* associated candidate branch datum;
3. $2 \mapsto 4',\ 4', 4'' \mapsto 4''$ with associated candidate branch datum $(S, S, 4k+2, 3; [2\ldots,2], [4,\ldots,4,2], [4,\ldots,4,1,1])$;
4. $2, 4' \mapsto 4',\ 4'' \mapsto 4''$ with *no* associated candidate branch datum;
5. $2, 4', 4'' \mapsto 4'$ with associated candidate branch datum $(S, S, 4k+4, 3; [2\ldots,2], [4,\ldots,4], [4,\ldots,4,2,1,1])$.

For the first case, the result we get is the following:

Theorem 3.13. $(S, S, d, 3; [2\ldots,2,1], [4,\ldots,4,1], [4,\ldots,4,1])$ *is realizable if and only if* $d = x^2 + y^2$ *with* $x, y \in \mathbb{Z}$ *of different parity.*

Proof. Let f realize the corresponding candidate orbifold cover and take $\Gamma, \widetilde{\Gamma}, \widetilde{f}$ as in Proposition 3.11. Since the structure of $S(2,4,4)$ is unique up to scaling we can assume $\widetilde{\Gamma} = \Gamma$ is the orientation-preserving subgroup of the group generated by the reflections in the sides of the triangle with vertices $0, 1, i$ (so that X and $\widetilde{X}$ have area 1). Then $\widetilde{\Lambda} = \Lambda = 2\mathbb{Z} \oplus 2i\mathbb{Z}$, so $2\lambda = 2x + 2iy$ for $x, y \in \mathbb{Z}$, and $d = |\lambda|^2 = x^2 + y^2$. Since d is odd, x and y have different parity. Conversely, if $d = x^2 + y^2$ we can define $\widetilde{f}(z) = (x + iy) \cdot z$ and we get f. □

A remarkable fact about Theorem 3.13 is that the set of odd integers of the form $x^2 + y^2$ has asymptotic zero density, namely

$$\lim_{M\to\infty} \frac{1}{M} \# \left\{ d \leqslant M :\ d \text{ odd } d = x^2 + y^2,\ x, y \in \mathbb{Z} \right\} = 0.$$

However an old theorem of Fermat says that an odd prime is always the sum of two squares. So the candidate branch datum

$$(S, S, d, 3; [2\ldots,2,1], [4,\ldots,4,1], [4,\ldots,4,1])$$

is exceptional with probability 1 but realizable when the degree is a prime.

We would also like to mention that a connection of Theorem 3.13 to the theory of elliptic curves and universal ramified covers with signature was developed in [6].

An argument similar to that proving Theorem 3.13 shows that

$$(S, S, 4k+2, 3; [2\ldots,2], [4,\ldots,4,2], [4,\ldots,4,1,1])$$

is realizable precisely if $d = 2(x^2 + y^2)$, and

$$(S, S, 4k+4, 3; [2\ldots,2], [4,\ldots,4], [4,\ldots,4,2,1,1])$$

is realizable precisely if $d = 4(x^2 + y^2)$. In fact, we still have $d = n^2 + m^2$ for $n, m \in \mathbb{Z}$, and the extra information about which cone points are mapped to which gives conditions on the parity of n and m that lead to the conclusion.

In [27] we have carried out a complete analysis of the candidate branch data with associated Euclidean candidate orbifold cover, getting various realizability results, often in terms of integer quadratic forms. This is another sample of our achievements:

Theorem 3.14. *The candidate branch data*

$$(S, S, d, 3; [2 \ldots, 2, 1], [3, \ldots, 3, 1], [6, \ldots, 6, 1])$$
$$(S, S, d, 3; [3 \ldots, 3, 1], [3, \ldots, 3, 1], [3, \ldots, 3, 1])$$

are realizable if and only if $d = x^2 + xy + y^2$ *with* $x, y \in \mathbb{Z}$*.*

A comment similar to that made for Theorem 3.13 applies here: the integers of the form $x^2 + xy + y^2$ have asymptotic zero density, but a prime of the form $3k + 1$ (or equivalently $6k + 1$) can always be written as $x^2 + xy + y^2$ by a result of Gauss.

Before concluding with the Euclidean case, we would like to mention that the case where one of the involved orbifolds is $S(2, 2, 2, 2)$ is rather more complicated, because $S(2, 2, 2, 2)$ does *not* have a unique geometric structure up to scaling.

3.9. The prime-degree conjecture

Theorems 3.13 and 3.14 and the comments accompanying them give a strong supporting evidence to the conjecture made in [8] that any candidate branch datum with a prime degree is realizable. We mention here that more recently, Zieve [39] conjectured that a candidate branch datum

$$\left(\widetilde{\Sigma}, S, d, n; \pi_1, \ldots, \pi_n\right)$$

is realizable provided that

- $\mathrm{GCD}(\pi_j) = 1$ for $j = 1, \ldots, n$, and

- $\sum_{j=1}^{n} \left(1 - \frac{1}{\mathrm{lcm}(\pi_j)}\right) \neq 2$.

As one easily sees, the candidate branch data with $\sum_{j=1}^{n} \left(1 - \frac{1}{\mathrm{lcm}(\pi_j)}\right) = 2$ are precisely those whose associated candidate orbifold cover is of Euclidean type. The results in [27], including those stated above, show that indeed some of these data are exceptional (even with $\mathrm{GCD}(\pi_j) = 1$ for $j = 1, \ldots, n$ in some cases). So an equivalent way of expressing Zieve's conjecture is to say that a branch datum is realizable if $\mathrm{GCD}(\pi_j) = 1$ for $j = 1, \ldots, n$ and the datum is not one of the exceptional ones found in [27]. This would imply the prime-degree conjecture, because:

- If one of the π_i's reduces to $[d]$ only then the branch datum is realizable by [8];

- All the exceptional data of [27] occur when the degree is composite.

3.10. The hyperbolic case

We conclude by mentioning more results from [27] and from [28], where the realizability of certain candidate orbifold covers $\widetilde{X} \overset{d:1}{\dashrightarrow} S(p_1, p_2, p_3)$ of hyperbolic type was analyzed. To describe the choice we have made of what candidates to study, we recall that the space of hyperbolic structures on a surface of genus g with k cone points is an analytic space of complex dimension $3(g-1)+k$, so in particular any triangular hyperbolic $S(p_1, p_2, p_3)$ is rigid, but in all other cases there are continuous deformations. We have then concentrated on the cases where $\widetilde{X}$ has deformation space of dimension at most 1, getting the results summarized in the next table. For each possible type of $\widetilde{X}$ we indicate the number of candidate branch data for which there exists a corresponding $\widetilde{X} \overset{d:1}{\dashrightarrow} S(p_1, p_2, p_3)$, and among these the number of exceptional ones:

$\widetilde{X}$	candidates	exceptions
$S(q_1, q_2, q_3)$	11	2
$S(q_1, q_2, q_3, q_4)$	146	29
$T(q)$	22	5

Remark 3.15. The fact that for a given $\widetilde{\Sigma}$ and k there exist only finitely many candidate branch data inducing some $\widetilde{\Sigma}(q_1, \ldots, q_k) \to S(p_1, p_2, p_3)$ is true but not completely obvious.

Remark 3.16. As opposed to what happened for the spherical and the Euclidean case, the hyperbolic candidates analyzed in [27, 28] were selected using geometry, but their realizability or exceptionality was mostly discussed using combinatorial tools.

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