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Continuity of spectrum and spectral radius in algebras of operators

LAURA BURLANDO[(1)]

RÉSUMÉ. — Dans cet article nous étudions la continuité spectrale dans l'algèbre des opérateurs linéaires et continus sur un espace de Banach. Plusieurs nouvelles conditions suffisantes (la plupart d'elles sont aussi nécessaires au moins dans le cas particulier d'un espace de Hilbert séparable) pour la continuité des fonctions spectre et rayon spectrale sont étudiées et comparées à celles déjà connues. Des exemples sont donnés aussi pour étudier les liaisons parmi toutes ces conditions. Deux sous-ensembles (qui au moins dans le cas d'un espace de Hilbert séparable ne sont pas propres) de l'ensemble des points de continuité ds fonctions spectre et rayon spectrale sont définis. Nous étudions les propriétés algébriques et topologiques de ces ensembles.

ABSTRACT. — This paper deals with spectral continuity in the algebra of linear and continuous operators on a Banach space. Several new sufficient conditions (most of which are also necessary, at least in the particular case of a separable Hilbert space) for the continuity of the spectrum and spectral radius functions are studied and compared with the already known ones. Examples are given to specify the connections between these conditions. Two subsets (which, at least in the case of a separable Hilbert space, are not proper) of the sets of the continuity points of the spectrum and spectral radius functions are introduced, by means of two of the conditions above, and algebraic and topological properties of theirs are studied.

Introduction

The continuity points of the spectrum and spectral radius functions in the algebra of linear and continuous operators on a separable Hilbert space have been recently characterized by CONWAY and MORREL ([CM]).

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A different characterization of the continuity points of spectrum has been given afterwards by the authors of [AFHV].

Both the conditions given by [CM] and [AFHV] for the continuity of spectrum with respect to the Hausdorff metric at the operator A require that the union of the union of all trivial components of a convenient subset of the spectrum and the set $\rho^{\pm}_{s-F}(A)$ of the semi-Fredholm points of A with nonzero index is dense in the spectrum of A. The two characterizations differ in the subset of the spectrum whose trivial components are considered (see [CM], 3.1 and [AFHV], Th. 14. 15).

Both the conditions given by [CM] for the continuity of the spectral radius function at the operator A require that the maximum of the supremum $\beta(A)$ of the modulus function on $\rho^{\pm}_{s-F}(A)$ and $\sup\{\inf\{|\lambda|: \lambda \in \omega\}: \omega$ is a component of $\tau\}$ (where τ is a convenient subset of the spectrum) coincides with the spectral radius of A. The two characterizations differ in the subset of the spectrum whose components are considered (see [CM], 2.5 and 2.6).

These four conditions, which characterize the continuity points of spectrum and spectral radius in the case of a separable Hilbert space, are at least sufficient for the continuity of the two spectral functions for any Banach space (see [CM], §4, [AFHV], page 277, and Corollary 1. 15 and Corollary 2.13 of this paper). The authors of [AFHV] suspect that they are also necessary for any Banach space (see [AFHV], page 313).

In this paper I give new conditions which ensure the continuity of the spectrum and spectral radius functions in the algebra of all linear and continuous operators on a Banach space and I investigate systematically their reciprocal connnections and their relationships with the conditions of [CM] and [AFHV].

In Section 1 I introduce a subset $\sigma_m(A)$ of the spectrum $\sigma(A)$ of an operator A whose union with the set $\sigma_p^0(A)$ of all normal eigenvalues of A is contained in the subset defined by [CM] and coincides with it in the particular case of a Hilbert space (see Definition 1.2). I study the connections between the unions of all trivial components of seven different subsets of $\sigma(A)$ (among which $\sigma(A)\backslash\overline{\rho^{\pm}_{s-F}(A)}$, $\sigma_m(A) \cup \sigma_p^0(A)$ and the two sets introduced in [CM] and [AFHV]) and prove that there are some relationships of inclusion and three chains can be constructed (see Theorem 1.6). In particular, if $\Gamma_1(A)$ denotes the union of all trivial components of $\sigma(A)\backslash\overline{\rho^{\pm}_{s-F}(A)}$, $\Gamma_0(A)$ denotes the union of all trivial components of the set introduced in [AFHV], $\Gamma_2(A)$ denotes the union of all trivial components of the set introduced in [CM] and $\Gamma_4(A)$ denotes the union of all trivial components of $\sigma_m(A) \cup \sigma_p^0(A)$, $\Gamma_1(A) \subset \Gamma_0(A) \subset \Gamma_2(A) \subset \Gamma_4(A)$ and all the other three unions of trivial components contain $\Gamma_1(A)$ and are contained in $\Gamma_4(A)$. By means of several examples, I show that none of the inclusions enunciated in Theorem 1.6 can be inverted. Moreover, I prove that, for any of the seven subsets above of $\sigma(A)$, the union of the union of its trivial components and $\overline{\rho_{s-F}^{\pm}(A)}$ coincides with $\Gamma_1(A) \cup \overline{\rho_{s-F}^{\pm}(A)}$ or with $\Gamma_4(A) \cup \overline{\rho_{s-F}^{\pm}(A)}$ and, if $\Gamma_4(A) \cup \overline{\rho_{s-F}^{\pm}(A)} = \sigma(A)$, also $\Gamma_1(A) \cup \overline{\rho_{s-F}^{\pm}(A)}$ and $\Gamma_4(A) \cup \overline{\rho_{s-F}^{\pm}(A)}$ coincide (see Theorem 1.13). Hence I have obtained five new sufficient conditions for the continuity of spectrum at the point A, which are equivalent to the two already known ones and therefore are also necessary, at least in the case of a separable Hilbert space (see Corollary 1.15).

In Section 2 I study the connections between the suprema of the sets of the infima of the modulus function on the components of the seven subsets above of $\sigma(A)$. Several inequalities can be established and three chains can be constructed (see Theorem 2.2). In particular, if $\delta_5(A)$ and $\delta_2(A)$ denote the suprema introduced in [CM] (where $\delta_5(A)$ coincides with the supremum $\delta(A)$ defined in [CM]) and $\delta_2(A)$ coincides with the supremum $\delta_0(A)$ defined in [CM]) and if we put $\delta_4(A) = \sup\{\inf\{|\lambda|: \lambda \in \omega\} : \omega$ is a component of $\sigma_m(A) \cup \sigma_p^0(A)\}$, $\delta_5(A) \leq \delta_2(A) \leq \delta_4(A)$. By means of several examples, I prove that none of the inequalities enunciated in Theorem 2.2 can be inverted. Moreover, I prove that the maxima of $\beta(A)$ and any of the seven suprema above can assume at most two values, the maximum of which is $\delta_2(A) \vee \beta(A) = \delta_4(A) \vee \beta(A) = \delta_5(A) \vee \beta(A)$ (see Theorem 2.12). In this way, I have obtained five new sufficient conditions for the continuity of the spectral radius function at the point A , two of which are equivalent to the two already known ones and therefore are also necessary, at least in the case of a separable Hilbert space (see Corollary 2.13). An example proves that the remaining three conditions are not necessary for the continuity of spectral radius, even in a separable Hilbert space (see Example 2.14).

In Section 3 I introduce a subset $\sum_0(X)$ (defined by the equivalent conditions of Corollary 1.15) of the continuity points of spectrum and a subset $R_0(X)$ (defined by the equivalent conditions iv), v), vi) and vii) of Corollary 2.13) of the continuity points of spectral radius in the algebra of all linear and continuous operators on a Banach space X, with $\sum_0(X) \subset R_0(X)$. Both subsets are not proper, at least in the case of a Hilbert separable space. I study algebraic and topological properties of $\sum_0(X)$ and $R_0(X)$ and compare them with another sets, a subset $\tau(X)$ of $\sum_0(X)$ and a subset $\pi(X)$ of $R_0(X)$, that I have already defined in a previous paper ([B]) by means of topological conditions on the spectrum.

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We shall denote the norm by the symbol $\| \ \|$ in any normed space.

If X is a complex nonzero Banach space, for any $x \in X$ and for any $\varepsilon > O$ let $B_X(x, \varepsilon)$ denote the set of all points of X whose distance from x is smaller than ε. We shall denote by $L_c(X)$ the algebra of all linear and continuous operators on X, by $L_{cc}(X)$ the ideal of compact operators of $L_c(X)$ and by I_X the identity of $L_c(X)$.

By a continuous projection on X we shall mean an operator $P \in L_c(X)$ such that $P^2 = P$. Obviously, $I_X - P$ is a projetion too, Im $P = \ker(I_X - P)$ (we shall always use the symbol Im to denote the range of a function), so that Im P is closed, and $X =$ Im $P \oplus \ker P$ (where the symbol $\oplus$ means algebraic direct sum). For any $A \in L_c(X)$, we shall denote the spectrum of A by $\sigma(A)$, the spectral radius of A by $r(A)$ and $\mathbf{C} \backslash \sigma(A)$ (where the symbol $\mathbf{C}$ denotes the complex plane) by $\rho(A)$. We recall that: $\sigma(A)$ is compact and nonempty, the resolvent function:

$$\lambda \in \rho(A) \longrightarrow R(\lambda, A) = (\lambda I_X - A)^{-1} \in L_c(X)$$

is analytic, $r(A) = \lim_{n\to\infty} \|A^n\|^{1/n}$, the peripheral spectrum of A is the set of all points of $\sigma(A)$ whose modulus is equal to $r(A)$ and a spectral set of A is a subset of its spectrum that is both open and closed in the relative topology of $\sigma(A)$.

Let $\rho_{s-F}(A)$ denote the semi-Fredholm domain of A, that is the set of all points $\lambda \in \mathbf{C}$ such that $\lambda I_X - A$ is a semi-Fredholm operator (see [Ka], page 230). For any $n \in \mathbf{Z} \cup \{-\infty, +\infty\}$, let $\rho^n_{s-F}(A)$ denote the set of all points $\lambda \in \rho_{s-F}(A)$ such that ind $(\lambda I_X - A) = n$ (where, for any semi-Fredholm operator $T \in L_c(X)$, ind T denotes the semi-Fredholm index of T, see [Ka], IV, (5.1)). From the stability of semi- Fredholm index (see [Ka], IV, 5.17) it follows that $\rho^n_{s-F}(A)$ is open for any $n \in \mathbf{Z} \cup \{-\infty, +\infty\}$. Consequently, if we put

$$\rho^{\pm}_{s-F}(A) = \{\lambda \in \rho_{s-F}(A) : \text{ ind } (\lambda I_X - A) \neq 0\},$$

also $\rho^{\pm}_{s-F}(A)$ and $\rho_{s-F}(A)$ are open subsets of $\mathbf{C}$. It is immediate to remark that $\rho(A) \subset \rho^0_{s-F}(A)$ and, consequently, $\rho^{\pm}_{s-F}(A) \subset \sigma(A)$ (so that, as $\sigma(A)$ is closed, also $\overline{\rho^{\pm}_{s-F}(A)} \subset \sigma(A)$).

We shall denote by $\sigma^0_p(A)$ the set of all normal eigenvalues of A, that is the set of all isolated points of $\sigma(A)$ for which the corresponding spectral projection (see [TL], page 321) has finite-dimentional range. From [Ka], IV, 5.28 it follows that $\sigma^0_p(A) \subset \rho_{s-F}(A)$.

We recall that, for any $\lambda \in \rho_{s-F}(A)$, there exists a neighborhood U of λ, contained in $\rho_{s-F}(A)$, such that dim $\ker(\mu I_X - A)$ and dim $X/\mathrm{Im}\ (\mu I_X - A)$ are constant for $\mu \in U\backslash\{\lambda\}$ (see [Ka], IV, 5.31). Therefore, since $\rho^n_{s-F}(A)$ is open for any $n \in \mathbf{Z} \cup \{-\infty, +\infty\}$, $\sigma^0_p(A) \subset \rho^0_{s-F}(A)$.

We also recall that any isolated point of $\sigma(A)$ which belongs to $\rho_{s-F}(A)$ belongs to $\sigma^0_p(A)$, too (see [Ka], IV, 5.28 and 5.10). Hence $\sigma(A) \cap \rho^0_{s-F}(A) = \sigma^0_p(A) \cup (\overset{\circ}{\sigma}(A) \cap \rho^0_{s-F}(A))$.

We shall denote $\mathbf{C}\backslash\rho_{s-F}(A)$ with $\sigma_{s-F}(A)$. Since $\rho_{s-F}(A)$ is open and $\rho_{s-F}(A) \supset \rho(A)$, it follows that $\sigma_{s-F}(A)$ is closed and $\sigma_{s-F}(A) \subset \sigma(A)$. Since $\rho_{s-F}(A) \cap \partial\sigma(A) \subset \rho^0_{s-F}(A)$ by [Ka], IV, 5.31, it follows that $\partial\sigma(A) \subset \sigma_{s-F}(A) \cup \sigma^0_p(A)$. Hence $\sigma_{s-F}(A) \cup \sigma^0_p(A)$ is nonempty.

If $Q : L_c(X) \longrightarrow L_c(X)/L_{cc}(X)$ is the canonical map, we shall denote by $\sigma_e(A)$ the essential spectrum of A (that is the spectrum of $Q(A)$ in the Calkin algebra $L_c(X)/L_{cc}(X)$), by $\sigma_{\mathrm{le}}(A)$ the left essential spectrum of A (that is the left spectrum of $Q(A)$ in $L_c(X)/L_{cc}(X)$ and by $\sigma_{\mathrm{re}}(A)$ the right essential spectrum of A (that is the right spectrum of $Q(A)$ in $L_c(X)/L_{cc}(X)$). Obviously, $\sigma_e(A)$, $\sigma_{\mathrm{le}}(A)$ and $\sigma_{\mathrm{re}}(A)$ are compact subsets of $\mathbf{C}$ and $\sigma_e(A) = \sigma_{\mathrm{le}}(A) \cup \sigma_{\mathrm{re}}(A)$. We recall that $\mathbf{C}\backslash\sigma_e(A)$ coincides with the Fredholm domain of A, that is $\cup_{n\in\mathbf{Z}}\rho^n_{s-F}(A)$ (see [CPY], (3.2.8)). Hence $\sigma_e(A) = \sigma_{s-F}(A) \cup \rho^{-\infty}_{s-F}(A) \cup \rho^{+\infty}_{s-F}(A)$. Since $\rho^{-\infty}_{s-F}(A)$ and $\rho^{+\infty}_{s-F}(A)$ are open subsets of $\mathbf{C}$, it follows that $\partial\sigma_e(A) \subset \sigma_{s-F}(A)$. From [CPY], (4.3.4) it follows that $(\mathbf{C}\backslash\sigma_{\mathrm{le}}(A)) \cup (\mathbf{C}\backslash\sigma_{\mathrm{re}}(A)) \subset \rho_{s-F}(A)$. Hence $\sigma_{s-F}(A) \subset \sigma_{\mathrm{le}}(A) \cap \sigma_{\mathrm{re}}(A)$. The opposite inclusion is not always satisfied in a generic Banach space (see for instance, in this paper, Example 1.1), whereas from [CPY], (4.3.4) it follows immediatly that $\sigma_{s-F}(A) = \sigma_{\mathrm{le}}(A) \cap \sigma_{\mathrm{re}}(A)$ if X is a Hilbert space.

Since

$$\sigma_{s-F}(A) \subset \sigma_{\mathrm{le}}(A) \cap \sigma_{\mathrm{re}}(A) \subset \sigma_e(A) = \sigma_{s-F}(A) \cup \rho^{-\infty}_{s-F}(A) \cup \rho^{+\infty}_{s-F}(A),$$

it follows that

$$\sigma_e(A) = (\sigma_{\mathrm{le}}(A) \cap \sigma_{\mathrm{re}}(A)) \cup \rho^{-\infty}_{s-F}(A) \cup \rho^{+\infty}_{s-F}(A)$$

and

$$(\sigma_{\rm le}(A) \cap \sigma_{\rm re}(A)) \setminus \sigma_{s-F}(A) \subset \rho_{s-F}^{-\infty}(A) \cup \rho_{s-F}^{+\infty}(A).$$

Let $\mathbf{K_C}$ denote the set of all compact nonempty subsets of the complex plane, endowed with the Hausdorff metric. We put

$\Sigma(X) = \{T \in L_c(X)$: the spectrum function $\sigma : L_c(X) \longrightarrow \mathbf{K_C}$ is continuous at $T\}$,

$R(X) = \{T \in L_c(X)$: the spectral radius function $r : L_c(X) \longrightarrow [0, +\infty)$ is continuous at $T\}$,

$\pi(X) = \{T \in L_c(X)$: any neighborhood of the peripheral spectrum of T contains a nonempty spectral set of $T\}$ (see [B], 1.1) and

$\tau(X) = \{T \in L_c(X)$: any open set in the relative topology of $\sigma(T)\}$ contains a nonempty spectral set of $T\}$

(see [B], 2.1). Obviously, $\Sigma(X) \subset R(X)$ and $\tau(X) \subset \pi(X)$.

If, for any topological space W and for any $w \in W$, $C_w(W)$ denotes the component of W which contains w, for any $A \in L_c(X)$ we put $\psi(A) = \{\lambda \in \sigma(A) : C_\lambda(\sigma(A)) = \{\lambda\}\}$ (see [B], 2.3) and $\varphi(A) = \{\lambda \in \sigma(A) : C_\lambda(\sigma(A)) \subset \{\mu \in \mathbf{C} : |\lambda| \leq |\mu| \leq r(A)\}\}$ (see [B], 1.3). We recall that $\tau(X) = \{A \in L_c(X) : \sigma(A) = \overline{\psi(A)}\}$ (see [B], 2.4) and $\pi(X) = \{A \in L_c(X) : \sup\{|\mu| : \mu \in \varphi(A)\} = r(A)\}$ (see [B], 1.5). From [M], 2.1 and 2.2 it follows that $\pi(X) \subset R(X)$ and $\tau(X) \subset \Sigma(X)$.

Finally we recall that, if X is complex infinite-dimentional Hilbert space, for any compact nonempty subset K of the complex plane and for any orthonormal basis E of X there exists $A \in L_c(X)$, diagonal with respect to E (and therefore normal), such that $\sigma(A) = K$ (see [Ha], Prob.46, in which only the case of a separable Hilbert space is treated; the general case follows immediately).

1.

Let X be a complex nonzero Banach space and let $A \in L_c(X)$. From [CPY], (4.3.4) it follows that

$$\mathbf{C} \setminus \sigma_{\rm re}(A) \subset \{\lambda \in \rho_{s-F}(A) : \dim X/\mathrm{Im}\,(\lambda I_X - A) < +\infty\}$$

and

$$\mathbf{C} \setminus \sigma_{\rm le}(A) \subset \{\lambda \in \rho_{s-F}(A) : \dim \ker(\lambda I_X - A) < +\infty\}.$$

Pietsch, in the example at pages 366 and 367 of [P], has shown that in a non-reflexive Banach space there may exist a semi-Fredholm operator with finite-dimensional null space (resp., finite-codimensional range) whose range (resp., null space) is not the range of a continuous projection. Hence, by [CPY], (4.3.4), none of the two inclusions above can be inverted. The following example, inspired by the one of Pietsch, shows that even in a reflexive space none of the two inclusions above can be inverted and $\sigma_{\text{le}}(A) \cap \sigma_{\text{re}}(A)$ is not always contained in $\sigma_{s-F}(A)$.

EXAMPLE 1.1 .— Let us consider the complex Banach space ℓ_p (where $p \in (0,2) \cup (2,+\infty)$). Since $p \neq 2$, there exists a closed subspace X of ℓ_p such that X is not the range of any continuous projection on ℓ_p (see [Kö], 31.3, (6)).

$\ell_p \times \ell_p$ is a Banach space with respect to the canonical norm defined by $\|(x,y)\| = \|x\| + \|y\|$ for any $(x,y) \in \ell_p \times \ell_p$, and obviously $X \times \ell_p$ is a closed subspace of $\ell_p \times \ell_p$.

If $\{e_n\}_{n\in \mathbf{N}}$ denotes the canonical basis of ℓ_p, that is $e_n = (\delta_{nk})_{k\in\mathbf{N}}$ for any $n \in \mathbf{N}$, it is not difficult to verify that the operator $T : \ell_p \longrightarrow \ell_p \times \ell_p$ (where, for any $x = (x_n)_{n\in\mathbf{N}} \in \ell_p$, $Tx = \left(\sum_{n\in\mathbf{N}} x_{2n}e_n, \sum_{n\in\mathbf{N}} x_{2n+1}e_n\right)$) is an isomorphism of Banach spaces. If we define $A(x,y) = \left(0, T^{-1}(x,y)\right)$ for any $(x,y) \in X \times \ell_p$, it follows that the operator $A : X \times \ell_p \longrightarrow X \times \ell_p$ is linear and continuous, $\ker A = \{0\}$ and

$$\text{Im } A = \{0\} \times T^{-1}(X \times \ell_p) = \overline{\text{Im } A}.$$

Consequently A is a semi-Fredholm operator and, moreover,

$$\begin{aligned} 0 \in \{\lambda \in \rho_{s-F}(A) \,:\, \dim\ker(\lambda I_{X\times\ell_p} - A) < +\infty\} = \\ = \{\lambda \in \rho_{s-F}(A^*) \,:\, \dim\,(X \times \ell_p)^*/\text{Im}\,(\lambda I_{(X\times\ell_p)^*} - A^*) < +\infty\} \end{aligned}$$

(where A^* is the adjoint of A, see [Ka], IV, 5.13).

Nevertheless, we prove that $0 \in \sigma_{\text{le}}(A)$ (so that, as, since $p \in (1,+\infty)$, $X \times \ell_p$ is reflexive, $0 \in \sigma_{\text{re}}(A^*)$, too, and none of the two inclusions above can be inverted).

We prove that Im A is not the range of any continuous projection on $X \times \ell_p$. Suppose that there exists a projection $P \in L_c(X \times \ell_p)$ such that

$$\text{Im } P = \text{Im } A = \{0\} \times T^{-1}(X \times \ell_p).$$

For any $k = 1, 2$, we define the linear and continuous operators

$$P_k \,:\, \ell_p \times \ell_p \longrightarrow \ell_p \quad \text{and} \quad J_k \,:\, \ell_p \longrightarrow \ell_p \times \ell_p$$

in the following way:

$$P_k(x_1, x_2) = x_k \text{ for any } (x_1, x_2) \in \ell_p \times \ell_p$$

and

$$J_k x = (\delta_{1k} x, \delta_{2k} x) \text{ for any } x \in \ell_p.$$

If we put $Q = P_1 T P_2 P J_2 T^{-1} J_1$, it follows that $Q \in L_c(\ell_p)$ and

$$\text{Im } Q \subset P_1 T P_2(\text{ImP}) = P_1 T(T^{-1}(X \times \ell_p)) = P_1(X \times \ell_p) = X.$$

Moreover, for any $x \in X$, as $(0, T^{-1}(x, 0)) \in \text{ImP}$ it follows that

$$\begin{aligned} Qx &= P_1 T P_2 P J_2 T^{-1}(x, 0) = \\ &= P_1 T P_2 P(0, T^{-1}(x, 0)) = \\ &= P_1 T P_2(0, T^{-1}(x, 0)) = \\ &= P_1 T T^{-1}(x, 0) = \\ &= P_1(x, 0) = \\ &= x. = \end{aligned}$$

Consequently, since $Qz \in X$ for any $z \in \ell_p$ and $Qx = x$ for any $x \in X$, Q is a continuous projection on ℓ_p and Im Q $= X$. This is a contradiction, as X is not the range of any continuous projection on ℓ_p. Therefore Im A is not the range of any continuous projection on $X \times \ell_p$.

From [CPY], (4.3.4) it follows that $0 \in \sigma_{\text{le}}(A)$.

Obviously, since Im A has not complementary closed subspace, $X \times \ell_p$ / ImA is infinite-dimensional, so that, by the inclusions above, $0 \in \sigma_{\text{re}}(A)$, too.

We have thus proved that

$$0 \in (\sigma_{\text{le}}(A) \cap \sigma_{\text{re}}(A)) \backslash \sigma_{s-F}(A). \square$$

If X is a complex nonzero Banach space and $A \in L_c(X)$, let $\Gamma_0(A)$ denote the set of all points $\lambda \in \sigma(A)$ such that $\{\lambda\}$ is a component of

$$(\sigma(A) \backslash \overline{\rho_{s-F}^{\pm}(A)}) \cup \left(\cup_{n \in \mathbf{Z}} \left(\overset{o}{\overline{\rho_{s-F}^{n}(A)}} \backslash \rho_{s-F}^{n}(A) \right) \right)$$

and let $\Gamma_2(A)$ denote the set of all points $\lambda \in \sigma(A)$ such that λ is a component of $(\sigma_{\text{le}}(A) \cap \sigma_{\text{re}}(A)) \cup \sigma_p^0(A)$. We recall two recent characterizations of $\sum(X)$ for a separable Hilbert space, that will be useful afterwards.

THEOREM (1) ([CM], 3.1). — *Let X be a complex nonzero separable Hilbert space and let $A \in L_c(X)$; then $A \in \sum(X)$ iff $\sigma(A) = \overline{\rho_{s-F}^{\pm}(A) \cup \Gamma_2(A)}$.* $\square$

The condition $\sigma(A) = \overline{\rho_{s-F}^{\pm}(A) \cup \Gamma_2(A)}$ is at least sufficient for membership in $\sum(X)$ for any complex nonzero Banach space X. In fact, even if $\rho_{s-F}(A)$ does not always coincide with $\sigma_{\mathrm{le}}(A) \cap \sigma_{\mathrm{re}}(A)$, the equality

$$\sigma_e(A) = (\sigma_{\mathrm{le}}(A) \cap \sigma_{\mathrm{re}}(A)) \cup \rho_{s-F}^{-\infty}(A) \cup \rho_{s-F}^{+\infty}(A)$$

is anyway satisfied in any Banach space, so that Conway and Morrel's proof of the sufficiency of the condition above for membership in $\sum(X)$ can be repeated without alterations in the general case of a complex nonzero Banach space.

THEOREM (2) ([AFHV], Th. 14.15). — *Let X be a complex nonzero separable Hilbert space and let $A \in L_c(X)$; then $A \in \sum(X)$ iff $\sigma(A) = \overline{\rho_{s-F}^{\pm}(A) \cup \Gamma_0(A)}$.* $\square$

We take the opportunity of remarking that in [AFHV], Th. 14.15 the proof the sufficiency of the condition above for membership in $\sum(X)$, in order to avoid pathological examples like $S \oplus 0$ on $l_2 \oplus l_2$, should be stated more exactly by observing that, if $D(\lambda_j, \epsilon/2)$ does not contain any component of $\sigma(A)$, it contains anyway a component of $\sigma_e(A)$, so that Corollary 1.6 can still be applied (see [He], Cor.16). This part of the proof of Theorem 14.15 can be extended, without further alterations, to the general case of a complex nonzero Banach space. Therefore, if X is a complex nonzero Banach space and $A \in L_c(X)$, the condition $\sigma(A) = \overline{\rho_{s-F}^{\pm}(A) \cup \Gamma_0(A)}$ is at least sufficient membership in $\sum(X)$.

DEFINITION 1.2. — *Let X be a complex nonzero Banach space and let $A \in L_c(X)$. We define*

$$\chi(A) = \left\{\lambda \in \sigma_{\mathrm{le}}(A) \cap \sigma_{\mathrm{re}}(A) : C_\lambda\big(\sigma_{\mathrm{le}}(A) \cap \sigma_{\mathrm{re}}(A)\big) \cap \sigma_{s-F}(A) = \emptyset\right\}$$

and $\sigma_m(A) = \sigma_{s-F}(A) \cup \chi(A)$.

We remark that $\chi(A) \cap \sigma_{s-F}(A) = \emptyset$ and that $\sigma_{s-F}(A) \subset \sigma_m(A) \subset \sigma_{\mathrm{le}}(A) \cap \sigma_{\mathrm{re}}(A)$ (so that, if X is a Hilbert space, $\sigma_{s-F}(A) = \sigma_m(A) = \sigma_{\mathrm{le}}(A) \cap \sigma_{\mathrm{re}}(A)$.

THEOREM 1.3. — *Let X be a complex nonzero Banach space and let $A \in L_c(X)$; then $C_\lambda(\sigma_m(A)) = C_\lambda(\sigma_{s-F}(A))$ for any $\lambda \in \sigma_{s-F}(A)$ and $C_\lambda(\sigma_m(A)) = C_\lambda(\sigma_{\mathrm{le}}(A) \cap \sigma_{\mathrm{re}}(A))$ for any $\lambda \in \chi(A)$.*

Proof. — For any $\lambda \in \chi(A)$, since $\sigma_m(A) \subset C_\lambda(\sigma_{\text{le}}(A) \cap \sigma_{\text{re}}(A))$ it follows that

$$C_\lambda(\sigma_m(A)) \in C_\lambda(\sigma_{le}(A) \cap \sigma_{re}(A)).$$

In addition, since $\lambda \in \chi(A)$ and $C_\mu(\sigma_{\text{le}}(A) \cap \sigma_{\text{re}}(A)) = C_\lambda(\sigma_{\text{le}}(A) \cap \sigma_{\text{re}}(A))$ for any $\mu \in C_\lambda(\sigma_{\text{le}}(A) \cap \sigma_{\text{re}}(A))$, it follows that $C_\lambda(\sigma_{\text{le}}(A) \cap \sigma_{\text{re}}(A)) \subset \chi(A) \subset \sigma_m(A)$, so that $C_\lambda(\sigma_{\text{le}}(A) \cap \sigma_{\text{re}}(A)) \subset C_\lambda(\sigma_m(A))$. Therefore

$$C_\lambda(\sigma_{\text{le}}(A) \cap \sigma_{\text{re}}(A)) = C_\lambda(\sigma_m(A)) \text{ for any } \lambda \in \chi(A).$$

For any $\lambda \in \sigma_{s-F}(A)$, since $\sigma_{s-F}(A) \subset \sigma_m(A) \subset \sigma_{\text{le}}(A) \cap \sigma_{\text{re}}(A)$ it follows that

$$C_\lambda(\sigma_{s-F}(A)) \subset C_\lambda(\sigma_m(A)) \subset C_\lambda(\sigma_{\text{le}}(A) \cap \sigma_{\text{re}}(A)).$$

Moreover, since $\lambda \in \sigma_{s-F}(A)$, $\sigma_{s-F}(A) \cap C_\lambda(\sigma_{\text{le}}(A) \cap \sigma_{\text{re}}(A)) \neq \emptyset$, so that $C_\lambda(\sigma_{\text{le}}(A) \cap \sigma_{\text{re}}(A)) \cap \chi(A) = \emptyset$. It follows that $C_\lambda(\sigma_m(A)) \cap \chi(A) = \emptyset$. Hence $C_\lambda(\sigma_m(A)) \subset \sigma_{s-F}(A)$ and, consequently, $C_\lambda(\sigma_m(A)) \subset C_\lambda(\sigma_{s-F}(A))$. Therefore $C_\lambda(\sigma_m(A)) = C_\lambda(\sigma_{s-F}(A))$ for any $\lambda \in \sigma_{s-F}(A)$.

DEFINITION 1.4. — *Let X be a complex nonzero Banach space and let $A \in L_c(X)$. We define*

$$\Gamma_1(A) = \{\lambda \in \sigma(A)\backslash\overline{\rho^{\pm}_{s-F}(A)} : C_\lambda(\sigma(A)\backslash\overline{\rho^{\pm}_{s-F}(A)}) = \{\lambda\}\},$$
$$\Gamma_3(A) = \{\lambda \in \sigma_{s-F}(A) \cup \sigma^0_p(A) : C_\lambda(\sigma_{s-F}(A) \cup \sigma^0_p(A)) = \{\lambda\}\},$$
$$\Gamma_4(A) = \{\lambda \in \sigma_m(A) \cup \sigma^0_p(A) : C_\lambda(\sigma_m(A) \cup \sigma^0_p(A)) = \{\lambda\}\},$$

and

$$\Gamma_5(A) = \{\lambda \in \sigma_e(A) \cup \sigma^0_p(A) : C_\lambda(\sigma_e(A) \cup \sigma^0_p(A)) = \{\lambda\}\}. \square$$

We remark that $\Gamma_j(A) \subset \sigma(A)$ for any $j = 0, \ldots, 5$ and, if X is a Hilbert space, $\Gamma_2(A) = \Gamma_3(A) = \Gamma_4(A)$ for any $A \in L_c(X)$ (because $\sigma_{s-F}(A) = \sigma_{\text{le}}(A) \cap \sigma_{\text{re}}(A) = \sigma_m(A)$ in a Hilbert space).

LEMMA 1.5. — *Let X be a locally compact Hausdorff space and let C be a nonempty connected compact subset of X. Then C is a component of X iff there exists an open subset U of X such that C is a component of U.*

Proof. — Obviously, if C is a component of X and we put $U = X$, it follows that U is an open subset of X and C is a component of U.

Conversely, suppose that there exist an open subset U of X such that C is a component of U. Since X is a locally compact Hausdorff space and

C is compact, compact neighborhoods of C are a neighborhood base of C. Therefore there exists a compact subset V of X such that $C \subset \overset{o}{V} \subset V \subset U$. Since C is a component of U, C is also a component of V. Since V is a compact Hausdorff space and $C \subset \overset{o}{V}$, from [Hy], 2.4, Th. 2.15 it follows that there exists an open and closed set W in the relative topology of V such that $C \subset W \subset \overset{o}{V}$.

Since V is compact and X is a Hausdorff space, V is closed, so that W is a closed subset of X. Moreover, since W is open in the relative topology of V and is contained in $\overset{o}{V}$, W is open also in the relative topology of $\overset{o}{V}$, so that, as obviously $\overset{o}{V}$ is open in X, W is an open subset of X.

Hence there exists an open and closed subset W of X such that $C \subset W \subset U$. If D denotes the component of X that contains C, it follows that $D \subset W \subset U$. Consequently, D is a component of U, so that, as also C is a component of U, $D = C$. Therefore C is a component of X. □

THEOREM 1.6.— *Let X be a complex nonzero Banach space and let $A \in L_c(X)$. Then*

$$\Gamma_1(A) \subset \psi(A) \subset \Gamma_5(A) \subset \Gamma_j(A) \subset \Gamma_4(A) \quad \textit{for any} \quad j = 2, 3$$

and

$$\Gamma_1(A) \subset \Gamma_0(A) \subset \Gamma_5(A);$$

moreover,

$$\Gamma_4(A) = \Gamma_2(A) \cup \Gamma_3(A) \quad \textit{and} \quad \Gamma_1(A) = \psi(A) \cap \Gamma_0(A).$$

Proof.— Since $\sigma(A) \backslash \overline{\rho^{\pm}_{s-F}(A)}$ is open in the relative topology of $\sigma(A)$, from Lemma 1.5 it follows that $\Gamma_1(A) \subset \psi(A)$.

We prove that $\psi(A) \subset \sigma_e(A) \cup \sigma^0_p(A)$.

Since $\rho^{\pm}_{s-F}(A) \subset \overset{o}{\sigma}(A)$, it follows that

$$\psi(A) \cap \rho_{s-F}(A) \subset \rho^0_{s-F}(A) \cap \sigma(A) = \sigma^0_p(A) \cup \left(\rho^0_{s-F}(A) \cap \overset{o}{\sigma}(A)\right).$$

Hence $\psi(A) \cap \rho_{s-F}(A) = \sigma^0_p(A)$, and therefore $\psi(A) \subset \sigma_{s-F}(A) \cup \sigma^0_p(A) \subset \sigma_e(A) \cup \sigma^0_p(A)$. Consequently, for any $\lambda \in \psi(A)$,

$$\lambda \in C_\lambda(\sigma_e(A) \cup \sigma^0_p(A)) \subset C_\lambda(\sigma(A)) = \{\lambda\},$$

so that $\lambda \in \Gamma_5(A)$.

Hence $\psi(A) \subset \Gamma_5(A)$.

We prove that

$$\Gamma_5(A) \subset \Gamma_j(A) \quad \text{for any} \quad j = 2,3.$$

Let $\lambda \in \Gamma_5(A)$; then $C_\lambda(\sigma_e(A) \cup \sigma_p^0(A)) = \{\lambda\}$, so that $\lambda \in \partial\sigma_e(A) \cup \sigma_p^0(A) \subset \sigma_{s-F}(A) \cup \sigma_p^0(A)$. Since $\sigma_{s-F}(A) \subset \sigma_{\mathrm{le}}(A) \cap \sigma_{\mathrm{re}}(A) \subset \sigma_e(A)$ and $C_\lambda(\sigma_e(A) \cup \sigma_p^0(A)) = \{\lambda\}$, it follows that

$$C_\lambda(\sigma_{s-F}(A) \cup \sigma_p^0(A)) = C_\lambda((\sigma_{\mathrm{le}}(A) \cap \sigma_{\mathrm{re}}(A)) \cup \sigma_p^0(A)) = \{\lambda\},$$

and therefore $\lambda \in \Gamma_2(A) \cap \Gamma_3(A)$.

We prove that

$$\Gamma_2(A) \cup \Gamma_3(A) = \Gamma_4(A).$$

For any $\lambda \in \sigma_p^0(A)$, λ is isolated in $\sigma(A)$, so that, as $\sigma_m(A) \cup \sigma_p^0(A) \subset \sigma(A)$, λ is isolated in $\sigma_m(A) \cup \sigma_p^0(A)$, too. Consequently,

$$C_\lambda(\sigma_m(A) \cup \sigma_p^0(A)) = \{\lambda\}$$

and therefore

$$\lambda \in \Gamma_4(A).$$

For any $\lambda \in \Gamma_3(A) \backslash \sigma_p^0(A)$, $\lambda \in \sigma_{s-F}(A)$ and $C_\lambda(\sigma_{s-F}(A)) = \{\lambda\}$. Consequently, by Theorem 1.3, $C_\lambda(\sigma_m(A)) = \{\lambda\}$, so that also

$$C_\lambda(\sigma_m(A) \cup \sigma_p^0(A)) = \{\lambda\}.$$

Therefore

$$\lambda \in \Gamma_4(A).$$

For any $\lambda \in \Gamma_2(A) \backslash \Gamma_3(A)$, $\lambda \in (\sigma_{\mathrm{le}}(A) \cap \sigma_{\mathrm{re}}(A) \backslash \sigma_{s-F}(A)$ and

$$C_\lambda((\sigma_{\mathrm{le}}(A) \cap \sigma_{\mathrm{re}}(A)) \cup \sigma_p^0(A)) = \{\lambda\},$$

so that, obviously, $C_\lambda(\sigma_{\mathrm{le}}(A) \cap \sigma_{\mathrm{re}}(A)) = \{\lambda\}$. Since $\lambda \notin \sigma_{s-F}(A)$, it follows that $\lambda \in \chi(A) \subset \sigma_m(A) \subset \sigma_{\mathrm{le}}(A) \cap \sigma_{\mathrm{re}}(A)$ and hence $C_\lambda(\sigma_m(A)) = \{\lambda\}$, so that also $C_\lambda(\sigma_m(A) \cup \sigma_p^0(A)) = \{\lambda\}$.

Therefore

$$\lambda \in \Gamma_4(A).$$

We have thus proved that

$$\Gamma_2(A) \cup \Gamma_3(A) \subset \Gamma_4(A).$$

Conversely, for any $\lambda \in \Gamma_4(A)$, if $\lambda \in \sigma_{s-F}(A) \cup \sigma_p^0(A)$ it follows that

$$\lambda \in C_\lambda(\sigma_{s-F}(A) \cup \sigma_p^0(A)) \subset C_\lambda(\sigma_m(A) \cup \sigma_p^0(A)) = \{\lambda\}.$$

Consequently,

$$C_\lambda(\sigma_{s-F}(A) \cup \sigma_p^0(A)) = \{\lambda\}$$

and therefore

$$\lambda \in \Gamma_3(A).$$

If, instead, $\lambda \in \chi(A)$, from Theorem 1.3 it follows that

$$\begin{aligned} \{\lambda\} &= C_\lambda(\sigma_m(A) \cup \sigma_p^0(A)) = \\ &= C_\lambda(\sigma_m(A)) = \\ &= C_\lambda\left(\sigma_{\mathrm{le}}(A) \cap \sigma_{\mathrm{re}}(A)\right) = \\ &= C_\lambda\left((\sigma_{\mathrm{le}}(A) \cap \sigma_{\mathrm{re}}(A)) \cup \sigma_p^0(A)\right), \end{aligned}$$

so that $\lambda \in \Gamma_2(A)$.

Therefore

$$\Gamma_4(A) \subset \Gamma_2(A) \cup \Gamma_3(A).$$

We have thus proved that

$$\Gamma_1(A) \subset \psi(A) \subset \Gamma_5(A) \subset \Gamma_j(A) \subset \Gamma_4(A) \quad \text{for any} \quad j = 2, 3$$

and

$$\Gamma_4(A) = \Gamma_2(A) \cup \Gamma_3(A).$$

Now we prove that $\Gamma_1(A) \subset \Gamma_0(A)$.

We define

$$s(A) = \left(\sigma(A) \backslash \overline{\rho_{s-F}^{\pm}(A)}\right) \cup \left(\bigcup_{n \in \mathbf{Z}} \left(\overset{o}{\overline{\rho_{s-F}^{n}(A)}} \backslash \rho_{s-F}^{n}(A)\right)\right).$$

Since $\sigma(A)\backslash\overline{\rho^{\pm}_{s-F}(A)} \subset s(A) \subset \sigma(A)$, it follows that $\sigma(A)\backslash\overline{\rho^{\pm}_{s-F}(A)} = s(A)\backslash\overline{\rho^{\pm}_{s-F}(A)}$. Therefore $\sigma(A)\backslash\overline{\rho^{\pm}_{s-F}(A)}$ is open in the relative topology of $s(A)$.

We remark that

$$s(A) = \left(\left(\mathbf{C}\backslash\overline{\rho^{\pm}_{s-F}(A)}\right) \cup \left(\bigcup_{n\in\mathbf{Z}} \overline{\overset{o}{\rho^{n}_{s-F}(A)}}\right)\right) \cap \\ \cap \left(\sigma(A)\backslash\left(\bigcup_{n\in\mathbf{Z}\backslash\{0\}} \rho^{n}_{s-F}(A)\right)\right)$$

(because $\overline{\rho^{0}_{s-F}(A)}\backslash\rho^{0}_{s-F}(A) \subset \sigma(A)\backslash\overline{\rho^{\pm}_{s-F}(A)}$); therefore $s(A)$ is the intersection between an open and a closed subset of the complex plane and, consequently, it is a Hausdorff locally compact space. Hence from Lemma 1.5 it follows that $\Gamma_1(A) \subset \Gamma_0(A)$.

We have proved that $\Gamma_1(A) \subset \Gamma_0(A) \cap \psi(A)$. Since $\overline{\overset{o}{\rho^{0}_{s-F}(A)}}\backslash\rho^{0}_{s-F}(A) \subset \sigma(A)\backslash\overline{\rho^{\pm}_{s-F}(A)}$ and $\overline{\overset{o}{\rho^{n}_{s-F}(A)}} \subset \overset{o}{\sigma}(A)$ (so that $\overline{\overset{o}{\rho^{n}_{s-F}(A)}} \cap \psi(A) = \emptyset$) for any $n \in \mathbf{Z}\backslash\{0\}$, it follows that

$$\Gamma_0(A) \cap \psi(A) \subset \sigma(A)\backslash\overline{\rho^{\pm}_{s-F}(A)}.$$

Consequently,

$$C_\lambda\left(\sigma(A)\backslash\overline{\rho^{\pm}_{s-F}(A)}\right) = \{\lambda\} \quad \text{for any} \quad \lambda \in \Gamma_0(A) \cap \psi(A),$$

so that $\Gamma_0(A) \cap \psi(A) \subset \Gamma_1(A)$. Hence $\Gamma_1(A) = \Gamma_0(A) \cap \psi(A)$.

Finally we prove that

$$\Gamma_0(A) \subset \Gamma_5(A).$$

Obviously

$$\Gamma_0(A) \cap \left(\sigma(A)\backslash\overline{\rho^{\pm}_{s-F}(A)}\right) \subset \Gamma_1(A) \subset \Gamma_5(A).$$

For any $n \in \mathbf{Z}$,

$$\overline{\overset{o}{\rho^{n}_{s-F}(A)}} \cap \sigma_e(A) = \overline{\overset{o}{\rho^{n}_{s-F}(A)}}\backslash\rho^{n}_{s-F}(A);$$

consequently, $\overline{\overset{o}{\rho^{n}_{s-F}(A)}}\backslash\rho^{n}_{s-F}(A)$ is open in the relative topology of $\sigma_e(A)$, so that any trivial component of $\overline{\overset{o}{\rho^{n}_{s-F}(A)}}\backslash\rho^{n}_{s-F}(A)$ is also a component of

$\sigma_e(A)$ (and therefore it is a component of $\sigma_e(A) \cup \sigma_p^0(A)$, too) by Lemma 1.5.

Hence

$$\Gamma_0(A) \cap \left(\bigcup_{n \in \mathbf{Z}} \left(\overline{\overset{o}{\rho^n_{s-F}(A)}} \setminus \rho^n_{s-F}(A) \right) \right) \subset \Gamma_5(A).$$

We have thus proved that

$$\Gamma_0(A) \subset \Gamma_5(A). \quad \square$$

We shall prove that none of the inclusions enunciated in Theorem 1.6 can be inverted.

First of all, we prove that, in general, there is not any relationship of inclusion between $\psi(A)$ and $\Gamma_0(A)$.

LEMMA 1.7.— *Let X and Y be Banach spaces, let $S \in L_c(X)$ and let $T \in L_c(Y)$. Then the operator $S \oplus T \in L_c(X \oplus Y)$ is semi-Fredholm iff S and T are semi-Fredholm and $\{$ ind S, ind $T\} \neq \{-\infty, +\infty\}$; moreover, if $S \oplus T$ is a semi-Fredholm operator, ind $(S \oplus T) =$ ind S + ind T.*

Proof.— We define $A = S \oplus T$. From [TL], V, 5.2 it follows that $\ker A = \ker S \oplus \ker T$ and Im $A =$ Im $S \oplus$ Im T (so that the vector space $(X \oplus Y)/\text{Im } A$ is algebrically isomorphic to the vector space $X/\text{Im } S \oplus Y/\text{Im } T$). It is not difficult to verify that Im A is closed if and only if both Im S and Im T are closed. Therfore A is a semi-Fredholm operator if and only if Im S and Im T are closed and either $\ker S$ and $\ker T$ are finite-dimensional or $X/\text{Im } S$ and $Y/\text{Im } T$ are finite-dimensional. Hence A is a semi-Fredholm operator if and only if S and T are semi-Fredholm operators and $\{\text{ind } S, \text{ind } T\} \neq \{-\infty, +\infty\}$.

In addition, from the equalities enunciated above it follows that, if A is a semi-Fredholm operator, ind A $=$ ind $S +$ ind T. $\square$

The following example shows that $\Gamma_0(A)$ is not always contained in $\psi(A)$.

Example 1.8 .— We denote by S the unilateral left shift operator on ℓ_2 and by 0 the null operator on ℓ_2. Let us consider the operator $A = S \oplus 0 \in L_c(\ell_2 \oplus \ell_2)$. From [TL], V, 5.4 it follows that $\sigma(A) = \sigma(S) \cup \sigma(0) = \overline{B_{\mathbf{C}}(0,1)}$ (see [Ha], Sol. 67). Therefore, since obviously $\overline{B_{\mathbf{C}}(0,1)}$ is connected, $\psi(A) = \emptyset$.

We prove that $\Gamma_0(A) \neq \emptyset$.

We recall that $\mathbf{C}\backslash\partial B_{\mathbf{C}}(0,1) = \rho_{s-F}(S)$ and ind $(\lambda I_{\ell_2} - S) = 1$ for any $\lambda \in B_{\mathbf{C}}(0,1)$ (see [Ka], IV, 5.24). Since $\lambda I_{\ell_2\oplus\ell_2} - A = (\lambda I_{\ell_2} - S) \oplus \lambda I_{\ell_2}$ for any $\lambda \in \mathbf{C}$ and 0 is not a semi-Fredholm operator, from Lemma 1.7 it follows that $\sigma_{s-F}(A) = \mathbf{C}\backslash(\{0\} \cup \partial B_{\mathbf{C}}(0,1))$ and ind $(\lambda I_{\ell_2\oplus\ell_2} - A) =$ ind $(\lambda I_{\ell_2} - S) = 1$ for any $\lambda \in B_{\mathbf{C}}(0,1)\backslash\{0\}$.

Therefore $\rho^{\pm}_{s-F}(A) = \rho^{1}_{s-F}(A) = B_{\mathbf{C}}(0,1)\backslash\{0\}$ and $\rho^{0}_{s-F}(A) = \rho(A) = \mathbf{C}\backslash\overline{B_{\mathbf{C}}(0,1)}$, so that

$$\begin{aligned}
&\left(\sigma(A)\backslash\overline{\rho^{\pm}_{s-F}(A)}\right) \cup \left(\bigcup_{n\in\mathbf{Z}} \left(\overset{o}{\overline{\rho^{n}_{s-F}(A)}}\backslash\rho^{n}_{s-F}(A)\right)\right) = \\
&= \left(\overset{o}{\overline{\rho^{0}_{s-F}(A)}}\backslash\rho^{0}_{s-F}(A)\right) \cup \left(\overset{o}{\overline{\rho^{1}_{s-F}(A)}}\backslash\rho^{1}_{s-F}(A)\right) = \\
&= \overset{o}{\overline{\rho^{1}_{s-F}(A)}}\backslash\rho^{1}_{s-F}(A) = \\
&= B_{\mathbf{C}}(0,1)\backslash(B_{\mathbf{C}}(0,1)\backslash\{0\}) = \\
&= \{0\}.
\end{aligned}$$

Consequently, $\Gamma_0(A) = \{0\} \neq \emptyset$. □

Since $\Gamma_0(A)$ is not always contained in $\psi(A)$, $\Gamma_1(A) \subset \psi(A)$ and $\Gamma_0(A) \subset \Gamma_5(A)$, it follows that $\Gamma_0(A)$ is not always contained in $\Gamma_1(A)$ and $\Gamma_5(A)$ is not always contained in $\psi(A)$.

The following example proves that $\psi(A)$ is not always contained in $\Gamma_0(A)$.

EXAMPLE 1.9. — We denote by S unilateral left shift on ℓ_2. Let us consider the complex Hilbert space

$$X = \{(x_n)_{n\in\mathbf{N}} \,:\, x_n \in \ell_2 \text{ for any } n \in \mathbf{N} \text{ and } (\|x_n\|)_{n\in\mathbf{N}} \in \ell_2\}\,;$$

the norm in X is defined in the following way :

$$\|(x_n)_{n\in\mathbf{N}}\| = \left(\sum_{n\in\mathbf{N}} \|x_n\|^2\right)^{1/2}$$

for any $(x_n)_{n\in\mathbf{N}} \in X$. Since $\|S\| = 1$, the sequence

$$\left((1/2^n)I_{\ell_2} + (1/2^{n+2})S^{n+1}\right)_{n\in\mathbf{N}}$$

of linear and continuous operators on ℓ_2 is bounded in norm, so that

$$\left(\left((1/2^n)I_{\ell_2} + (1/2^{n+2})S^{n+1}\right) x_n\right)_{n\in\mathbf{N}} \in X$$

for any $(x_n)_{n\in\mathbf{N}} \in X$ and the linear operator

$$A : X \longrightarrow X$$

(where $A(x_n)_{n\in\mathbf{N}} = \left(\left((1/2^n)I_{\ell_2} + (1/2^{n+2})S^{n+1}\right)x_n\right)_{n\in\mathbf{N}}$ for any $(x_n)_{n\in\mathbf{N}} \in X$) is continuous.

For any $n \in \mathbf{N}$, we define two closed subspaces of X,

$$Y_n = \{(x_k)_{k\in\mathbf{N}} \in X \,:\, x_k = 0 \quad \text{if } k \neq n\}$$

(which obviously is isomorphic to ℓ_2) and

$$X_n = \{(x_k)_{k\in\mathbf{N}} \in X \,:\, x_k = 0 \quad \text{for any } k = 0, \dots, n\},$$

and two linear and continuous operators,

$$T_n : Y_n \longrightarrow Y_n$$

(where $T_n(\delta_{nk}x)_{k\in\mathbf{N}} = \left(\delta_{nk}\left((1/2^n)I_{\ell_2} + (1/2^{n+2})S^{n+1}\right)x\right)_{k\in\mathbf{N}}$ for any $x \in \ell_2$) and

$$A_n : X_n \longrightarrow X_n$$

(where $A_n z = Az$ for any $z \in X_n$).

Obviously, for any $n \in \mathbf{N}$, $X = \left(\overset{n}{\underset{k=0}{\oplus}} Y_k\right) \oplus X_n$ and $A = \left(\overset{n}{\underset{k=0}{\oplus}} T_k\right) \oplus A_n$.

Moreover, since $\sigma(S) = \overline{B_{\mathbf{C}}(0,1)}$ and $\rho^1_{s-F}(S) = B_{\mathbf{C}}(0,1)$ (see [Ka], IV, 5.24), it follows that

$$\sigma(T_n) = \sigma\left((1/2^n)I_{\ell_2} + (1/2^{n+2})S^{n+1}\right) = \overline{B_{\mathbf{C}}(1/2^n, 1/2^{n+2})}$$

and, as

$$S^{n+1} - \delta\exp^{i\theta} I_{\ell_2} = \overset{n}{\underset{k=0}{\pi}}\left(S - \delta^{1/(n+1)}\exp^{i(\theta+2k\pi)/(n+1)} I_{\ell_2}\right)$$

for any $\delta \geq 0$ and for any $\theta \in [0, 2\pi)$, $\rho^{n+1}_{s-F}(S^{n+1}) = B_{\mathbf{C}}(0,1)$ (see [CPY], (3.2.7)) (so that $\rho^{n+1}_{s-F}(T_n) = \rho^{n+1}_{s-F}\left((1/2^n)I_{\ell_2} + (1/2^{n+2})S^{n+1}\right) = B_{\mathbf{C}}(1/2^n, 1/2^{n+2})$.

It is not difficult to verify that the balls $\overline{B_{\mathbf{C}}(1/2^n, 1/2^{n+2})}$, $n \in \mathbf{N}$, are pairwise disjoint.

For any $n \in \mathbf{N}$, since $x_k = 0$ for any $k = 0,\ldots,n$ and for any $(x_k)_{k\in\mathbf{N}} \in X_n$ it follows that

$$\|A(x_k)_{k\in\mathbf{N}}\| = \left(\sum_{k=n+1}^{+\infty} \|((1/2^k)I_{\ell_2} + (1/2^{k+2})S^{k+1})x_k\|^2\right)^{1/2} \leq$$
$$\leq \left(\sup\{\|(1/2^k)I_{\ell_2} + (1/2^{k+2})S^{k+1}\| \,:\, k > n\}\right) \|(x_k)_{k\in\mathbf{N}}\| \leq$$
$$\leq \left((1/2^{n+1}) + (1/2^{n+3})\right)\|(x_k)_{k\in\mathbf{N}}\| =$$
$$= (5/2^{n+3})\|(x_k)_{k\in\mathbf{N}}\|$$

for any $(x_k)_{k\in\mathbf{N}} \in X_n$.

Therfore $\|A_n\| \leq 5/2^{n+3}$ and, consequently, $\sigma(A_n) \subset \overline{B_{\mathbf{C}}(0,5/2^{n+3})}$.

It is not difficult to verify that $\overline{B_{\mathbf{C}}(0,5/2^{n+3})} \cap \overline{B_{\mathbf{C}}(1/2^n,1/2^{n+2})} = \emptyset$; therefore $\overline{B_{\mathbf{C}}(1/2^n,1/2^{n+2})} \subset \rho(A_n) \cap \left(\bigcap_{k=0}^{n-1}\rho(T_k)\right)$. From Lemma 1.7 it follows that $B_{\mathbf{C}}(1/2^n,1/2^{n+2}) \subset \rho_{s-F}^{n+1}(A)$.

By [TL], V,5.4,

$$\sigma(A) = \left(\bigcup_{k=0}^{n}\sigma(T_k)\right)\bigcup\sigma(A_n) = \left(\bigcup_{k=0}^{n}\overline{B_{\mathbf{C}}(1/2^k,1/2^{k+2})}\right)\bigcup\sigma(A_n)$$

for any $n \in \mathbf{N}$; therefore

$$\bigcup_{k\in\mathbf{N}}\overline{B_{\mathbf{C}}(1/2^k,1/2^{k+2})} \subset \sigma(A) \subset \overline{B_{\mathbf{C}}(0,5/2^{n+3})}\bigcup$$
$$\bigcup\left(\bigcup_{k\in\mathbf{N}}\overline{B_{\mathbf{C}}(1/2^k,1/2^{k+2})}\right)$$

for any $n \in \mathbf{N}$. Since $5/2^{n+3} \underset{n\to+\infty}{\longrightarrow} 0$, it follows that

$$\sigma(A) = \{0\}\bigcup\left(\bigcup_{n\in\mathbf{N}}\overline{B_{\mathbf{C}}(1/2^n,1/2^{n+2})}\right) = \overline{\bigcup_{n\in\mathbf{N}}B_{\mathbf{C}}(1/2^n,1/2^{n+2})}.$$

Consequently, $\rho_{s-F}^n(A) = \emptyset$ for any negative integer n,

$$\rho_{s-F}^n(A) = B_{\mathbf{C}}(1/2^{n-1},1/2^{n+1}) = \overset{o}{\overline{\rho_{s-F}^n(A)}}$$

for any positive integer n and

$$\rho_{s-F}^0(A) = \rho(A) = \mathbf{C}\setminus\left(\{0\}\bigcup\left(\bigcup_{n\in\mathbf{N}}\overline{B_{\mathbf{C}}(1/2^n,1/2^{n+2})}\right)\right) = \overset{o}{\overline{\rho_{s-F}^0(A)}}.$$

Therefore

$$\rho^{\pm}_{s-F}(A) = \bigcup_{n\in\mathbf{N}} B_{\mathbf{C}}(1/2^n, 1/2^{n+2}),$$

so that $\sigma(A) = \overline{\rho^{\pm}_{s-F}(A)}$ and

$$\left(\sigma(A)\backslash\overline{\rho^{\pm}_{s-F}(A)}\right) \bigcup \left(\bigcup_{n\in\mathbf{Z}} \left(\overset{o}{\overline{\rho^{n}_{s-F}(A)}}\backslash\rho^{n}_{s-F}(A)\right)\right) = \emptyset.$$

Hence $\Gamma_0(A) = \emptyset$, whereas, obviously, $\psi(A) = \{0\}$. Therefore $\psi(A)$ is not contained in $\Gamma_0(A)$.

Since $\psi(A)$ is not always contained in $\Gamma_0(A)$, $\Gamma_1(A) \subset \Gamma_0(A)$ and $\psi(A) \subset \Gamma_5(A)$, it follows that $\psi(A)$ is not always contained in $\Gamma_1(A)$ and $\Gamma_5(A)$ is not always contained in $\Gamma_0(A)$.

We prove that, generally speaking, neither $\Gamma_2(A)$ nor $\Gamma_3(A)$ are contained in $\Gamma_5(A)$.

EXAMPLE 1.10 . — We consider the complex Hilbert space ℓ_2 and the linear and continuous operator $A : \ell_2 \longrightarrow \ell_2$ (where, if $\{e_n\}_{n\in\mathbf{N}}$ denotes the canonical basis of ℓ_2, that is $e_n = (\delta_{nk})_{k\in\mathbf{N}}$ for any $n \in \mathbf{N}$, $A(x_n)_{n\in\mathbf{N}} = \sum_{n\in\mathbf{N}} x_n e_{2n}$ for any $(x_n)_{n\in\mathbf{N}} \in \ell_2$).

Obviously, $\|Ax\| = \|x\|$ for any $x \in \ell_2$, so that $\|A\| = 1$ and $\sigma(A) \subset \overline{B_{\mathbf{C}}(0,1)}$. In addition, for any $\lambda \in B_{\mathbf{C}}(0,1)$ and for any $x \in \ell_2$,

$$\|(\lambda I_{\ell_2} - A)x\| \geq \|Ax\| - |\lambda|\|x\| = (1 - |\lambda|)\|x\|.$$

Consequently, $\lambda I_{\ell_2} - A$ is one-to-one and $\mathrm{Im}(\lambda I_{\ell_2} - A)$ is closed for any $\lambda \in B_{\mathbf{C}}(0,1)$ (see [TL],IV,5.9), so that $B_{\mathbf{C}}(0,1) \subset \rho_{s-F}(A)$.

Since the function:

$$\lambda \in \rho_{s-F}(A) \longrightarrow \text{ind } (\lambda I_{\ell_2} - A) \in [-\infty, +\infty]$$

is locally constant (see [Ka], IV, 5.17) and $B_{\mathbf{C}}(0,1)$ is connected, it follows that

$$\text{ind } (\lambda I_{\ell_2} - A) = \text{ind A for any } \lambda \in B_{\mathbf{C}}(0,1).$$

We consider two closed subspaces of ℓ_2,

$$X_1 = \{(x_n)_{n\in\mathbf{N}} \in \ell_2 : x_{2n+1} = 0 \text{ for any } n \in \mathbf{N}\}$$

and

$$X_2 = \{(x_n)_{n\in\mathbf{N}} \in \ell_2 : x_{2n} = 0 \text{ for any } n \in \mathbf{N}\}.$$

It is not difficult to verify that Im A $= X_1$ and $\ell_2 = X_1 \oplus X_2$. Since obviously $\ell_2/\mathrm{Im}\ \mathrm{A} = \ell_2/X_1$ is isomorphic to X_2, which is infinite-dimensional, it follows that ind A $= -\infty$.

Therefore $\sigma(A) = \overline{B_{\mathbf{C}}(0,1)}$, $\rho_{s-F}^{-\infty}(A) = B_{\mathbf{C}}(0,1)$, $\sigma_{s-F}(A) = \partial B_{\mathbf{C}}(0,1)$ and $\sigma_e(A) = \overline{B_{\mathbf{C}}(0,1)}$.

Let us consider the operator $A \oplus 0$ on $\ell_2 \oplus \ell_2$ (where 0 is the null operator on ℓ_2). By [TL], V,5.4, $\sigma(A \oplus 0) = \sigma(A) \bigcup \{0\} = \overline{B_{\mathbf{C}}(0,1)}$. Since $\rho_{s-F}(0) = \mathbf{C} \backslash \{0\} = \rho(0)$, from Lemma 1.7 it follows that

$$\rho_{s-F}^{-\infty}(A \oplus 0) = B_{\mathbf{C}}(0,1) \backslash \{0\}$$

and

$$\sigma_{s-F}(A \oplus 0) = \{0\} \bigcup \partial B_{\mathbf{C}}(0,1).$$

Therfore $\sigma_e(A \oplus 0) = \overline{B_{\mathbf{C}}(0,1)}$, so that $\Gamma_5(A \oplus 0) = \emptyset$.

Since $\ell_2 \oplus \ell_2$ is a Hilbert space, it follows that $\Gamma_2(A \oplus 0) = \Gamma_3(A \oplus 0) = \{0\}$. Hence neither $\Gamma_2(A \oplus 0)$ nor $\Gamma_3(A \oplus 0)$ are contained in $\Gamma_5(A \oplus 0)$.

We prove that, in general, there is not any relationship of inclusion between $\Gamma_2(A)$ and $\Gamma_3(A)$.

The following example shows that $\Gamma_2(A)$ is not always contained in $\Gamma_3(A)$.

Example 1.11 .— Let us consider the complex Banach space $\ell_\infty \times \ell_\infty$, with the canonical norm of the product ($\|(x,y)\| = \|x\| + \|y\|$ for any $(x,y) \in \ell_\infty \times \ell_\infty$), and the operator

$$H : \ell_\infty \longrightarrow \ell_\infty \times \ell_\infty$$

(where $H(x_n)_{n\in\mathbf{N}} = ((x_{2n})_{n\in\mathbf{N}}, (x_{2n+1})_{n\in\mathbf{N}})$ for any $(x_n)_{n\in\mathbf{N}} \in \ell_\infty$). Obviously H is an isomorphism of Banach spaces, and

$$\|H^{-1}(x,y)\| = \|x\| \vee \|y\| \text{ for any } (x,y) \in \ell_\infty \times \ell_\infty.$$

If we consider the complex Banach space $c_0 \times \ell_\infty$ and define the linear and continuous operator

$$A : c_0 \times \ell_\infty \longrightarrow c_0 \times \ell_\infty$$

(where $A(x,y) = (0, H^{-1}(x,y))$ for any $(x,y) \in c_0 \times \ell_\infty$), it follows that

$$\|A(x,y)\| = \|H^{-1}(x,y)\| \leq \|(x,y)\|$$

for any $(x,y) \in c_0 \times \ell_\infty$, so that $\|A\| \leq 1$. Consequently, $\sigma(A) \subset \overline{B_{\mathbb{C}}(0,1)}$.

In addition, for any $\lambda \in \overline{B_{\mathbb{C}}(0,1)}$,

$$\begin{aligned}
&\|(\lambda I_{c_0 \times \ell_\infty} - A)(x,y)\| = \|(\lambda x, \lambda y - H^{-1}(x,y))\| \geq \\
&\quad \geq \|\lambda y - H^{-1}(x,y)\| \geq \|H^{-1}(x,y)\| - |\lambda| \|y\| \geq \\
&\quad \geq \|H^{-1}(x,y)\| - |\lambda| (\|x\| \vee \|y\|) = \\
&\quad = (1 - |\lambda|)(\|x\| \vee \|y\|) \geq ((1 - |\lambda|)/2)(\|x\| + \|y\|) = \\
&\quad = ((1 - |\lambda|)/2) \|(x,y)\|
\end{aligned}$$

for any $(x,y) \in c_0 \times \ell_\infty$. Therefore $\ker(\lambda I_{c_0 \times \ell_\infty} - A) = \{0\}$ and Im $(\lambda I_{c_0 \times \ell_\infty} - A)$ is closed for any $\lambda \in B_{\mathbb{C}}(0,1)$ (see [TL], IV, 5.9) and, consequently, $B_{\mathbb{C}}(0,1) \subset \rho_{s-F}(A)$.

Since the function:

$$\lambda \in \rho_{s-F}(A) \longrightarrow \text{ind } (\lambda I_{c_0 \times \ell_\infty} - A) \in [-\infty, +\infty]$$

is locally constant (see [Ka], IV, 5.17) and $B_{\mathbb{C}}(0,1)$ is connected, it follows that

$$\text{ind } (\lambda I_{c_0 \times \ell_\infty} - A) = \text{ ind A for any } \lambda \in B_{\mathbb{C}}(0,1).$$

We remark that Im A is not the range of any continuous projection on $c_0 \times \ell_\infty$ (see [P], example at pages 366 and 367); consequently, $(c_0 \times \ell_\infty)/\text{Im A}$ is infinite-dimensional.

Therefore $\sigma(A) = \overline{B_{\mathbb{C}}(0,1)}$ and $\rho_{s-F}^{-\infty}(A) = B_{\mathbb{C}}(0,1)$.

It follows that $\sigma_{s-F}(A) \bigcup \sigma_p^0(A) = \partial B_{\mathbb{C}}(0,1)$, and therefore $\Gamma_3(A) = \emptyset$.

We prove that $\Gamma_2(A) \neq \emptyset$.

Since ImA is not the range of any continuous projection on $c_0 \times \ell_\infty$ and $c_0 \times \ell_\infty/\mathrm{Im}\ A$ is infinite-dimentional, from [CPY], (4.3.4) it follows that $0 \in \sigma_{\mathrm{le}}(A) \bigcap \sigma_{\mathrm{re}}(A)$, so that

$$\sigma_{\mathrm{le}}(A) \bigcap \sigma_{\mathrm{re}}(A) \supset \{0\} \bigcup \partial B_{\mathbf{C}}(0,1).$$

We prove that $\sigma_{\mathrm{le}}(A) \bigcap \sigma_{\mathrm{re}}(A) = \{0\} \bigcup \partial B_{\mathbf{C}}(0,1)$.

For any $\lambda \in B_{\mathbf{C}}(0,1) \backslash \{0\}$, $\mathrm{Im}\ (\lambda I_{c_0 \times \ell_\infty} - A) = \{(x,y) \in c_0 \times \ell_\infty :$ there exist $u \in c_0$ and $v \in \ell_\infty$ such that $\lambda u = x$ and $\lambda v - H^{-1}(u,v) = y\} =$ $= \{(x,y) \in c_0 \times \ell_\infty :$ there exists $v \in \ell_\infty$ such that $\lambda v - H^{-1}(x/\lambda, v) = y\} =$ $= \{((x_n)_{n\in\mathbf{N}}, (y_n)_{n\in\mathbf{N}}) \in c_0 \times \ell_\infty :$ there exists $(v_n)_{n\in\mathbf{N}} \in \ell_\infty$ such that $\lambda v_{2n} - x_n/\lambda = y_{2n}$ and $\lambda v_{2n+1} - v_n = y_{2n+1}$ for any $n \in \mathbf{N}\}$.

We remark that, for any positive odd integer h, there exist $k \in \mathbf{N}\backslash\{0\}$ and $n \in \mathbf{N}$ such that $h = 2^{k+1}n + 2^k - 1$.

In fact, if we put $k = \max\{m \in \mathbf{N} : h+1 \text{ is divided by } 2^m\}$, it follows that $k \in \mathbf{N}\backslash\{0\}$ (because $h+1$ is even) and there exists $n \in \mathbf{N}$ such that $h + 1 = 2^k(2n+1)$; consequently, $h = 2^{k+1}n + 2^k - 1$. It is not difficult to verify that the function f from $(\mathbf{N}\backslash\{0\}) \times \mathbf{N}$ onto $2\mathbf{N}+1$, defined by $f((k,n)) = 2^{k+1}n + 2^k - 1$ for any $(k,n) \in (\mathbf{N}\backslash\{0\}) \times \mathbf{N}$, is one-to-one. Obviously, also the function $g : (k,n) \in \mathbf{N} \times \mathbf{N} \longrightarrow 2^{k+1}n + 2^k - 1 \in \mathbf{N}$ is one-to-one and onto, and $2\mathbf{N} = g(\{0\} \times \mathbf{N})$.

By induction on k, it is not difficult to verify that, if $\lambda v_{2n} - x_{n/\lambda} = y_{2n}$ and $\lambda v_{2n+1} - v_n = y_{2n+1}$ for any $n \in \mathbf{N}$ (where $(x_n)_{n\in\mathbf{N}} \in c_0$ and $(y_n)_{n\in\mathbf{N}}$, $(v_n)_{n\in\mathbf{N}} \in \ell_\infty$), $v_{2^{k+1}n+2^k-1} = (1/\lambda^{k+2})\left(x_n + \lambda \sum_{j=0}^{k} \lambda^j y_{2^{j+1}n+2^j-1}\right)$ for any $(k,n) \in \mathbf{N} \times \mathbf{N}$.

We remark that, as $|\lambda| < 1$, $\left(\sum_{j=0}^{k} \lambda^j y_{g(j,n)}\right)_{k\in\mathbf{N}}$ converges for any $n \in \mathbf{N}$ and for any $(y_n)_{n\in\mathbf{N}} \in \ell_\infty$.

If $(x_m)_{m\in\mathbf{N}} \in c_0$, $(y_m)_{m\in\mathbf{N}} \in \ell_\infty$ and also the sequence $(v_m)_{m\in\mathbf{N}}$ (defined by : $v_{g(k,n)} = (1/\lambda^{k+2})\left(x_n + \lambda \sum_{j=0}^{k} \lambda^j y_{g(j,n)}\right)$ for any $(k,n) \in \mathbf{N} \times \mathbf{N}$) belongs to ℓ_∞, since $1/|\lambda|^{k+2} \underset{k\to+\infty}{\longrightarrow} +\infty$ it follows that $\sum_{j=0}^{+\infty} \lambda^j y_{g(j,n)} = -x_n/\lambda$ for any $n \in \mathbf{N}$.

Conversely, if $(x_n)_{n\in\mathbf{N}} \in c_0$ and $(y_n)_{n\in\mathbf{N}} \in \ell_\infty$ are such that

$$\sum_{j=0}^{+\infty} \lambda^j y_{g(j,n)} = -x_n/\lambda$$

for any $n \in \mathbf{N}$ and the sequence $(v_n)_{n\in\mathbf{N}}$ is defined as above, it is not difficult to verify that $\lambda v_{2n} - x_n/\lambda = y_{2n}$ and $\lambda v_{2n+1} - v_n = y_{2n+1}$ for any $n \in \mathbf{N}$.

Moreover, for any $(k,n) \in \mathbf{N} \times \mathbf{N}$,

$$\begin{aligned}
|v_{g(k,n)}| &= (1/|\lambda|^{k+2}) \left| x_n + \lambda \sum_{j=0}^{k} \lambda^j y_{g(j,n)} \right| = \\
&= (1/|\lambda|^{k+2}) \left| -\lambda \sum_{j=k+1}^{+\infty} \lambda^j y_{g(j,n)} \right| \leq \\
&\leq (1/|\lambda|^{k+1}) \sum_{j=k+1}^{+\infty} |\lambda|^j |y_{g(j,n)}| \leq \\
&\leq \|(y_m)_{m\in\mathbf{N}}\| \left(\sum_{j=k+1}^{+\infty} |\lambda|^j \right) /|\lambda|^{k+1} = \\
&= \|(y_m)_{m\in\mathbf{N}}\| \left(|\lambda|^{k+1}/(1-|\lambda|)\right) /|\lambda|^{k+1} = \\
&= \|(y_m)_{m\in\mathbf{N}}\|/(1-|\lambda|).
\end{aligned}$$

We have thus proved that $\mathrm{Im}\,(\lambda I_{c_0\times\ell_\infty} - A) = \{((x_n)_{n\in\mathbf{N}}, (y_n)_{n\in\mathbf{N}}) \in c_0 \times \ell_\infty : \sum_{j=0}^{+\infty} \lambda^j y_{g(j,n)} = -x_n/\lambda$ for any $n \in \mathbf{N}\}$ for any $\lambda \in B_{\mathbf{C}}(0,1)\backslash\{0\}$.

Let $\lambda \in B_{\mathbf{C}}(0,1)\backslash\{0\}$. For any $x = (x_n)_{n\in\mathbf{N}} \in c_0$ and for any $y = (y_n)_{n\in\mathbf{N}} \in \ell_\infty$, we define $p_{2n}^{(\lambda)}(x,y) = -x_n/\lambda - \sum_{j=1}^{+\infty} \lambda^j y_{g(j,n)}$ and $p_{2n+1}^{(\lambda)}(x,y) = y_{2n+1}$ for any $n \in \mathbf{N}$.

Since $|p_{2n}^{(\lambda)}(x,y)| \leq \|x\|/|\lambda| + \lambda\|\|y\|/(1-|\lambda|)$ for any $n \in \mathbf{N}$, it follows that

$$(p_n^{(\lambda)}(x,y))_{n\in\mathbf{N}} \in \ell_\infty$$

and

$$\|(p_n^{(\lambda)}(x,y))_{n\in\mathbf{N}}\| \leq (\|x\| + \|y\|)/|\lambda|(1-|\lambda|).$$

Consequently, the linear operator $P_\lambda : c_0 \times \ell_\infty \longrightarrow c_0 \times \ell_\infty$ (where $P_\lambda(x,y) = \left(x, \left(p_n^{(\lambda)}(x,y)\right)_{n\in\mathbf{N}}\right)$ for any $(x,y) \in c_0 \times \ell_\infty$) is continuous and $\|P_\lambda\| \leq 1 + 1/|\lambda|(1-|\lambda|)$. Moreover, since, for any $(x,y) \in c_0 \times \ell_\infty$,

$$\sum_{j=0}^{+\infty} \lambda^j p_{g(j,n)}^{(\lambda)}(x,y) = p_{2n}^{(\lambda)}(x,y) + \sum_{j=1}^{+\infty} \lambda^j p_{g(j,n)}^{(\lambda)}(x,y) =$$

$$= -x_n/\lambda - \sum_{j=1}^{+\infty} \lambda^j y_{g(j,n)} + \sum_{j=1}^{+\infty} \lambda^j y_{g(j,n)} = -x_n/\lambda$$

for any $n \in \mathbf{N}$, it follows that $\operatorname{Im} P_\lambda \subset \operatorname{Im}(\lambda I_{c_0\times\ell_\infty} - A)$. Therefore, since, for any $(x,y) \in \operatorname{Im}(\lambda I_{c_0\times\ell_\infty} - A)$ and for any $n \in \mathbf{N}$, $p_{2n}^{(\lambda)}(x,y) = -x_n/\lambda - \sum_{j=1}^{+\infty} \lambda^j y_{g(j,n)} = y_{g(0,n)} = y_{2n}$, $P_\lambda(x,y) = (x,y)$ for any $(x,y) \in \operatorname{Im}\,(\lambda I_{c_0\times\ell_\infty} - A)$. Consequently, P_λ is a continuous projection on $c_0 \times \ell_\infty$ and $\mathrm{Im}\ \mathrm{P}_\lambda = \mathrm{Im}\ (\lambda I_{c_0\times\ell_\infty} - A)$, so that $\lambda \notin \sigma_{\mathrm{le}}(A)$ (see [CPY], (4.3.4)).

We have thus proved that

$$\{0\} \cup \partial B_{\mathbf{C}}(0,1) = \sigma_{\mathrm{le}}(A) \cap \sigma_{\mathrm{re}}(A) = \left(\sigma_{\mathrm{le}}(A) \cap \sigma_{\mathrm{re}}(A)\right) \cup \sigma_p^0(A).$$

Hence

$$\Gamma_2(A) = \{0\} \neq \emptyset. \quad \square$$

Since $\Gamma_2(A)$ is not always contained in $\Gamma_3(A)$ and $\Gamma_2(A) \subset \Gamma_4(A)$, it follows that $\Gamma_4(A)$ is not always contained in $\Gamma_3(A)$.

The following example shows that $\Gamma_3(A)$ is not always contained in $\Gamma_2(A)$.

EXAMPLE 1.12 .— Let K be a compact nonempty connected subset of the complex plane and let $(\lambda_n)_{n\in\mathbf{N}}$ be a sequence of elements of K such that $\overline{\{\lambda_n\}_{n\in\mathbf{N}}} = K$.

We consider the complex Banach space $\ell_\infty(\mathbf{N}, c_0 \times \ell_\infty) = \{(x_n)_{n\in\mathbf{N}} : x_n \in c_0 \times \ell_\infty$ for any $n \in \mathbf{N}$ and $\sup\{\|x_n\| : n \in \mathbf{N}\} < +\infty\}$, with the canonical supremum norm. Since K is compact, the sequence $(\lambda_n)_{n\in\mathbf{N}}$ is bounded, so that, if A is the operator of Example 1.11 and $\delta \in \mathbf{R}_+$ is such that $\delta > \operatorname{diam} K$,
$((\lambda_n I_{c_0\times\ell_\infty} + \delta A)x_n)_{n\in\mathbf{N}} \in \ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)$
and $\|((\lambda_n I_{c_0\times\ell_\infty} + \delta A)x_n)_{n\in\mathbf{N}}\| \leq (\|A\| + \sup_{n\in\mathbf{N}} |\lambda_n|)\|(x_n)_{n\in\mathbf{N}}\|$ for any $(x_n)_{n\in\mathbf{N}} \in \ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)$.
Consequently, the linear operator $T : \ell_\infty(\mathbf{N}, c_0 \times \ell_\infty) \longrightarrow \ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)$ (where $T(x_n)_{n\in\mathbf{N}} = ((\lambda_n I_{c_0\times\ell_\infty} + \delta A)x_n)_{n\in\mathbf{N}}$ for any $(x_n)_{n\in\mathbf{N}} \in \ell_\infty(\mathbf{N}, c_0 \times \ell_\infty))$ is continuous.

For any $n \in \mathbf{N}$, we consider two closed subspaces of $\ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)$,

$$X_n = \{(x_k)_{k\in\mathbf{N}} \in \ell_\infty(\mathbf{N}, c_0 \times \ell_\infty) : x_k = 0 \text{ if } k \neq n\}$$

and

$$Y_n = \{(x_k)_{k\in\mathbf{N}} \in \ell_\infty(\mathbf{N}, c_0 \times \ell_\infty) : x_n = 0\}.$$

Obviously X_n is isomorphic to $c_0 \times \ell_\infty$ and, since both X_n and Y_n are invariant under T, from [TL], V, 5.4 it follows that, if T_n denotes the restriction of T to X_n,

$$\sigma(T) \supset \sigma(T_n) = \sigma\left(\lambda_n I_{c_0\times\ell_\infty} + \delta A\right) = \overline{B_{\mathbf{C}}(\lambda_n, \delta)},$$

(see Example 1.11 and [TL], V, 3.4).

Therefore $\sigma(T) \supset \overline{\bigcup_{n\in\mathbf{N}} \overline{B_{\mathbf{C}}(\lambda_n, \delta)}}$.

We put $K_1 = \{\lambda \in \mathbf{C} : \operatorname{dist}(\lambda, K) \leq \delta\}$; it is not difficult to verify that $K_1 = \overline{\bigcup_{n\in\mathbf{N}} \overline{B_{\mathbf{C}}(\lambda_n, \delta)}} \subset \sigma(T)$ and that K_1 is connected.

We prove that

$$\mathbf{C}\backslash K_1 = \{\lambda \in \mathbf{C} : \operatorname{dist}(\lambda, K) > \delta\} \subset \rho(T).$$

For any $\lambda \in \mathbf{C}\backslash K_1$, there exists $\varepsilon > 0$ such that $|\lambda - \lambda_n| \geq \delta(1+\varepsilon)$ for any $n \in \mathbf{N}$. Since $|\lambda - \lambda_n|/\delta \geq 1+\varepsilon$ for any $n \in \mathbf{N}$, it follows that dist $((\lambda-\lambda_n)/\delta,\, \sigma(A)) =$ dist $((\lambda-\lambda_n)/\delta,\, \overline{B_{\mathbf{C}}(0,1)}) \geq \varepsilon$ for any $n \in \mathbf{N}$.

Therefore, by [DS], VII, 6.11, $\sup\{\|R((\lambda-\lambda_n)/\delta,\, A)\| \,:\, n \in \mathbf{N}\} < +\infty$, so that $((1/\delta)R((\lambda-\lambda_n)/\delta,\, A)x_n)_{n\in\mathbf{N}} \in \ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)$ for any $(x_n)_{n\in\mathbf{N}} \in \ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)$ and the linear operator

$$F_\lambda \,:\, \ell_\infty(\mathbf{N}, c_0 \times \ell_\infty) \longrightarrow \ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)$$

(where$F_\lambda(x_n)_{n\in\mathbf{N}} = ((1/\delta)R((\lambda-\lambda_n)/\delta,\, A)x_n)_{n\in\mathbf{N}}$ for any $(x_n)_{n\in\mathbf{N}} \in \ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)$) is continuous.

It is not difficult to verify that $F_\lambda\left(\lambda I_{\ell_\infty(\mathbf{N},c_0\times\ell_\infty)} - T\right) = I_{\ell_\infty(\mathbf{N},c_0\times\ell_\infty)} = \left(\lambda I_{\ell_\infty(\mathbf{N},c_0\times\ell_\infty)} - T\right) F_\lambda$ for any $\lambda \in \mathbf{C}\backslash K_1$.

Therefore $\mathbf{C}\backslash K_1 \subset \rho(T)$ and $R(\lambda, T) = F_\lambda$ for any $\lambda \in \mathbf{C}\backslash K_1$.

We have thus proved that

$$\sigma(T) = K_1.$$

We put $K_0 = \{\lambda \in \mathbf{C} \,:\, \sup\{|\lambda-\mu| \,:\, \mu \in K\} < \delta\}$; obviously, K_0 is open and contained in K_1 and, as diam $K < \delta$, $K \subset K_0$. It is not difficult to verify that K_0 is a convex set.

We prove that

$$K_0 \subset \rho_{s-F}(T).$$

We recall that $\|(\mu I_{c_0\times\ell_\infty} - A)x\| \geq (1-|\mu|)\|x\|/2$ for any $x \in c_0 \times \ell_\infty$ and for any $\mu \in B_{\mathbf{C}}(0,1)$ (see Example 1.11). For any $\lambda \in K_0$, there exists $\varepsilon \in (0,1)$ such that $\sup\{|\lambda-\lambda_n| \,:\, n \in \mathbf{N}\} \leq \delta(1-\varepsilon)$; consequently, for any $(x_n)_{n\in\mathbf{N}} \in \ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)$,

$$\begin{aligned}
&\left\| \left(\lambda I_{\ell_\infty(\mathbf{N},c_0\times\ell_\infty)} - T\right)(x_n)_{n\in\mathbf{N}}\right\| = \\
&\quad = \delta\|((((\lambda-\lambda_n)/\delta)I_{c_0\times\ell_\infty} - A)x_n)_{n\in\mathbf{N}}\| = \\
&\quad = \delta\sup\{\|(((\lambda-\lambda_n)/\delta)I_{c_0\times\ell_\infty} - A)x_n\| \,:\, n \in \mathbf{N}\} \geq \\
&\qquad \geq \delta\sup\{((1-|\lambda-\lambda_n|/\delta)/2)\|x_n\| \,:\, n \in \mathbf{N}\} = \\
&\quad = \sup\{((\delta-|\lambda-\lambda_n|)/2)\|x_n\| \,:\, n \in \mathbf{N}\} \geq \\
&\qquad \geq (\delta\varepsilon/2)\sup\{\|x_n\| \,:\, n \in \mathbf{N}\} = \\
&\quad = (\delta\varepsilon/2)\|(x_n)_{n\in\mathbf{N}}\|.
\end{aligned}$$

Therefore $\ker\left(\lambda I_{\ell_\infty(\mathbf{N},c_0\times\ell_\infty)} - T\right) = \{0\}$ and $\mathrm{Im}(\lambda I_{\ell_\infty(\mathbf{N},c_0\times\ell_\infty)} - T)$ is closed for any $\lambda \in K_0$ and, consequently, $K_0 \subset \rho_{s-F}(T)$.

For any $\lambda \in K_0$, since $(c_0 \times \ell_\infty)/\mathrm{Im}\,(((\lambda - \lambda_n)/\delta)I_{c_0 \times \ell_\infty} - A)$ is infinite-dimensional (see Example 1.11) and obviously it is isomorphic to a subspace of

$$\ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)/\mathrm{Im}\left(\lambda I_{\ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)} - T\right)$$

for any $n \in \mathbf{N}$, it follows that

$$\ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)/\mathrm{Im}\left(\lambda I_{\ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)} - T\right)$$

is infinite-dimensional, so that $\mathrm{ind}\left(\lambda I_{\ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)} - T\right) = -\infty$.

Hence

$$K_0 \subset \rho_{s-F}^{-\infty}(T).$$

We prove that

$$K \subset \sigma_{\mathrm{le}}(A) \cap \sigma_{\mathrm{re}}(A).$$

For any $\lambda \in \mathbf{C}$, we prove that, if $\mathrm{Im}\left(\lambda I_{\ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)} - T\right)$ is the range of a continuous projection on $\ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)$, $\mathrm{Im}\,(((\lambda - \lambda_n)/\delta)I_{c_0 \times \ell_\infty} - A)$ is the range of a continuous projection on $c_0 \times \ell_\infty$ for any $n \in \mathbf{N}$.

Suppose that $\mathrm{Im}\left(\lambda I_{\ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)} - T\right)$ is the range of a continuous projection P on $\ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)$.

For any $n \in \mathbf{N}$, we define the operators

$$P_n \,:\, \ell_\infty(\mathbf{N}, c_0 \times \ell_\infty) \longrightarrow c_0 \times \ell_\infty$$

and

$$J_n \,:\, c_0 \times \ell_\infty \longrightarrow \ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)$$

in the following way: $P_n(x_k)_{k\in\mathbf{N}} = x_n$ for any $(x_k)_{k\in\mathbf{N}} \in \ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)$ and $J_n x = (\delta_{kn} x)_{k\in\mathbf{N}}$ for any $x \in c_0 \times \ell_\infty$.

Obviously, $P_n J_n = I_{c_0 \times \ell_\infty}$,

$$P_n(\mathrm{Im}\left(\lambda I_{\ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)} - T\right)) \subset \mathrm{Im}\,(((\lambda - \lambda_n)/\delta)I_{c_0 \times \ell_\infty} - A)$$

and

$$J_n(\mathrm{Im}\,(((\lambda - \lambda_n)/\delta)I_{c_0 \times \ell_\infty} - A)) \subset \mathrm{Im}\left(\lambda I_{\ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)} - T\right).$$

If we consider the linear and continuous operator $P_n P J_n$ on $c_0 \times \ell_\infty$, it follows that

$$\mathrm{Im} P_n P J_n \subset P_n(\,\mathrm{Im} P) \subset \mathrm{Im}\,(((\lambda - \lambda_n)/\delta)I_{c_0 \times \ell_\infty} - A).$$

Moreover, for any $x \in \mathrm{Im}\,(((\lambda - \lambda_n)/\delta)I_{c_0 \times \ell_\infty} - A)$,

$$J_n x \in \mathrm{Im}\left(\lambda I_{\ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)} - T\right),$$

so that

$$P_n P J_n x = P_n J_n x = x.$$

Therefore $P_n P J_n$ is a continuous projection and

$$\mathrm{Im} P_n P J_n = \mathrm{Im}\,(((\lambda - \lambda_n)/\delta)I_{c_0 \times \ell_\infty} - A).$$

We have thus proved that, if $\mathrm{Im}\left(\lambda I_{\ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)} - T\right)$ is the range of a continuous projection on $\ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)$, $\mathrm{Im}\,(((\lambda - \lambda_n)/\delta)I_{c_0 \times \ell_\infty} - A)$ is the range of a continuous projection on $c_0 \times \ell_\infty$ for any $n \in \mathbf{N}$.

Since Im A is not the range of any continuous projection on $c_0 \times \ell_\infty$ (see Example 1.11), it follows that $\mathrm{Im}\left(\lambda_n I_{\ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)} - T\right))$ is not the range of any continuous projection on $\ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)$ for any $n \in \mathbf{N}$, so that, since $\{\lambda_n\}_{n \in \mathbf{N}} \subset K_0 \subset \rho_{s-F}^{-\infty}(T)$, $\{\lambda_n\}_{n \in \mathbf{N}} \subset \sigma_{\mathrm{le}}(T) \cap \sigma_{\mathrm{re}}(T)$ by [CPY], (4.3.4).

Therefore

$$\sigma_{\mathrm{le}}(T) \cap \sigma_{\mathrm{re}}(T) \supset \overline{\{\lambda_n\}_{n \in \mathbf{N}}} = K.$$

We prove that

$$K_0 \cap \sigma_{\mathrm{le}}(T) \cap \sigma_{\mathrm{re}}(T) = K.$$

Let $\lambda \in K_0 \backslash K$. For any $n \in \mathbf{N}$, we put $\mu_n = (\lambda - \lambda_n)/\delta$. Since $\lambda \in K_0 \backslash K$, it follows that $|\mu_n| \geq (1/\delta)\,\mathrm{dist}\,(\lambda, K) > 0$ for any $n \in \mathbf{N}$ and there exists $\varepsilon \in (0,1)$ such that $|\mu_n| \leq 1 - \varepsilon$ for any $n \in \mathbf{N}$.

Obviously,

$$\begin{aligned}
&\mathrm{Im}\left(\lambda I_{\ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)} - T\right) \subset \\
&\subset \{(x_n)_{n \in \mathbf{N}} \in \ell_\infty(\mathbf{N}, c_0 \times \ell_\infty) \,:\, x_n \in \mathrm{Im}\,(\mu_n I_{c_0 \times \ell_\infty} - A) \text{ for any } n \in \mathbf{N}\};
\end{aligned}$$

we prove that the equality holds.

Let $(x_n)_{n \in \mathbf{N}} \in \ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)$ be such that $x_n \in \mathrm{Im}\,(\mu_n I_{c_0 \times \ell_\infty} - A)$ for any $n \in \mathbf{N}$.

For any $n \in \mathbf{N}$, there exists $y_n \in c_0 \times \ell_\infty$ such that

$$x_n = (\mu_n I_{c_0 \times \ell_\infty} - A)\, y_n,$$

so that (see Example 1.11)

$$\|x_n\| \geq ((1-|\mu_n|)/2)\|y_n\| \geq (\varepsilon/2)\|y_n\|$$

and, consequently, $\|y_n\| \leq (2/\varepsilon)\|(x_k)_{k\in\mathbf{N}}\|$.

Therefore $(y_n)_{n\in\mathbf{N}} \in \ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)$.

In addition,

$$\left(\lambda I_{\ell_\infty(\mathbf{N},c_0\times\ell_\infty)} - T\right)(y_n/\delta)_{n\in\mathbf{N}} = ((\mu_n I_{c_0\times\ell_\infty} - A)y_n)_{n\in\mathbf{N}} = (x_n)_{n\in\mathbf{N}},$$

so that

$$(x_n)_{n\in\mathbf{N}} \in \operatorname{Im}\left(\lambda I_{\ell_\infty(\mathbf{N},c_0\times\ell_\infty)} - T\right).$$

Therefore

$$\begin{aligned}&\operatorname{Im}\left(\lambda I_{\ell_\infty(\mathbf{N},c_0\times\ell_\infty)} - T\right) = \\ &= \{(x_n)_{n\in\mathbf{N}} \in \ell_\infty(\mathbf{N}, c_0 \times \ell_\infty) \,:\, x_n \in \operatorname{Im}(\mu_n I_{c_0\times\ell_\infty} - A) \text{ for any } n \in \mathbf{N}\}.\end{aligned}$$

For any $n \in \mathbf{N}$, we define the continuous projection P_{μ_n} on $c_0 \times \ell_\infty$ like in Example 1.11. We recall that

$$\|P_{\mu_n}\| \leq 1 + 1/|\mu_n|(1-|\mu_n|) \leq 1 + \delta/\varepsilon \text{ dist } (\lambda, K)$$

for any $n \in \mathbf{N}$ (see Example 1.11).

Consequently, $(P_{\mu_n}x_n)_{n\in\mathbf{N}} \in \ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)$ for any $(x_n)_{n\in\mathbf{N}} \in \ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)$ and the linear operator

$$P \,:\, \ell_\infty(\mathbf{N}, c_0 \times \ell_\infty) \longrightarrow \ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)$$

(where $P(x_n)_{n\in\mathbf{N}} = (P_{\mu_n}x_n)_{n\in\mathbf{N}}$ for any $(x_n)_{n\in\mathbf{N}} \in \ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)$) is a continuous projection. Moreover, since obviously $Px = x$ for any $x \in \operatorname{Im}\left(\lambda I_{\ell_\infty(\mathbf{N},c_0\times\ell_\infty)} - T\right)$ and $P_{\mu_n}x_n \in \operatorname{Im}(\mu_n I_{c_0\times\ell_\infty} - A)$ for any $n \in \mathbf{N}$ and for any $(x_k)_{k\in\mathbf{N}} \in \ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)$ (see Example 1.11), it follows that $\operatorname{Im} P = \operatorname{Im}\left(\lambda I_{\ell_\infty(\mathbf{N},c_0\times\ell_\infty)} - T\right)$.

We have thus proved that $\operatorname{Im}\left(\lambda I_{\ell_\infty(\mathbf{N},c_0\times\ell_\infty)} - T\right)$ is the range of a continuous projection on $\ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)$ for any $\lambda \in K_0\backslash K$. Consequently, by [CPY], (4.3.4), $K_0\backslash K \subset \mathbf{C}\backslash\sigma_{\mathrm{le}}(T)$.

Therefore

$$K_0 \cap \sigma_{\mathrm{le}}(T) \cap \sigma_{\mathrm{re}}(T) = K.$$

We prove that

$$K_1 \backslash K_0 \subset \sigma_{s-F}(T).$$

Let $\lambda \in K_1 \backslash K_0$. If there exists $n \in \mathbf{N}$ such that $|\lambda - \lambda_n| = \delta$, since $(\lambda - \lambda_n)/\delta \in \partial B_{\mathbf{C}}(0,1) = \sigma_{s-F}(A)$ (see Example 1.11) it follows that the operator T_n defined before is not a semi-Fredholm element of $L_c(X_n)$. Consequently, $\lambda \in \sigma_{s-F}(T)$ by Lemma 1.7.

If, instead, $|\lambda - \lambda_n| \neq \delta$ for any $n \in \mathbf{N}$, since $\inf\{|\lambda - \mu| : \mu \in K\} \leq \delta \leq \sup\{|\lambda - \mu| : \mu \in K\}$ and K is compact and connected, it follows that there exists $\mu_\lambda \in K \backslash \{\lambda_n\}_{n \in \mathbf{N}}$ such that $|\lambda - \mu_\lambda| = \delta$. Since $(\lambda - \mu_\lambda)/\delta \in \partial B_{\mathbf{C}}(0,1) = \partial\sigma(A)$ (see Example 1.11), by [TL], V, 4.1 there exists a sequence $(\omega_n)_{n \in \mathbf{N}}$ of elements of $c_0 \times \ell_\infty$ such that $\|\omega_n\| = 1$ for any $n \in \mathbf{N}$ and $(((\lambda - \mu_\lambda)/\delta) I_{c_0 \times \ell_\infty} - A)\omega_n$ converges to 0 as $n \to +\infty$. Since $\mu_\lambda \in K$, there exists a subsequence $(\lambda_{n_j})_{j \in \mathbf{N}}$ of $(\lambda_n)_{n \in \mathbf{N}}$ such that $\lambda_{n_j} \underset{n \to +\infty}{\longrightarrow} \mu_\lambda$.

Hence

$$\begin{aligned}\left(\lambda I_{\ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)} - T\right) (\delta_{kn_j}\omega_j)_{k \in \mathbf{N}} &= \\ &= \left(\delta_{kn_j}((\mu_\lambda - \lambda_{n_j}) I_{c_0 \times \ell_\infty} + \delta(((\lambda - \mu_\lambda)/\delta) I_{c_0 \times \ell_\infty} - A))\omega_j\right)_{k \in \mathbf{N}}\end{aligned}$$

converges to zero as $j \to +\infty$.

Besides, $\|(\delta_{kn_j}\omega_j)\|_{k \in \mathbf{N}} = \|\omega_j\| = 1$ for any $j \in \mathbf{N}$ and, as $|\lambda - \lambda_n| \neq \delta$ for any $n \in \mathbf{N}$, $((\lambda - \lambda_n)/\delta) I_{c_0 \times \ell_\infty} - A$ is one-to-one for any $n \in \mathbf{N}$ (see Example 1.11). Consequently, $\lambda I_{\ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)} - T$ is one-to-one, so that $\mathrm{Im}(\lambda I_{\ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)} - T)$ is not closed (see [TL], IV, 5.9).

Therefore

$$\lambda \in \sigma_{s-F}(T)$$

We have thus proved that

$$K_1 \backslash K_0 \subset \sigma_{s-F}(T).$$

Hence $\sigma(T) = K_1$, $\rho_{s-F}^{-\infty}(T) = K_0$, $\sigma_{s-F}(T) = K_1 \backslash K_0$, $\sigma_{\mathrm{le}}(T) \cap \sigma_{\mathrm{re}}(T) = K \cup (K_1 \backslash K_0)$.

Suppose now that K does not consists of a single point.

Let $\mu_0 \in K$ and let X be an infinite-dimensional Banach space. We consider the linear and continuous operator $T \oplus \mu_0 I_X$ on $\ell_\infty(\mathbf{N}, c_0 \times \ell_\infty) \oplus X$.

From [TL], V, 5.4 it follows that $\sigma(T \oplus \mu_0 I_X) = \sigma(T) \cup \sigma(\mu_0 I_X) = K_1$. Moreover, by Lemma 1.7, $\rho_{s-F}^{-\infty}(T \oplus \mu_0 I_X) = K_0 \backslash \{\mu_0\}$ and $\sigma_{s-F}(T \oplus \mu_0 I_X) = \{\mu_0\} \cup (K_1 \backslash K_0)$.

We prove that, for any $\lambda \in K_0 \backslash \{\mu_0\}$, $\operatorname{Im}\left(\lambda I_{\ell_\infty(\mathbf{N}, c_0 \times \ell_\infty) \oplus X} - T \oplus \mu_0 I_X\right)$ is the range of a continuous projection on $\ell_\infty(\mathbf{N}, c_0 \times \ell_\infty) \oplus X$ if and only if $\operatorname{Im}\left(\lambda I_{\ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)} - T\right)$ is the range of a continuous projection on $\ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)$.

Let $\lambda \in K_0 \backslash \{\mu_0\}$. From [TL], V, 5.2 it follows that

$$\operatorname{Im}\left(\lambda I_{\ell_\infty(\mathbf{N}, c_0 \times \ell_\infty) \oplus X} - T \oplus \mu_0 I_X\right) = \operatorname{Im}\left(\lambda I_{\ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)} - T\right) \oplus X.$$

Therefore, obviously, if

$$\operatorname{Im}\left(\lambda I_{\ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)} - T\right)$$

is the range of a continuous projection Q on $\ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)$,

$$\operatorname{Im}\left(\lambda I_{\ell_\infty(\mathbf{N}, c_0 \times \ell_\infty) \oplus X} - T \oplus \mu_0 I_X\right)$$

is the range of the continuous projection $Q \oplus I_X$ on $\ell_\infty(\mathbf{N}, c_0 \times \ell_\infty) \oplus X$.

Conversely, it is not difficult to verify that, if

$$\operatorname{Im}\left(\lambda I_{\ell_\infty(\mathbf{N}, c_0 \times \ell_\infty) \oplus X} - T \oplus \mu_0 I_X\right)$$

is the range of a continuous projection P on $\ell_\infty(\mathbf{N}, c_0 \times \ell_\infty) \oplus X$ and

$$J : \ell_\infty(\mathbf{N}, c_0 \times \ell_\infty) \longrightarrow \ell_\infty(\mathbf{N}, c_0 \times \ell_\infty) \oplus X$$

and

$$Q : \ell_\infty(\mathbf{N}, c_0 \times \ell_\infty) \oplus X \longrightarrow \ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)$$

are the natural maps,

$$\operatorname{Im}\left(\lambda I_{\ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)} - T\right)$$

is the range of the continuous projection QPJ on $\ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)$.

Therefore, since $K_0 \backslash \{\mu_0\} = \rho_{s-F}^{-\infty}(T \oplus \mu_0 I_X) \subset \rho_{s-F}^{-\infty}(T)$, from [CPY], (4.3.4) it follows that

$$\begin{aligned}
&(K_0 \backslash \{\mu_0\}) \cap \sigma_{\mathrm{le}}(T \oplus \mu_0 I_X) \cap \sigma_{\mathrm{re}}(T \oplus \mu_0 I_X) = \\
&\quad = (K_0 \backslash \{\mu_0\}) \cap \sigma_{\mathrm{le}}(T \oplus \mu_0 I_X) = \\
&\quad = (K_0 \backslash \{\mu_0\}) \cap \sigma_{\mathrm{le}}(T) = \\
&\quad = (K_0 \backslash \{\mu_0\}) \cap \sigma_{\mathrm{le}}(T) \cap \sigma_{\mathrm{re}}(T) = \\
&\quad = K_0 \backslash \{\mu_0\}.
\end{aligned}$$

Hence $\sigma(T \oplus \mu_0 I_X) = K_1$, $\rho_{s-F}^{-\infty}(T \oplus \mu_0 I_X) = K_0 \backslash \{\mu_0\}$, $\sigma_{s-F}(T \oplus \mu_0 I_X) = \{\mu_0\} \cup (K_1 \backslash K_0)$ and $\sigma_{\mathrm{le}}(T \oplus \mu_0 I_X) \cap \sigma_{\mathrm{re}}(T \oplus \mu_0 I_X) = K \cup (K_1 \backslash K_0)$.

Since $\sigma_{s-F}(T \oplus \mu_0 I_X) \cap K_0 = \{\mu_0\}$ and K_0 is an open set, it follows that $\{\mu_0\}$ is a component of $\sigma_{s-F}(T \oplus \mu_0 I_X)$. Hence $\mu_0 \in \Gamma_3(T \oplus \mu_0 I_X)$.

Since $\mu_0 \in K \subset \sigma_{\mathrm{le}}(T \oplus \mu_0 I_X) \cap \sigma_{\mathrm{re}}(T \oplus \mu_0 I_X)$, K is connected and does not consist of a single point, it follows that

$$C_{\mu_0}((\sigma_{\mathrm{le}}(T \oplus \mu_0 I_X) \cap \sigma_{\mathrm{re}}(T \oplus \mu_0 I_X)) \cup \sigma_p^0(T \oplus \mu_0 I_X)) \supsetneqq \{\mu_0\},$$

so that $\mu_0 \notin \Gamma_2(T \oplus \mu_0 I_X)$.

Therefore $\Gamma_3(T \oplus \mu_0 I_X)$ is not contained in $\Gamma_2(T \oplus \mu_0 I_X)$. □

Since $\Gamma_3(A)$ is not always contained in $\Gamma_2(A)$ and $\Gamma_3(A) \subset \Gamma_4(A)$, it follows that $\Gamma_4(A)$ is not always contained in $\Gamma_2(A)$.

We remark that Example 1.8, Example 1.9 and Example 1.10 have been given in Hilbert spaces, whereas necessarily the spaces of Example 1.11 and Example 1.12 are not Hilbert, as $\Gamma_2(A)$, $\Gamma_3(A)$ and $\Gamma_4(A)$ coincide in a Hilbert space.

THEOREM 1.13.— *Let X be a complex nonzero Banach space and let $A \in L_c(X)$. Then $\psi(A) \cup \overline{\rho_{s-F}^{\pm}(A)} = \Gamma_0(A) \cup \overline{\rho_{s-F}^{\pm}(A)} = \Gamma_1(A) \cup \overline{\rho_{s-F}^{\pm}(A)} \subset \Gamma_2(A) \cup \overline{\rho_{s-F}^{\pm}(A)} = \Gamma_3(A) \cup \overline{\rho_{s-F}^{\pm}(A)} = \Gamma_4(A) \cup \overline{\rho_{s-F}^{\pm}(A)} = \Gamma_5(A) \cup \overline{\rho_{s-F}^{\pm}(A)}$; moreover, if $\sigma(A) = \Gamma_3(A) \cup \overline{\rho_{s-F}^{\pm}(A)}$, all the sets above coincide.*

Proof.— From Theorem 1.6 it follows that $\Gamma_1(A) \cup \overline{\rho_{s-F}^{\pm}(A)} \subset \psi(A) \cup \cup \overline{\rho_{s-F}^{\pm}(A)} \subset \Gamma_5(A) \cup \overline{\rho_{s-F}^{\pm}(A)} \subset \Gamma_j(A) \cup \overline{\rho_{s-F}^{\pm}(A)} \subset \Gamma_4(A) \cup \overline{\rho_{s-F}^{\pm}(A)}$ for any $j = 2, 3$ and $\Gamma_1(A) \cup \overline{\rho_{s-F}^{\pm}(A)} \subset \Gamma_0(A) \cup \overline{\rho_{s-F}^{\pm}(A)} \subset \Gamma_5(A) \cup \overline{\rho_{s-F}^{\pm}(A)}$.

We prove that

$$\Gamma_4(A) \backslash \overline{\rho_{s-F}^{\pm}(A)} \subset \Gamma_5(A).$$

Since $\sigma_{s-F}(A) \subset \sigma_m(A) \subset \sigma_e(A) = \sigma_{s-F}(A) \cup \rho_{s-F}^{-\infty}(A) \cup \rho_{s-F}^{+\infty}(A)$, it follows that $\sigma_m(A) \backslash \overline{\rho_{s-F}^{\pm}(A)} = \sigma_{s-F}(A) \backslash \overline{\rho_{s-F}^{\pm}(A)} = \sigma_e(A) \backslash \overline{\rho_{s-F}^{\pm}(A)}$. Therefore, by Lemma 1.5, $C_\lambda(\sigma_e(A)) = \{\lambda\}$ for any $\lambda \in \sigma_m(A) \backslash \overline{\rho_{s-F}^{\pm}(A)}$ such that $C_\lambda(\sigma_m(A)) = \{\lambda\}$. It follows immediately that

$$\Gamma_4(A) \backslash \overline{\rho_{s-F}^{\pm}(A)} \subset \Gamma_5(A).$$

Therefore $\Gamma_4(A) \cup \overline{\rho_{s-F}^{\pm}(A)} \subset \Gamma_5(A) \cup \overline{\rho_{s-F}^{\pm}(A)}$ and, consequently, $\Gamma_2(A) \cup \cup \overline{\rho_{s-F}^{\pm}(A)} = \Gamma_3(A) \cup \overline{\rho_{s-F}^{\pm}(A)} = \Gamma_4(A) \cup \overline{\rho_{s-F}^{\pm}(A)} = \Gamma_5(A) \cup \overline{\rho_{s-F}^{\pm}(A)}$.

Obviously, $\psi(A)\backslash\overline{\rho^{\pm}_{s-F}(A)} \subset \Gamma_1(A)$.

Since $\overline{\overset{o}{\rho^{n}_{s-F}(A)}} \subset \overline{\rho^{\pm}_{s-F}(A)}$ for any $n \in \mathbf{Z}\backslash\{0\}$ and $\overline{\overset{o}{\rho^{0}_{s-F}(A)}} \subset \sigma(A)\backslash \backslash\overline{\rho^{\pm}_{s-F}(A)}$, it follows that

$$\left(\left(\sigma(A)\backslash\overline{\rho^{\pm}_{s-F}(A)}\right) \bigcup \left(\bigcup_{n\in\mathbf{Z}} \left(\overline{\overset{o}{\rho^{n}_{s-F}(A)}}\backslash\rho^{n}_{s-F}(A)\right)\right)\right)\backslash\overline{\rho^{\pm}_{s-F}(A)} =$$
$$= \sigma(A)\backslash\overline{\rho^{\pm}_{s-F}(A)}$$

and, consequently, $\Gamma_0(A)\backslash\overline{\rho^{\pm}_{s-F}(A)} \subset \Gamma_1(A)$.

Therefore

$$\psi(A) \cup \overline{\rho^{\pm}_{s-F}(A)} = \Gamma_0(A) \cup \overline{\rho^{\pm}_{s-F}(A)} = \Gamma_1(A) \cup \overline{\rho^{\pm}_{s-F}(A)}.$$

We prove that, if

$$\sigma(A) = \overline{\Gamma_3(A) \cup \rho^{\pm}_{s-F}(A)}, \quad \Gamma_3(A)\backslash\overline{\rho^{\pm}_{s-F}(A)} \subset \Gamma_1(A).$$

We remark that $\left(\overset{o}{\sigma}(A) \cap \rho^{0}_{s-F}(A)\right) \cap \left(\rho^{\pm}_{s-F}(A) \cup \Gamma_3(A)\right) = \emptyset$. Consequently, if $\Gamma_3(A) \cup \rho^{\pm}_{s-F}(A)$ is dense in $\sigma(A)$, $\overset{o}{\sigma}(A) \cap \rho^{0}_{s-F}(A) = \emptyset$ and therefore, since $\sigma(A) \cap \rho^{0}_{s-F}(A) = \left(\overset{o}{\sigma}(A) \cap \rho^{0}_{s-F}(A)\right) \cup \sigma^{0}_{p}(A)$, $\sigma(A) = \sigma_{s-F}(A) \cup \sigma^{0}_{p}(A) \cup \rho^{\pm}_{s-F}(A)$. It follows immediately that $\sigma(A)\backslash\overline{\rho^{\pm}_{s-F}(A)} = \left(\sigma_{s-F}(A) \cup \sigma^{0}_{p}(A)\right)\backslash\overline{\rho^{\pm}_{s-F}(A)}$, and therefore $\Gamma_3(A)\backslash\overline{\rho^{\pm}_{s-F}(A)} \subset \Gamma_1(A)$.

We have thus proved that, if $\sigma(A) = \overline{\Gamma_3(A) \cup \rho^{\pm}_{s-F}(A)}$ (which is equivalent to $\sigma(A) = \overline{\Gamma_2(A) \cup \rho^{\pm}_{s-F}(A)}$, $\sigma(A) = \overline{\Gamma_4(A) \cup \rho^{\pm}_{s-F}(A)}$, $\sigma(A) = \overline{\Gamma_5(A) \cup \rho^{\pm}_{s-F}(A)}$, $\psi(A) \cup \overline{\rho^{\pm}_{s-F}(A)} = \Gamma_j(A) \cup \overline{\rho^{\pm}_{s-F}(A)}$ for any $j = 0, \ldots, 5$. $\square$

We remark that, if $\overline{\Gamma_3(A) \cup \rho^{\pm}_{s-F}(A)} \subsetneq \sigma(A)$, $\overline{\psi(A) \cup \rho^{\pm}_{s-F}(A)}$ may be strictly contained in $\Gamma_5(A)$. The following is an example.

EXAMPLE 1.14 .— Let us consider the complex Hilbert space $\ell_2(\mathbf{Z})$ and the linear and continuous operator

$$U : \sum_{n\in\mathbf{Z}} x_n e_n \in \ell_2(\mathbf{Z}) \longrightarrow \sum_{n\in\mathbf{Z}\backslash\{0\}} x_n e_{n-1} \in \ell_2(\mathbf{Z})$$

(where $\{e_n\}_{n\in\mathbf{N}}$ denotes the canonical basis of $\ell_2(\mathbf{Z})$). We recall that $\sigma(U) = \overline{B_{\mathbf{C}}(0,1)}$, $\sigma_{s-F}(U) = \partial B_{\mathbf{C}}(0,1)$ and $B_{\mathbf{C}}(0,1) \subset \rho^0_{s-F}(U)$ (see [Ka], IV, 5.25).

If 0 denotes the null operator on $\ell_2(\mathbf{Z})$, we consider the linear and continuous operator $U \oplus 0$ on $\ell_2(\mathbf{Z}) \oplus \ell_2(\mathbf{Z})$. From [TL], V, 5.4 and Lemma 1.7 it follows that $\sigma(U \oplus 0) = \overline{B_{\mathbf{C}}(0,1)}$, $\sigma_{s-F}(U \oplus 0) = \{0\} \cup \partial B_{\mathbf{C}}(0,1)$ and $B_{\mathbf{C}}(0,1)\setminus\{0\} \subset \rho^0_{s-F}(U \oplus 0)$. Therefore $\rho^{\pm}_{s-F}(U \oplus 0) = \emptyset$ and $\sigma_e(U \oplus 0) = \{0\} \cup \partial B_{\mathbf{C}}(0,1)$. Consequently, $\Gamma_5(U \oplus 0) = \{0\}$ and $\psi(U \oplus 0) = \emptyset$, so that $\overline{\psi(U \oplus 0) \cup \rho^{\pm}_{s-F}(U \oplus 0)} = \emptyset \subsetneq \{0\} = \Gamma_5(U \oplus 0)$. □

The following result is an immediate consequence of Theorem 1.13, Theorem (1), Theorem (2) and of the remarks after Theorem (1) and Theorem (2).

COROLLARY 1.15.— *Let X be a complex nonzero Banach space and let $A \in L_C(X)$. Then the following conditions are equivalent:*

i) $\sigma(A) = \overline{\rho^{\pm}_{s-F}(A) \cup \psi(A)}$;

ii) $\sigma(A) = \overline{\rho^{\pm}_{s-F}(A) \cup \Gamma_0(A)}$;

iii) $\sigma(A) = \overline{\rho^{\pm}_{s-F}(A) \cup \Gamma_1(A)}$;

iv) $\sigma(A) = \overline{\rho^{\pm}_{s-F}(A) \cup \Gamma_2(A)}$;

v) $\sigma(A) = \overline{\rho^{\pm}_{s-F}(A) \cup \Gamma_3(A)}$;

vi) $\sigma(A) = \overline{\rho^{\pm}_{s-F}(A) \cup \Gamma_4(A)}$;

vii) $\sigma(A) = \overline{\rho^{\pm}_{s-F}(A) \cup \Gamma_5(A)}$.

The equivalent conditions i), ii), iii), iv), v), vi) *and* vii) *imply the following condition :*

viii) $A \in \Sigma(X)$.

Besides, if X is a Hilbert separable space, all the conditions i), ii), iii), iv), v), vi), vii) *and* viii) *are equivalent.* □

2.

If X is a complex nonzero Banach space and $A \in L_c(X)$, we put

$\delta_5(A) = \sup\{\inf\{|\lambda| : \lambda \in \omega\} : \omega$ is a component of $\sigma_e(A) \cup \sigma^0_p(A)\}$,

$\delta_2(A) = \sup\{\inf\{|\lambda| : \lambda \in \omega\} : \omega$ is a component of $\big(\sigma_{\mathrm{le}}(A) \cap \sigma_{\mathrm{re}}(A)\big) \cup$ $\cup \sigma_p^0(A)\}$ and $\beta(A) = \sup\{|\lambda| : \lambda \in \rho_{s-F}^{\pm}(A)\}$ (if $\rho_{s-F}^{\pm}(A) = \emptyset$, we put $\beta(A) = 0$).

We recall two recent characterizations of $R(X)$ for a separable Hilbert space, that will be useful afterwards.

THEOREM (3) ([CM], 2.6 and 2.5).— *Let X be a complex nonzero separable Hilbert space and let $A \in L_c(X)$; then $A \in R(X)$ iff $r(A) =$ $= \beta(A) \vee \delta_2(A)$.* □

THEOREM (4) ([CM], 2.6 and [AFHV], Th. 14.1).— *Let X be a complex nonzero separable Hilbert space and let $A \in L_c(X)$; then $A \in R(X)$ iff $r(A) = \beta(A) \vee \delta_5(A)$.* □

We remark that the proof of the sufficiency of the condition $r(A) =$ $= \beta(A) \vee \delta_5(A)$ for membership in $R(X)$, given in [AFHV], Th. 14.1, can be repeated without alterations in the general case of a complex nonzero Banach space. Therefore the condition $r(A) = \beta(A) \vee \delta_5(A)$ is at least sufficient for membership in $R(X)$ for any complex nonzero Banach space X.

DEFINITION 2.1.— *Let X be a complex nonzero Banach space and let $A \in L_c(X)$. We define:*

$\delta_*(A) = \sup\{|\lambda| : \lambda \in \varphi(A)\}$,

$\delta_0(A) = \sup\{\inf\{|\lambda| : \lambda \in \omega\}$: *$\omega$ is a component of* $\Big(\sigma(A) \backslash \overline{\rho_{s-F}^{\pm}(A)}\Big) \cup$ $\Big(\bigcup_{n \in \mathbf{Z}} \Big(\overset{o}{\overline{\rho_{s-F}^{n}(A)}} \backslash \rho_{s-F}^{n}(A)\Big)\Big)\}$

(if $\Big(\sigma(A) \backslash \overline{\rho_{s-F}^{\pm}(A)}\Big) \cup \Big(\bigcup_{n \in \mathbf{Z}} \Big(\overset{o}{\overline{\rho_{s-F}^{n}(A)}} \backslash \rho_{s-F}^{n}(A)\Big)\Big) = \emptyset$, *we put* $\delta_0(A) = 0$*),*

$\delta_1(A) = \sup\{\inf\{|\lambda| : \lambda \in \omega\}$: *$\omega$ is a component of* $\sigma(A) \backslash \overline{\rho_{s-F}^{\pm}(A)}\}$

(if $\sigma(A) \backslash \overline{\rho_{s-F}^{\pm}(A)} = \emptyset$, *we put* $\delta_1(A) = 0$*),*

$\delta_3(A) = \sup\{\inf\{|\lambda| : \lambda \in \omega\}$: *$\omega$ is a component of* $\sigma_{s-F}(A) \cup \sigma_p^0(A)\}$ *and*

$\delta_4(A) = \sup\{\inf\{|\lambda| : \lambda \in \omega\}$: *$\omega$ is a component of* $\sigma_m(A) \cup \sigma_p^0(A)\}$. □

We remark that $\delta_*(A) = \sup\{\inf\{|\lambda| : \lambda \in \omega\}$: ω is a component of $\sigma(A)\}$, $\delta_*(A) \leq r(A)$, $\delta_j(A) \leq r(A)$ for any $j = 0, \ldots, 5$ and, if X is a Hilbert space, $\delta_2(A) = \delta_3(A) = \delta_4(A)$ (because $\sigma_{s-F}(A) = \sigma_m(A) = \sigma_{\mathrm{le}}(A) \cap \sigma_{\mathrm{re}}(A)$ in a Hilbert space).

THEOREM 2.2.— *Let X be a complex nonzero Banach space and let $A \in L_c(X)$. Then $\delta_*(A) \leq \delta_5(A) \leq \delta_j(A) \leq \delta_4(A)$ for any $j = 2,3$ and $\delta_1(A) \leq \delta_0(A)$; moreover, $\delta_4(A) = \delta_2(A) \vee \delta_3(A)$.*

Proof.— In [CM], 2.4 the inequality $\delta_*(A) \leq \delta_5(A)$ is proved in the case of a separable Hilbert space. Since $\partial\sigma(A) \subset \sigma_{s-F}(A) \cup \sigma_p^0(A) \subset (\sigma_{\mathrm{le}}(A) \cap \sigma_{\mathrm{re}}(A)) \cup \sigma_p^0(A)$ in any Banach space, the proof of [CM], 2.4 can be extended to the general case of a complex nonzero Banach space, without alterations. Therefore $\delta_*(A) \leq \delta_5(A)$.

We prove that $\delta_5(A) \leq \delta_j(A)$ for any $j = 2,3$.

Let C be a component of $\sigma_e(A) \cup \sigma_p^0(A)$. If $C \cap \sigma_p^0(A) \neq \emptyset$, it follows that C consists of a single point of $\sigma_p^0(A)$, so that C is also a component of $\sigma_{s-F}(A) \cup \sigma_p^0(A)$ and $(\sigma_{\mathrm{le}}(A) \cap \sigma_{\mathrm{re}}(A)) \cup \sigma_p^0(A)$ and, consequently, $\delta_j(A) \geq \inf\{|\mu| : \mu \in C\}$ for any $j = 2,3$. If, instead, $C \subset \sigma_e(A)$, C is a component of $\sigma_e(A)$ and therefore, by [Č], III, 21 B.8, $\partial C \subset \partial\sigma_e(A) \subset \sigma_{s-F}(A) \subset \sigma_{\mathrm{le}}(A) \cap \sigma_{\mathrm{re}}(A)$. Hence there exists $\lambda \in C \cap \sigma_{s-F}(A) \subset C \cap \sigma_{\mathrm{le}}(A) \cap \sigma_{\mathrm{re}}(A)$. Since $\sigma_{s-F}(A) \subset \sigma_{\mathrm{le}}(A) \cap \sigma_{\mathrm{re}}(A) \subset \sigma_e(A)$, it follows that $C_\lambda(\sigma_{s-F}(A) \cup \sigma_p^0(A)) \subset C_\lambda((\sigma_{\mathrm{le}}(A) \cap \sigma_{\mathrm{re}}(A)) \cup \sigma_p^0(A)) \subset C$ and, consequently, as $\delta_3(A) \geq \inf\{|\mu| : \mu \in C_\lambda(\sigma_{s-F}(A) \cup \sigma_p^0(A))\}$ and $\delta_2(A) \geq \inf\{|\mu| : \mu \in C_\lambda((\sigma_{\mathrm{le}}(A) \cap \sigma_{\mathrm{re}}(A)) \cup \sigma_p^0(A))\}$, $\delta_j(A) \geq \inf\{|\mu| : \mu \in C\}$ for any $j = 2,3$.

Hence $\delta_j(A) \geq \sup\{\inf\{|\mu| : \mu \in \omega\}$: ω is a component of $\sigma_e(A) \cup \sigma_p^0(A)\} = \delta_5(A)$ for any $j = 2,3$.

Since any component of $\sigma_{s-F}(A)$ is a componet of $\sigma_m(A)$ (see Theorem 1.3) and the points of $\sigma_p^0(A)$ are isolated in $\sigma(A)$, it follows obviously that $\delta_3(A) \leq \delta_4(A)$.

We prove that

$$\delta_2(A) \leq \delta_4(A).$$

Let C be a component of $(\sigma_{\mathrm{le}}(A) \cap \sigma_{\mathrm{re}}(A)) \cup \sigma_p^0(A)$. Since, by Definition 1.2, any component of $(\sigma_{\mathrm{le}}(A) \cap \sigma_{\mathrm{re}}(A))$ has nonempty intersection with $\sigma_m(A)$, it follows that $C \cap (\sigma_m(A) \cup \sigma_p^0(A)) \neq \emptyset$. Consequently, C contains a component D of $\sigma_m(A) \cup \sigma_p^0(A)$, so that $\delta_4(A) \geq \inf\{|\lambda| : \lambda \in D\} \geq \inf\{|\lambda| : \lambda \in C\}$.

Hence $\delta_4(A) \geq \sup\{\inf\{|\lambda| : \lambda \in \omega\} : \omega$ is a component of $(\sigma_{\mathrm{le}}(A) \cap \sigma_{\mathrm{re}}(A)) \cup \sigma_p^0(A)\} = \delta_2(A)$.

We prove that

$$\delta_4(A) = \delta_2(A) \vee \delta_3(A).$$

We have proved that $\delta_2(A) \vee \delta_3(A) \leq \delta_4(A)$. Since any component of $\sigma_m(A)$ which is not a component of $\sigma_{s-F}(A)$ is a component of $\sigma_{\mathrm{le}}(A) \cap \sigma_{\mathrm{re}}(A)$ (see Theorem 1.3), and the points of $\sigma_p^0(A)$ are isolated in $\sigma(A)$, it follows that $\delta_4(A) = \sup\{\inf\{|\mu| : \mu \in \omega\} : \omega \text{ is a component of } \sigma_m(A) \cup \sigma_p^0(A)\} \leq \delta_2(A) \vee \delta_3(A)$.

Therefore

$$\delta_4(A) = \delta_2(A) \vee \delta_3(A).$$

We prove that

$$\delta_1(A) \leq \delta_0(A).$$

We put

$$s(A) = \left(\sigma(A) \backslash \overline{\rho_{s-F}^{\pm}(A)}\right) \bigcup \left(\bigcup_{n \in \mathbf{Z}} \left(\overline{\overset{o}{\rho_{s-F}^{n}(A)}} \backslash \rho_{s-F}^{n}(A)\right)\right)$$

and we recall that, as $\overline{\overset{o}{\rho_{s-F}^{0}(A)}} \bigcap \overline{\rho_{s-F}^{\pm}(A)} = \emptyset$,

$$s(A) = \left(\sigma(A) \backslash \overline{\rho_{s-F}^{\pm}(A)}\right) \bigcup \left(\bigcup_{n \in \mathbf{Z}\backslash\{0\}} \left(\overline{\overset{o}{\rho_{s-F}^{n}(A)}} \backslash \rho_{s-F}^{n}(A)\right)\right).$$

Hence, $s(A) \backslash \overline{\rho_{s-F}^{\pm}(A)} = \sigma(A) \backslash \overline{\rho_{s-F}^{\pm}(A)}$ and

$$s(A) \bigcap \left(\bigcup_{n \in \mathbf{Z}\backslash\{0\}} \overline{\overset{o}{\rho_{s-F}^{n}(A)}}\right) = \bigcup_{n \in \mathbf{Z}\backslash\{0\}} \left(\overline{\overset{o}{\rho_{s-F}^{n}(A)}} \backslash \rho_{s-F}^{n}(A)\right).$$

Therefore $\sigma(A) \backslash \overline{\rho_{s-F}^{\pm}(A)}$ is both open and closed in $s(A)$ and, consequently, any component of $\sigma(A) \backslash \overline{\rho_{s-F}^{\pm}(A)}$ is a component of $s(A)$. Since $\sigma(A) = \overline{\rho_{s-F}^{\pm}(A)}$ implies $\delta_1(A) = 0 \leq \delta_0(A)$ (see Definition 2.1), it follows that $\delta_1(A) \leq \delta_0(A)$. □

We shall prove that none of the inequalities enunciated in Theorem 2.2 can be inverted.

First of all we prove that, generally speaking, there is not any relationship between the chains $\delta_*(A) \leq \delta_5(A) \leq \delta_j(A) \leq \delta_4(A)$ $(j = 2, 3)$ and $\delta_1(A) \leq \delta_0(A)$.

The following example shows that $\delta_0(A)$ may be strictely smaller than $\delta_*(A)$.

EXAMPLE 2.3 .— We denote by S the unilateral left shift operator on ℓ_2 and by 0 the null operator on ℓ_2. Let us consider the operator $A = 0 \oplus (2I_{\ell_2} + S) \in L_c(\ell_2 \oplus \ell_2)$.

From [TL], V, 5.4 it follows that $\sigma(A) = \sigma(0) \cup (2I_{\ell_2} + S) = \{0\} \cup \overline{B_{\mathbf{C}}(2,1)}$ (see [Ha], Sol. 67 and [TL], V, 3.4). Hence the components of $\sigma(A)$ are $\{0\}$ and $\overline{B_{\mathbf{C}}(2,1)}$, so that $\delta_*(A) = \inf\{|\lambda| : \lambda \in \overline{B_{\mathbf{C}}(2,1)}\} = 1$.

Since $\rho_{s-F}(S) \cap \sigma(S) = \rho^1_{s-F}(S) = B_{\mathbf{C}}(0,1)$ (see [Ka], IV, 5.24), it follows that $\rho_{s-F}(2I_{\ell_2} + S) \cap \sigma(2I_{\ell_2} + S) = \rho^1_{s-F}(2I_{\ell_2} + S) = B_{\mathbf{C}}(2,1)$. Consequently, since 0 is not a semi-Fredholm operator, $\rho_{s-F}(A) \cap \sigma(A) = \rho^1_{s-F}(A) = B_{\mathbf{C}}(2,1)$ by Lemma 1.7 and therefore

$$\left(\sigma(A) \backslash \overline{\rho^{\pm}_{s-F}(A)}\right) \bigcup \left(\bigcup_{n \in \mathbf{Z}} \left(\overset{o}{\overline{\rho^n_{s-F}(A)}} \backslash \rho^n_{s-F}(A) \right) \right) = \{0\}.$$

Hence $\delta_0(A) = 0 < 1 = \delta_*(A)$.

The following example shows that $\delta_4(A)$ may be strictly smaller than $\delta_1(A)$.

EXAMPLE 2.4 .— Let $T \in L_c(\ell_2)$ be such that $\sigma(T) = [2,3]$. Since $\rho^{\pm}_{s-F}(T) \subset \overset{o}{\sigma}(T) = \emptyset$ and $\rho^0_{s-F}(T) \cap \sigma(T) = \left(\rho^0_{s-F}(T) \cap \overset{o}{\sigma}(T)\right) \cup \sigma^0_p(T) = \emptyset$, it follows that $\sigma(T) = \sigma_{s-F}(T)$.

We denote by S the unilateral left shift operator on ℓ_2. Let us consider the operator $A = (I_{\ell_2} + S) \oplus T \in L_c(\ell_2 \oplus \ell_2)$. From [TL], V, 5.4 it follows that $\sigma(A) = \sigma(I_{\ell_2} + S) \cup \sigma(T) = \overline{B_{\mathbf{C}}(1,1)} \cup [2,3]$ (see [Ha], Sol. 67 and [TL], V, 3.4).

Moreover, since $\rho_{s-F}(S) \cap \sigma(S) = \rho^1_{s-F}(S) = B_{\mathbf{C}}(0,1)$ (see [Ka], IV, 5.24), and therefore $\rho_{s-F}(I_{\ell_2} + S) \cap \sigma(I_{\ell_2} + S) = \rho^1_{s-F}(I_{\ell_2} + S) = B_{\mathbf{C}}(1,1)$, from Lemma 1.7 it follows that $\rho_{s-F}(A) \cap \sigma(A) = \rho^1_{s-F}(A) = B_{\mathbf{C}}(1,1)$. Hence $\sigma(A) \backslash \overline{\rho^{\pm}_{s-F}(A)} = \sigma(A) \backslash \overline{\rho^1_{s-F}(A)} = (2,3]$, so that $\delta_1(A) = \inf\{|\lambda| : \lambda \in (2,3]\} = 2$.

Since ℓ_2 is a Hilbert space, $\delta_4(A) = \delta_2(A) = \delta_3(A) = \sup\{\inf\{|\lambda| : \lambda \in \omega\} : \omega$ is a component of $\sigma_{s-F}(A) \cup \sigma^0_p(A)\}$. Since $\rho_{s-F}(A) \cap \sigma(A) = B_{\mathbf{C}}(1,1)$ and there are no isolated points in $\sigma(A)$, it follows that $\sigma_{s-F}(A) \cup \sigma^0_p(A) = \partial B_{\mathbf{C}}(1,1) \cup [2,3]$, which is connected. Hence $\delta_4(A) = \inf\{|\lambda| : \lambda \in \partial B_{\mathbf{C}}(1,1) \cup [2,3]\} = 0 < 2 = \delta_1(A)$. □

The following example proves that $\delta_1(A)$ does not always coincide with $\delta_0(A)$.

EXAMPLE 2.5 .— Let $A \in L_c(\ell_2 \oplus \ell_2)$ denote the operator of Example 2.3. We recall that $\sigma(A) = \{0\} \cup \overline{B_{\mathbf{C}}(2,1)}$ and $\rho_{s-F}(A) \cap \sigma(A) = \rho^1_{s-F}(A) = B_{\mathbf{C}}(2,1)$. Let us consider the operator

$$A \oplus 2I_{\ell_2} \in L_c(\ell_2 \oplus \ell_2 \oplus \ell_2).$$

From [TL], V, 5.4 and Lemma 1.7 it follows that $\sigma(A \oplus 2I_{\ell_2}) = \sigma(A) \cup \sigma(2I_{\ell_2}) = \{0\} \cup \overline{B_{\mathbf{C}}(2,1)}$ and $\rho_{s-F}(A \oplus 2I_{\ell_2}) \cap \sigma(A \oplus 2I_{\ell_2}) = \rho^1_{s-F}(A \oplus 2I_{\ell_2}) = B_{\mathbf{C}}(2,1) \backslash \{2\}$.

Hence

$$\sigma(A \oplus 2I_{\ell_2}) \backslash \overline{\rho^{\pm}_{s-F}(A \oplus 2I_{\ell_2})} = \{0\}$$

and

$$\overset{o}{\overline{\rho^1_{s-F}(A \oplus 2I_{\ell_2})}} \backslash \rho^1_{s-F}(A \oplus 2I_{\ell_2}) = \{2\},$$

so that

$$\left(\sigma(A \oplus 2I_{\ell_2}) \backslash \overline{\rho^{\pm}_{s-F}(A \oplus 2I_{\ell_2})}\right) \bigcup \bigcup \left(\bigcup_{n \in \mathbf{Z}} \overset{o}{\overline{(\rho^n_{s-F}(A \oplus 2I_{\ell_2})}} \backslash \rho^n_{s-F}(A \oplus 2I_{\ell_2})) \right) = \{0,2\}.$$

Consequently, $\delta_1(A \oplus 2I_{\ell_2}) = 0 < 2 = \delta_0(A \oplus 2I_{\ell_2})$. □

We remark that the direct sum with the null operator in Example 2.3 and Example 2.5 prevents the sets

$$\left(\sigma(A) \backslash \overline{\rho^{\pm}_{s-F}(A)}\right) \bigcup \left(\bigcup_{n \in \mathbf{Z}} \overset{o}{\overline{(\rho^n_{s-F}(A)}} \backslash \rho^n_{s-F}(A)) \right)$$

and, respectively, $\sigma(A \oplus 2I_{\ell_2}) \backslash \overline{\rho^{\pm}_{s-F}(A \oplus 2I_{\ell_2})}$ from being empty. Hence, the inequality $\delta_0(A) < \delta_*(A)$ in Example 2.3 and the inequality $\delta_1(A \oplus 2I_{\ell_2}) < \delta_0(A \oplus 2I_{\ell_2})$ in Example 2.5 are intrinsic and do not depend on the arbitrary definitions which make δ_1 and δ_0 equal to zero in the case of the related sets being empty (see Definition 2.1).

The following example proves that $\delta_*(A)$ does not always coincide with $\delta_5(A)$.

EXAMPLE 2.6 .— Let us consider the unilateral left shift operator S on ℓ_2. We recall that $\sigma(S) = \overline{B_{\mathbf{C}}(0,1)}$, $\rho_{s-F}(S) = \mathbf{C} \backslash \partial B_{\mathbf{C}}(0,1)$ and

$\rho^1_{s-F}(S) = B_{\mathbb{C}}(0,1)$ (see [Ka], IV, 5.24), so that $\sigma_e(S) = \partial B_{\mathbb{C}}(0,1)$ and $\sigma^0_p(S) = \emptyset$. Hence $\delta_*(S) = 0 < 1 = \delta_5(S)$. □

$\delta_5(A)$ may be strictly smaller than $\delta_2(A)$ and $\delta_3(A)$. The following is an example.

EXAMPLE 2.7 .— Let us consider the operator $A \in L_c(\ell_2)$ of Example 1.10. We recall that $\sigma(A) = \sigma_e(A) = \overline{B_{\mathbb{C}}(0,1)}$ and $\sigma_{s-F}(A) = \partial B_{\mathbb{C}}(0,1)$.

Therefore $\delta_5(A) = \inf\{|\lambda| : \lambda \in \overline{B_{\mathbb{C}}(0,1)}\} = 0$. Since ℓ_2 is a Hilbert space, it follows that $\delta_2(A) = \delta_3(A) = \inf\{|\lambda| : \lambda \in \partial B_{\mathbb{C}}(0,1)\} = 1 > 0 = \delta_5(A)$. □

An example of an operator A such that $\delta_*(A) < \delta_5(A) < \delta_2(A)$ is given in [CM], page 183.

Now we prove that, generally speaking, there is not any relationship of inequality between $\delta_2(A)$ and $\delta_3(A)$.

The following example proves that $\delta_3(A)$ may be strictly smaller than $\delta_2(A)$.

EXAMPLE 2.8 .— Let $A \in L_c(c_0 \times \ell_\infty)$ denote the operator of Example 1.11. We recall that $\sigma(A) = \overline{B_{\mathbb{C}}(0,1)}$, $\sigma_{s-F}(A) = \partial B_{\mathbb{C}}(0,1)$ and $\sigma_{\mathrm{le}}(A) \cap \sigma_{\mathrm{re}}(A) = \{0\} \cup \partial B_{\mathbb{C}}(0,1)$.

Let us consider the linear and continuous operator $I_{c_0 \times \ell_\infty} + A$. It is not difficult to verify that $\sigma(I_{c_0 \times \ell_\infty} + A) = 1 + \sigma(A) = \overline{B_{\mathbb{C}}(1,1)}$, $\sigma_{s-F}(I_{c_0 \times \ell_\infty} + A) = 1 + \sigma_{s-F}(A) = \partial B_{\mathbb{C}}(1,1)$ and $\sigma_{\mathrm{le}}(I_{c_0 \times \ell_\infty} + A) \cap \sigma_{\mathrm{re}}(I_{c_0 \times \ell_\infty} + A) = 1 + (\sigma_{\mathrm{le}}(A) \cap \sigma_{\mathrm{re}}(A)) = \{1\} \cup \partial B_{\mathbb{C}}(1,1)$. Hence $\delta_3(I_{c_0 \times \ell_\infty} + A) = \inf\{|\lambda| : \lambda \in \partial B_{\mathbb{C}}(1,1)\} = 0 < 1 = \delta_2(I_{c_0 \times \ell_\infty} + A)$.

LEMMA 2.9.— *Let X be a normed real space such that* $\dim X > 1$ *and let K be a convex subset of X, bounded if $\overset{o}{K} \neq \emptyset$. Then ∂K is path-connected.*

Proof.— If $\overset{o}{K} \neq \emptyset$, $\partial K = \overline{K}$ and consequently ∂K is convex. Hence, in particular, ∂K is path-connected.

If, instead, there exists $u_0 \in \overset{o}{K}$, K is bounded. We put $M = \sup\{\|x\| : x \in K\}$. For any $v \in \partial B_X(0,1)$ and for any $\lambda \in (M + \|u_0\|, +\infty)$, $\|u_0 + \lambda v\| \geq |\lambda| - \|u_0\| > M$, so that $u_0 + \lambda v \notin K$.

We define the function $\varepsilon : \partial B_X(0,1) \longrightarrow (0, M + \|u_0\|]$ (where $\varepsilon(v) = \sup\{\lambda \in (0,+\infty) : u_0 + \lambda v \in K\}$ for any $v \in \partial B_X(0,1)$). Obviously, $u_0 + \varepsilon(v)v \in \partial K$ for any $v \in \partial B_X(0,1)$.

Since $u_0 \in \overset{o}{K}$, it is not difficult to verify that, for any $u \in \overline{K}$, $u_0 + s(u - u_0) \in \overset{o}{K}$ for any $s \in [0, 1)$. Therefore $\{u_0 + \lambda v : \lambda \in [0, +\infty)\} \cap \partial K = \{u_0 + \varepsilon(v)v\}$ for any $v \in \partial B_X(0, 1)$ and, obviously, $\varepsilon((u - u_0)/\|u - u_0\|) = \|u - u_0\|$ for any $u \in \partial K$.

We prove that ε is a continuous function.

Suppose that $u \in \partial B_X(0, 1)$ and $(u_n)_{n\in\mathbf{N}}$ is a sequence of elements of $\partial B_X(0, 1)$ such that $u_n \underset{n\to+\infty}{\longrightarrow} u$. Since $\varepsilon(u_n) \in [0, M + \|u_0\|]$ for any $n \in \mathbf{N}$, there exists a subsequence $(u_{n_j})_{j\in\mathbf{N}}$ of $(u_n)_{n\in\mathbf{N}}$ and $\eta \in [0, M + \|u_0\|]$ such that $\varepsilon(u_{n_j})$ converges to η as $j \to +\infty$. Hence $u_0 + \varepsilon(u_{n_j})u_{n_j} \underset{n\to+\infty}{\longrightarrow} u_0 + \eta u$, so that, since $u_0 + \varepsilon(u_{n_j})u_{n_j} \in \partial K$ for any $j \in \mathbf{N}$, also $u_0 + \eta u \in \partial K$. Consequently, $0 < \eta = \varepsilon(u)$.

We have thus proved that, for any sequence $(u_n)_{n\in\mathbf{N}}$ in $\partial B_X(0, 1)$ which converges to $u \in \partial B_X(0, 1)$, there exists a subsequence $(u_{n_j})_{j\in\mathbf{N}}$ such that $\varepsilon(u_{n_j})$ converges to $\varepsilon(u)$ as $j \to +\infty$. Therefore ε is continuous.

Since $\dim X > 1$, $X\backslash\{0\}$ is path-connected. Hence, for any $(u, v) \in \partial K \times \partial K$, there exists a continuous function $\gamma : [0, 1] \longrightarrow X\backslash\{0\}$ such that $\gamma(0) = u - u_0$ and $\gamma(1) = v - u_0$. Since ε is continuous, also the function $p : [0, 1] \longrightarrow \partial K$ (where $p(t) = u_0 + \varepsilon(\gamma(t)/\|\gamma(t)\|) \subset /\|\gamma(t)\|$ for any $t\gamma(\epsilon)[0, 1]$) is continuous. Moreover, $p(0) = u_0 + \varepsilon((u - u_0)/\|u - u_0\|)(u - u_0)/\|u - u_0\| = u_0 + u - u_0 = u$ and $p(1) = u_0 + \varepsilon((v - u_0)/\|v - u_0\|)(v - u_0)/\|v - u_0\| = u_0 + v - u_0 = v$.

Hence ∂K is path-connected. □

LEMMA 2.10.— *Let X be a normed real space such that $\dim X > 1$, let K_1 be a connected subset of X and let K_0 be an open, bounded and convex subset of X, such that $\overline{K}_0 \subset K_1$. Then $K_1\backslash K_0$ is connected.*

Proof.— If $K_0 = \emptyset$, the thesis is immediate.

If $K_0 \neq \emptyset$, since K_0 is bounded and X is connected it follows that $\partial K_0 \neq \emptyset$. Moreover, ∂K_0 is path-connnected by Lemma 2.9 and, as K_0 is open and $\overline{K}_0 \subset K_1$, $\partial K_0 \subset K_1\backslash K_0$.

Let F_1 and F_2 be two closed sets in the relative topology of $K_1\backslash K_0$ such that $F_1 \cap F_2 = \emptyset$ and $F_1 \cup F_2 = K_1\backslash K_0$. Since ∂K_0 is connected and contained in $K_1\backslash K_0$, there exists $j \in \{1, 2\}$ such that $\partial K_0 \subset F_j$. It is not restrictive to suppose that $\partial K_0 \subset F_1$. We define $G_1 = F_1 \cup \overline{K}_0$ and $G_2 = F_2$. Since $K_1\backslash K_0$ is closed in the relative topology of K_1, also F_1 and F_2 are closed in the relative topology of K_1. Consequently, G_1 and G_2 are closed in the relative topology of K_1. Moreover, $G_1 \cap G_2 = \emptyset$ (because $F_1 \cap F_2 = \emptyset$, $F_2 \subset K_1 \backslash K_0$ and $\partial K_0 \subset F_1$) and $G_1 \cup G_2 = F_1 \cup F_2 \cup \overline{K}_0 = (K_1 \backslash K_0) \cup \overline{K}_0 = K_1$. Since $G_1 \neq \emptyset$ and K_1 is connected, it follows that $F_2 = G_2 = \emptyset$. Hence $K_1 \backslash K_0$ is connected. □

The following example shows that $\delta_2(A)$ may be strictly smaller than $\delta_3(A)$.

EXAMPLE 2.11 .— Let K be a nonempty connected compact subset of the complex plane such that $\mathrm{dist}(0, K) + \max\{|\lambda| : \lambda \in K\} > \mathrm{diam} K$ (which implies that $0 \notin K$) and let $\delta \in \mathbf{R}_+$ be such that $\mathrm{diam} K < \delta \leq \mathrm{dist}(0, K) + \max\{|\lambda| : \lambda \in K\}$.

We define $K_0 = \{\lambda \in \mathbf{C} : \sup\{|\lambda - \mu| : \mu \in K\} < \delta\}$ and $K_1 = \{\lambda \in \mathbf{C} : \mathrm{dist}(\lambda, K) \leq \delta\}$, like in Example 1.12. We recall that $K \subset K_0 \subset K_1$, K_0 is convex and K_1 is connected.

Hence, since K_0 is open, $K_0 \subset K_1$, K_1 is closed and the complex plane is a 2-dimensional real normed space, $K_1 \backslash K_0$ is connected by Lemma 2.10.

Since $K \subset K_0$, K is open and closed in the relative topology of $(K_1 \backslash K_0) \cup K$. Therefore the components of $K \cup (K_1 \backslash K_0)$ are K and $K_1 \backslash K_0$.

We prove that

$$\inf\{|\lambda| : \lambda \in K_1 \backslash K_0\} \leq \inf\{|\lambda| : \lambda \in K\}.$$

If $\mathrm{dist}(0, K) > \delta$, let $\lambda \in K$ be such that $|\lambda| = \mathrm{dist}(0, K)$. Then $|\lambda - (\lambda - \lambda\delta/|\lambda|)| = \delta$, so that $\lambda - \lambda\delta/|\lambda| \in K_1 \backslash K_0$. Moreover, $\inf\{|\mu| : \mu \in K_1 \backslash K_0\} \leq |\lambda - \lambda\delta/|\lambda|| = |\lambda| - \delta < |\lambda| = \mathrm{dist}(0, K) = \inf\{|\mu| : \mu \in K\}$.

If $\mathrm{dist}(0, K) \leq \delta \leq \max\{|\lambda| : \lambda \in K\}$, $0 \in K_1 \backslash K_0$, so that $\inf\{|\mu| : \mu \in K_1 \backslash K_0\} = 0 \leq \inf\{|\lambda| : \lambda \in K\}$.

If $\max\{|\mu| : \mu \in K\} < \delta$, let $\lambda \in K$ be such that $|\lambda| = \max\{|\mu| : \mu \in K\}$. Then $0 < \delta - |\lambda| \leq \mathrm{dist}(0, K)$ and $|\lambda + \lambda(\delta - |\lambda|)/|\lambda|| = \delta$, so that $\lambda - \lambda\delta/|\lambda| \in K_1 \backslash K_0$. Moreover, $\inf\{|\mu| : \mu \in K_1 \backslash K_0\} \leq |\lambda - \lambda\delta/|\lambda|| = \delta - |\lambda| \leq \mathrm{dist}(0, K) = \inf\{|\mu| : \mu \in K\}$.

Suppose now that K is not contained in $\partial B_{\mathbf{C}}(0, \mathrm{dist}(0, K))$, let $\mu_0 \in K$ be such that $|\mu_0| > \mathrm{dist}(0, K)$ and let X be an infinite-dimensional Banach space.

We define the operator

$$T : \ell_\infty(\mathbf{N}, c_0 \times \ell_\infty) \longrightarrow \ell_\infty(\mathbf{N}, c_0 \times \ell_\infty)$$

like in Example 1.12. We recall that $\sigma(T \oplus \mu_0 I_X) = K_1$, $\rho_{s-F}^{-\infty}(T \oplus \mu_0 I_X) = K_0 \backslash \{\mu_0\}$, $\sigma_{s-F}(T \oplus \mu_0 I_X) = \{\mu_0\} \cup (K_1 \backslash K_0)$ and

$$\sigma_{\mathrm{le}}(T \oplus \mu_0 I_X) \cap \sigma_{\mathrm{re}}(T \oplus \mu_0 I_X) = K \cup (K_1 \backslash K_0)$$

(see Example 1.12). Hence $\sigma_p^0(T \oplus \mu_0 I_X) = \emptyset$ and $\delta_2(T \oplus \mu_0 I_X) = \inf\{|\lambda| : \lambda \in K\} = \mathrm{dist}(0, K)$.

Since $\mu_0 \in K$, the components of $\sigma_{s-F}(T \oplus \mu_0 I_X)$ are $\{\mu_0\}$ and $K_1 \backslash K_0$. Hence $\delta_3(T \oplus \mu_0 I_X) = |\mu_0| > \mathrm{dist}(0, K) = \delta_2(T \oplus \mu_0 I_X)$. □

We remark that Example 2.3, Example 2.4, Example 2.5, Example 2.6 and Example 2.7 have been given in Hilbert spaces, whereas necessarily the spaces of Example 2.8 and Example 2.11 are not Hilbert, as $\delta_2(A)$, $\delta_3(A)$ and $\delta_4(A)$ coincide in a Hilbert space.

Obviously, Example 2.8 and Example 2.11 prove also that none of the two inequalities $\delta_2(A) \leq \delta_4(A)$ and $\delta_3(A) \leq \delta_4(A)$ can be inverted in a generic Banach space, since $\delta_4(A) = \delta_2(A) \vee \delta_3(A)$ (see Theorem 2.2).

THEOREM 2.12.— *Let X be a complex nonzero Banach space and let $A \in L_c(X)$. Then $\delta_*(A) \vee \beta(A) = \delta_0(A) \vee \beta(A) = \delta_1(A) \vee \beta(A) \leq \delta_2(A) \vee \beta(A) = \delta_3(A) \vee \beta(A) = \delta_4(A) \vee \beta(A) = \delta_5(A) \vee \beta(A)$.*

Proof.— From Theorem 2.2 it follows that $\delta_*(A) \vee \beta(A) \leq \delta_5(A) \vee \beta(A) \leq \delta_j(A) \vee \beta(A) \leq \delta_4(A) \vee \beta(A)$ for any $j = 2, 3$ and $\delta_1(A) \vee \beta(A) \leq \delta_0(A) \vee \beta(A)$.

We prove that, if $\delta_4(A) > \beta(A) \vee \sup\{|\lambda| : \lambda \in \sigma_p^0(A)\}$, $\delta_4(A) \leq \delta_5(A)$.

Let C be a component of $\sigma_m(A)$ such that $\inf\{|\lambda| : \lambda \in C\} > \beta(A) \vee \sup\{|\lambda| : \lambda \in \sigma_p^0(A)\}$. Hence $C \cap \overline{\rho_{s-F}^{\pm}(A)} = \emptyset$. Since $\sigma_{\mathrm{le}}(A) \cap \sigma_{\mathrm{re}}(A) \subset \sigma_{s-F}(A) \cup \rho_{s-F}^{+\infty}(A) \cup \rho_{s-F}^{-\infty}(A)$, it follows that $C \subset \sigma_{s-F}(A)$ and therefore C is a component of $\sigma_{s-F}(A)$ and of $\sigma_{s-F}(A) \backslash \overline{\rho_{s-F}^{\pm}(A)}$. Since $\sigma_e(A) \backslash \overline{\rho_{s-F}^{\pm}(A)} = \sigma_{s-F}(A) \backslash \overline{\rho_{s-F}^{\pm}(A)}$, C is a component of $\sigma_e(A)$ (and hence also of $\sigma_e(A) \cup \sigma_p^0(A)$) by Lemma 1.5, so that $\delta_5(A) \geq \inf\{|\lambda| : \lambda \in C\}$. Therefore $\delta_4(A) \leq \delta_5(A)$.

It follows immediately that

$$\delta_5(A) \vee \beta(A) = \delta_4(A) \vee \beta(A) = \delta_3(A) \vee \beta(A) = \delta_2(A) \vee \beta(A).$$

Since any component C of $\sigma(A)$ such that $\inf\{|\lambda| : \lambda \in C\} > \beta(A)$ is also a component of $\sigma(A) \backslash \overline{\rho_{s-F}^{\pm}(A)}$, it follows that $\delta_*(A) \vee \beta(A) \leq \delta_1(A) \vee \beta(A)$.

We prove that, if $\delta_0(A) > \beta(A)$, $\delta_0(A) \leq \delta_*(A)$.

Let C be a component of

$$\left(\sigma(A)\backslash\overline{\rho_{s-F}^{\pm}(A)}\right)\bigcup\left(\bigcup_{n\in\mathbf{Z}}\overset{o}{\overline{(\rho_{s-F}^{\pm}(A)}\backslash\rho_{s-F}^{n}(A))}\right)$$

such that $\inf\{|\lambda| : \lambda \in C\} > \beta(A)$. Then $C \subset \sigma(A)\backslash\{\lambda \in \mathbf{C} : \operatorname{dist}(\lambda, \overline{\rho_{s-F}^{\pm}(A)}) < \inf\{|\mu| : \mu \in C\} - \beta(A)\}$, which is closed and contained in $\sigma(A)\backslash\overline{\rho_{s-F}^{\pm}(A)}$. Consequently, C is closed and it is a component of $\sigma(A)\backslash\overline{\rho_{s-F}^{\pm}(A)}$, so that, by Lemma 1.5, C is a component of $\sigma(A)$. Hence $\delta_*(A) \geq \inf\{|\lambda| : \lambda \in C\}$, and therefore $\delta_*(A) \geq \delta_0(A)$.

It folllows immediately that $\delta_0(A) \vee \beta(A) \leq \delta_*(A) \vee \beta(A)$.

We have thus proved that

$$\begin{aligned}\delta_*(A) \vee \beta(A) = \delta_0(A) \vee \beta(A) = \delta_1(A) \vee \beta(A) \leq \delta_2(A) \vee \beta(A) = \\ = \delta_3(A) \vee \beta(A) = \delta_4(A) \vee \beta(A) = \delta_5(A) \vee \beta(A). \quad \square\end{aligned}$$

The following result is an immediate consequence of Theorem 2.12, Theorem (3), Theorem (4) and of the remarks after Theorem (4).

COROLLARY 2.13.— *Let X be a complex nonzero Banach space and let $A \in L_c(X)$. Then the following conditions are equivalent:*

i) $r(A) = \delta_*(A) \vee \beta(A)$;

ii) $r(A) = \delta_0(A) \vee \beta(A)$;

iii) $r(A) = \delta_1(A) \vee \beta(A)$.

The equivalent conditions i), ii) *and* iii) *imply the following conditions, which are equivalent:*

iv) $r(A) = \delta_2(A) \vee \beta(A)$;

v) $r(A) = \delta_3(A) \vee \beta(A)$;

vi) $r(A) = \delta_4(A) \vee \beta(A)$;

vii) $r(A) = \delta_5(A) \vee \beta(A)$.

The equivalent conditions iv), v), vi) *and* vii) *imply the following condition:*

viii) $A \in R(X)$.

Besides, if X is a Hilbert separable space, the conditions iv), v), vi), vii) *and* viii) *are equivalent.* $\square$

We remark that the conditions i), ii) and iii) are not equivalent to the conditions iv), v), vi), vii) and viii). Here is an example.

EXAMPLE 2.14 . — Let us consider the complex Banach space $\ell_2(\mathbf{Z})$ and the linear and continuous operator

$$U \,:\, \ell_2(\mathbf{Z}) \longrightarrow \ell_2(\mathbf{Z})$$

of Example 1.14. We recall that $\sigma(U) = \overline{B_{\mathbf{C}}(0,1)}$, $\sigma_{s-F}(U) = \partial B_{\mathbf{C}}(0,1)$ and $\rho^0_{s-F}(U) = \rho_{s-F}(U) = \mathbf{C}\backslash\partial B_{\mathbf{C}}(0,1)$.

Since $\rho^{\pm}_{s-F}(U) = \emptyset$, $\beta(U) = 0$. Moreover, $\delta_*(U) = 0$ and $\delta_3(U) = 1$. Hence $\delta_*(U) \vee \beta(U) = 0 < 1 = \delta_3(U) \vee \beta(U) = r(U)$, so that conditions iv), v), vi), vii) and viii) are satisfied, whereas conditions i), ii), and iii) are not satisfied. □

We have thus proved that the inequality of Theorem 2.12 may be strict and that the conditions i), ii) and iii) are not necessary for membership in $R(X)$, even if X is a separable Hilbert space.

3.

DEFINITION 3.1.— *Let X be a complex nonzero Banach space. We define*

$$R_0(X) = \{A \in L_c(X) \,:\, r(A) = \delta_3(A) \vee \beta(A)\}$$

and

$$\Sigma_0(X) = \{A \in L_c(X) \,:\, \sigma(A) = \overline{\rho^{\pm}_{s-F}(A) \cup \psi(A)}\}.$$

By Corollary 1.15 and Corollary 2.13, $R_0(X) \subset R(X)$, $\Sigma_0(X) \subset \Sigma(X)$ and , if X is a separable Hilbert space, $R_0(X) = R(X)$ and $\Sigma_0(X) = \Sigma(X)$. Therefore it is interesting to study algebraic and topological properties of $R_0(X)$ and $\Sigma_0(X)$.

We recall that, if X is a Hilbert space and $A \in L_c(X)$ is normal, $\ker(\lambda I_X - A) = \ker(\overline{\lambda} I_X - A^*)$ for any $\lambda \in \mathbf{C}$ (so that $\rho^{\pm}_{s-F}(A) = \emptyset$) and, for any $\lambda \in \sigma(A) \cap \rho^0_{s-F}(A)$, λ is a pole of first order of the resolvent (see [TL], VI.3, Prob.9), so that, since $\sigma(A) \cap \rho^0_{s-F}(A) = (\overset{o}{\sigma}(A) \cap \rho^0_{s-F}(A)) \cup \sigma^0_p(A)$, $\lambda \in \sigma^0_p(A)$. Therefore $\sigma(A) = \sigma_{s-F}(A) \cup \sigma^0_p(A)$. Hence, if $N(X)$ denotes the set of all linear, continuous and normal operators on X, $R_0(X) \cap N(X) = \pi(X) \cap N(X)$ and $\Sigma_0(X) \cap N(X) = \tau(X) \cap N(X)$ (see [B],1.5 and 2.4).

If X is a complex nonzero Banach space, since, for any $A \in L_c(X)$, $\psi(A) \subset \Gamma_3(A)$ (see Theorem 1.6) and $A \in \pi(X)$ if and only if $r(A) = \delta_*(A)$ (see [B], 1.5), which implies $r(A) = \delta_3(A)$ (see Corollary 2.13), it follows that $\tau(X) \subset \Sigma_0(X) \subset R_0(X)$ and $\pi(X) \subset R_0(X)$. It is immediate to remark that both $R_0(X)$ and $\Sigma_0(X)$ are closed with respect to the product with a complex number. The following example shows that, generally speaking, $\pi(X)$ and $R_0(X)$ are not invariant under translation (so that they are not always vector subspaces of $L_c(X)$).

EXAMPLE 3.2 .— Let X be a complex infinite-dimensional Hilbert space and let $A \in L_c(X)$ be such that $\sigma(A) = \partial B_{\mathbf{C}}(0,1)$. Then $\delta_*(A) = 1 = r(A)$, so that $A \in \pi(X) \subset R_0(X)$.

Nevertheless, we prove that $I_X + A \notin R_0(X)$.

Since $\rho^{\pm}_{s-F}(A) \subset \overset{o}{\sigma}(A) = \emptyset$ and $\rho^{0}_{s-F}(A) \cap \sigma(A) = \left(\rho^{0}_{s-F}(A) \cap \overset{o}{O}(A)\right) \cup \sigma^{0}_{p}(A) = \emptyset$, it follows that $\sigma_{s-F}(A) = \partial B_{\mathbf{C}}(0,1)$. Consequently, $\rho^{\pm}_{s-F}(I_X + A) = \emptyset$ and $\sigma_{s-F}(I_X + A) = \partial B_{\mathbf{C}}(1,1) = \sigma(I_X + A)$, so that $\beta(I_X + A) = 0 = \delta_3(I_X + A)$.

Therefore $\beta(I_X + A) \vee \delta_3(I_X + A) = 0 < 2 = r(I_X + A)$, so that $I_X + A \notin R_0(X)$ (and hence it does not belong to $\pi(X)$, either). □

Example 3.2 proves also that $\pi(X)$ is not always contained in $\Sigma_0(X)$. Hence, since, for instance, the unilateral left shift operator on ℓ_2 (see [Ka], IV, 5.24) belongs obviously to $\Sigma_0(\ell_2)\backslash\pi(\ell_2)$, there is not any relationship of inclusion, generally speaking, between $\Sigma_0(X)$ and $\pi(X)$. Since $\tau(X) \subset \pi(X)$, $\tau(X) \subset \Sigma_0(X)$, $\Sigma_0(X) \subset R_0(X)$ and $\pi(X) \subset R_0(X)$ it follows that, generally speaking, $\tau(X) \subsetneq \Sigma_0(X)$, $\tau(X) \subsetneq \pi(X)$, $\pi(X) \subsetneq R_0(X)$ and $\Sigma_0(X) \subsetneq R_0(X)$.

From [B], 2.13 it follows that, in particular, $\tau(X)$ is invariant under translation. It is easy to verify that $\Sigma_0(X)$ is invariant under translation, too. The following example shows that $\tau(X)$ and $\Sigma_0(X)$ are not always vector subspaces of $L_c(X)$ (a counterexample for $\tau(X)$ has been already given in [B], 3.1).

EXAMPLE 3.3 .— Let us consider the complex Hilbert space $\ell_2(\mathbf{Z})$. We denote by $\{e_n\}_{n\in\mathbf{Z}}$ the canonical basis of $\ell_2(\mathbf{Z})$. We define two linear and continuous operators on $\ell_2(\mathbf{Z})$,

$$T_0 : \sum_{n\in\mathbf{Z}} x_n e_n \in \ell_2(\mathbf{Z}) \longrightarrow \sum_{n\in\mathbf{Z}} x_{2n} e_{2n+1} \in \ell_2(\mathbf{Z})$$

and

$$T_1 : \sum_{n\in\mathbf{Z}} x_n e_n \in \ell_2(\mathbf{Z}) \longrightarrow \sum_{n\in\mathbf{Z}} x_{2n-1}e_{2n} \in \ell_2(\mathbf{Z}).$$

Since $T_0^2 = T_1^2 = 0$, it follows that $\sigma(T_0) = \sigma(T_1) = \{0\}$ and hence T_0 and T_1 belong to $\tau(\ell_2(\mathbf{Z})) \subset \Sigma_0(\ell_2(\mathbf{Z}))$.

We put $T = T_0 + T_1$. It is not difficult to verify that $T(x_n)_{n\in\mathbf{Z}} = \sum_{n\in\mathbf{Z}} x_n e_{n+1}$ for any $(x_n)_{n\in\mathbf{Z}} \in \ell_2(\mathbf{Z})$. We prove that $T \notin \Sigma_0(\ell_2(\mathbf{Z}))$ (so that it does not belong to $\tau(\ell_2(\mathbf{Z}))$, either).

We recall that $\sigma(T) = \partial B_{\mathbf{C}}(0,1)$ (see [Ha], Sol. 68). Consequently, $\psi(T) = \emptyset$ and, as $\overset{o}{\sigma}(T) = \emptyset$, $\rho^{\pm}_{s-F}(T) = \emptyset$. Therefore

$$T \notin \Sigma_0(\ell_2(\mathbf{Z})).\square$$

We remark that, genrally speaking, $\tau(X)$, $\pi(X)$, $\Sigma_0(X)$ and $R_0(X)$ are neither open nor closed subsets of $L_c(X)$. In [B], remarks after 3.3, we proved that, if X is an infinite-dimensional complex Hilbert space, $\pi(X)$ and $\tau(X)$ are not closed subsets of $L_c(X)$. Since $R_0(X) \cap N(X) = \pi(X) \cap N(X)$, such a proof can be used to show also that neither $\Sigma_0(X)$ nor $R_0(X)$ are closed subsets of $L_c(X)$.

[B], 3.4 proves that, if X is infinite-dimensional Hilbert space, $\tau(X)$ and $\pi(X)$ are not open subsets of $L_c(X)$. Since it is not restrictive to suppose that the operator A, defined in the proof of [B], 3.4, is diagonal, and $R_0(X) \cap N(X) = \pi(X) \cap N(X)$, it follows that neither $\Sigma_0(X)$ nor $R_0(X)$ are open subsets of $L_c(X)$.

Let X be a complex nonzero Banach space.

From [B], 3.8 and 3.11 it follows that $\pi(X)$ and, respectively, $\tau(X)$ are G_δ-sets.

We shall show that also $R_0(X)$ and $\Sigma_0(X)$ are G_δ-sets.

We recall that the function $\beta : L_c(X) \longrightarrow [0, +\infty)$ is lower semi-continuous (the proof follows easily from the stability of semi-Fredholm index, see [Ka], IV, 5.17). We recall also that the function $\delta_5 : L_c(X) \longrightarrow [0, +\infty)$ is lower semi-continuous (the proof of [CM], 2.2 can be extended without alterations to the general case of a complex nonzero Banach space).

THEOREM 3.4.— *Let X be a complex nonzero Banach space. Then $R_0(X)$ is a G_δ-set.*

Proof.— Since β and δ_5 are lower semi-continuous, also $\beta \vee \delta_5$ is a lower semi-continuous function. Hence $\beta \vee \delta_3 = \beta \vee \delta_5$ (see Theorem 2.12) is lower semi-continuous.

Since the spectral radius function is upper semi-continuous (see [R], (1.6.16)), it follows that the function $r - \beta \vee \delta_3 : L_c(X) \longrightarrow [0, +\infty)$ is upper semi-continuous. Therefore $(r - \beta \vee \delta_3)^{-1}([0, 1/n))$ is an open subset of $L_c(X)$ for any positive integer n and, consequently,

$$\bigcap_{n=1}^{+\infty} (r - \beta \vee \delta_3)^{-1}([0, 1/n)) = (r - \beta \vee \delta_3)^{-1}(\{0\}) = R_0(X)$$

(see Definition 3.1) is a G_δ-set. □

DEFINITION 3.5.— *Let X be a complex nonzero Banach space and let $\varepsilon \in \mathbf{R}_+$. We define $\sum_0^{(\varepsilon)}(X) = \{A \in L_c(X)$: for any $\lambda \in \sigma(A)$, if $B_{\mathbf{C}}(\lambda, \varepsilon) \cap \rho_{s-F}^{\pm}(A) = \emptyset$, $B_{\mathbf{C}}(\lambda, \varepsilon)$ contains a nonempty spectral set of $A\}$.*
□

LEMMA 3.6.— *Let X be a complex nonzero Banach space and let $\varepsilon \in \mathbf{R}_+$. Then $\Sigma_0^{(\varepsilon)}(X)$ is an open subset of $L_c(X)$.*

Proof.— Let $A \in \Sigma_0^{(\varepsilon)}(X)$. If we define

$$\sigma_\varepsilon = \{\lambda \in \sigma(A) : \operatorname{dist}\left(\lambda, \rho_{s-F}^{\pm}(A)\right) \geq \varepsilon\},$$

it follows that, for any $\lambda \in \sigma_\varepsilon$, $B_{\mathbf{C}}(\lambda, \varepsilon)$ contains a nonempty spectral set α_λ of A; since α_λ is closed, there exists $\eta(\lambda) \in (0, \varepsilon)$ such that $\alpha_\lambda \subset B_{\mathbf{C}}(\lambda, \eta(\lambda))$.

Since obviously

$$\sigma(A) \subset \left(\bigcup_{\lambda \in \rho_{s-F}^{\pm}(A)} B_{\mathbf{C}}(\lambda, \varepsilon) \right) \bigcup$$

$$\bigcup \sigma_\varepsilon \subset \left(\bigcup_{\lambda \in \rho_{s-F}^{\pm}(A)} B_{\mathbf{C}}(\lambda, \varepsilon) \right) \bigcup \left(\bigcup_{\lambda \in \sigma_\varepsilon} B_{\mathbf{C}}(\lambda, \varepsilon - \eta(\lambda)) \right)$$

and $\sigma(A)$ is compact, there exist $\lambda_1, \ldots, \lambda_n \in \rho_{s-F}^{\pm}(A)$ and $\mu_1, \ldots, \mu_m \in \sigma_\varepsilon$ such that

$$\sigma(A) \subset \left(\bigcup_{k=1}^{n} B_{\mathbf{C}}(\lambda_k, \varepsilon) \right) \bigcup \left(\bigcup_{k=1}^{m} B_{\mathbf{C}}(\mu_k, \varepsilon - \eta(\mu_k)) \right).$$

By [R], (1.6.16), [Ka], IV,5.17, [M], 1.3 and [Hy], Th. 2.15, there exists $\delta > 0$ such that, for any $T \in B_{L_c(X)}(A, \delta)$,

$$\sigma(T) \subset \left(\bigcup_{k=1}^{n} B_{\mathbf{C}}(\lambda_k, \varepsilon) \right) \bigcup \left(\bigcup_{k=1}^{m} B_{\mathbf{C}}(\mu_k, \varepsilon - \eta(\mu_k)) \right),$$

$\lambda_k \in \rho_{s-F}^{\pm}(T)$ for any $k = 1, \dots, n$ and $B_{\mathbf{C}}(\mu_j, \eta(\mu_j))$ contains a nonempty spectral set of T for any $j = 1, \dots, m$.

Let $T \in B_{L_c(X)}(A, \delta)$ and let $\lambda \in \sigma(T)$.

If $\lambda \in \bigcup_{k=1}^{n} B_{\mathbf{C}}(\lambda_k, \varepsilon)$, since $\lambda_k \in \rho_{s-F}^{\pm}(T)$ for any $k = 1, \dots, n$ it follows that $B_{\mathbf{C}}(\lambda, \varepsilon) \cap \rho_{s-F}^{\pm}(A) \neq \emptyset$. If there exists $k \in \{1, \dots, m\}$ such that $\lambda \in B_{\mathbf{C}}(\mu_k, \varepsilon - \eta(\mu_k))$, it follows that $B_{\mathbf{C}}(\mu_k, \eta(\mu_k)) \subset B_{\mathbf{C}}(\lambda, \varepsilon)$ and therefore, since $B_{\mathbf{C}}(\mu_k, \eta(\mu_k))$ contains a nonempty spectral set of T, $B_{\mathbf{C}}(\lambda, \varepsilon)$ contains a nonempty spectral set of T. Hence $T \in \Sigma_0^{(\varepsilon)}(X)$.

We have thus proved that $\Sigma_0^{(\varepsilon)}(X)$ is open.□

LEMMA 3.7.— *Let X be a complex nonzero Banach space. Then* $\Sigma_0(X) = \bigcap_{n=1}^{+\infty} \Sigma_0^{(1/n)}(X)$.

Proof.— From [HY], 2.4, Th. 2.15 it follows immediately that $\Sigma_0(X) \subset \bigcap_{n=1}^{+\infty} \Sigma_0^{(1/n)}(X)$.

Conversely, let $A \in \bigcap_{n=1}^{+\infty} \Sigma_0^{(1/n)}(X)$. Let U be an open subset of $\mathbf{C}$ such that $U \cap \sigma(A) \neq \emptyset$ and $U \cap \rho_{s-F}^{\pm}(A) = \emptyset$. It follows that $B_{\mathbf{C}}(\mu, \delta)$ contains a nonempty spectral set of A for any $\mu \in U \cap \sigma(A)$ and for any $\delta > 0$, so that, in particular, U contains a nonempty spectral set α of A. Let P denote the spectral projection associated with α (see [TL], page 321) and let A_0 denote the restriction of A to the invariant subspace Im P. Since $\sigma(A_0) = \alpha$ (see [TL],V,9.2), α is a spectral set of A and $\alpha \subset U$, it follows that $B_{\mathbf{C}}(\mu, \delta)$ contains a nonempty spectral set of A_0 for any $\mu \in \alpha$ and for any $\delta > 0$. Hence $A_0 \in \tau(\mathrm{Im}P)$ and, consequently, $\overline{\psi(A_0)} = \sigma(A_0) = \alpha$. Since α is open in $\sigma(A)$, from Lemma 1.5 it follows that $\psi(A_0) \subset \psi(A)$. Hence, since $\alpha \subset U$, $U \cap \psi(A) \neq \emptyset$.

Therefore $\rho_{s-F}^{\pm}(A) \cup \psi(A)$ is dense in $\sigma(A)$ and, consequently, $A \in \Sigma_0(X)$.□

The following result is an immediate consequence of Lemma 3.6 and Lemma 3.7.

THEOREM 3.8.— *Let X be a complex nonzero Banach space. Then $\Sigma_0(X)$ is a G_δ-set.* □

We remark that if, for any complex nonzero Banach space X, as the authors of [AFHV] suspect (see [AFHV], page 313), condition ii) of Corollary 1.15 is also necessary for membership in $\Sigma(X)$, so that $\Sigma(X) = \Sigma_0(X)$, and condition vii) of Corollary 2.13 is also necessary for membership in $R(X)$, so that $R_0(X) = R(X)$, Theorem 3.4 and Theorem 3.8 are an immediate consequence of [AFHV], Proposition 14.5. In [B] an example is given to prove that $\pi(X)$ is not always closed with respect to powers (see [B], 1.10). Since it is not restrictive to suppose the operator A of [B], 1.10 to be normal, it follows that, generally speaking, $R_0(X)$ is not closed with respect to powers, either (so that, in particular, $R_0(X)$ is not always closed with respect to the product of the algebra $L_c(X)$).

We recall that $\tau(X)$ is invariant under holomorphic functions (see [B], 2.13) and, consequently, it is also closed with respect to powers, whereas it is not always closed with respect to the product of the algebra $L_c(X)$ (see [B], 2.14). The behaviour of $\Sigma_0(X)$ with respect to powers and, more generally, holomorphic functions, will be the subject of a future paper.

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