

NEDELCHO VELEV MILEV

DRUMI DIMITROV BAINOV

**Ordinary dichotomy and perturbations of
the coefficient matrix of the linear impulsive
differential equation**

Annales de la faculté des sciences de Toulouse 5^e série, tome 12,
n° 1 (1991), p. 127-135

http://www.numdam.org/item?id=AFST_1991_5_12_1_127_0

© Université Paul Sabatier, 1991, tous droits réservés.

L'accès aux archives de la revue « Annales de la faculté des sciences de Toulouse » (<http://picard.ups-tlse.fr/~annales/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

**Ordinary dichotomy and perturbations
of the coefficient matrix
of the linear impulsive differential equation**

NEDELCHO VELEV MILEV and DRUMI DIMITROV BAINOV⁽¹⁾

RÉSUMÉ. — Dans cet article nous prouvons que, sous perturbation de la matrice des coefficients de l'équation différentielle autonome, la dichotomie ordinaire est préservée.

ABSTRACT. — In the present paper it is proved that the ordinary dichotomy is preserved under perturbations of the coefficient matrix of the linear impulsive differential equation.

1. Introduction

In relation to numerous applications in science and technology, in the last five years the theory of impulsive differential equations has developed intensively [3], [4], [5], [6], [10]. In the present paper one of the important properties of the ordinary dichotomy for linear impulsive differential equations is studied, namely that it is not destroyed under small perturbations of the coefficient matrix. We shall note that analogous questions about ordinary differential equations were considered in [7], [2], [1], [9].

2. Preliminary notes

Let $t_0 < t_1 < \dots < t_i < \dots$, $\lim t_i = \infty$ as $i \rightarrow \infty$, be a given sequence of real numbers. Consider the linear differential equation with impulses at

⁽¹⁾ Plovdiv University "Paissii Hilendarski".

The present investigation is supported by the Ministry of Culture, Science and Education of People's Republic of Bulgaria under Grant 61.

fixed moments

$$\begin{aligned} \frac{dx}{dt} &= A(t)x, \quad t \neq t_i; \\ x(t_i + 0) &= B_i x(t_i), \quad i = 1, 2, \dots \end{aligned} \tag{1}$$

where the $(n \times n)$ -coefficient matrix $A(t)$ is piecewise continuous in the interval $[t_0, +\infty)$ with points of discontinuity of the first kind at $t = t_i$ and the impulse matrices B_i , $i = 1, 2, \dots$ are constant. The underlying vector space E is \mathbb{R}^n or \mathbb{C}^n .

Remark 1. — For $t \in [t_i + 0, t_{i+1}]$ the fundamental matrix $X(t)$ of equation (1) admits the representation

$$X(t) = U(t)U^{-1}(t_i + 0)B_i U(t_i)U^{-1}(t_{i-1} + 0)B_{i-1} \cdots B_1 U(t_1)U^{-1}(t_0)$$

where $U(t)$ is the fundamental matrix of the equation $dx/dt = A(t)x$. The matrix $X(t)$ is continuously differentiable for $t \neq t_i$ with points of discontinuity of the first kind at $t = t_i$, i.e. $X(t_i + 0) = B_i X(t_i)$. The fundamental matrix $X(t)$ is invertible if and only if the impulse matrices B_i , $i = 1, 2, \dots$, are nonsingular.

Together with equation (1) we shall consider the perturbed equation

$$\begin{aligned} \frac{dx}{dt} &= (A(t) + \tilde{A}(t))x, \quad t \neq t_i; \\ x(t_i + 0) &= B_i x(t_i), \quad i = 1, 2, \dots \end{aligned} \tag{2}$$

where the matrix $\tilde{A}(t)$ is piecewise continuous on the interval $[t_0, +\infty)$ with points of discontinuity of the first kind for $t = t_i$, $i = 1, 2, \dots$

Let τ_0 be a fixed real number, $\tau_0 \geq t_0$.

DEFINITION 1 ([8]). — *The subspace Y of the underlying vector space E induces an ordinary dichotomy of the solutions of equation (1) on the interval $[\tau_0, +\infty)$ if for some subspace Z supplementary to Y there exists a constant N such that all solutions x, y, z of equation (1) for which $x = y + z$, $y(\tau_0) \in Y$ and $z(\tau_0) \in Z$ satisfy the conditions*

$$|y(t)| \leq N|x(s)| \text{ for } t \geq s \geq \tau_0 \quad \text{and} \quad |z(t)| \leq N|x(s)| \text{ for } t \geq s \geq \tau_0. \tag{3}$$

When the fundamental matrix $X(t)$ is invertible, Definition 1 can be written down in the following form.

Ordinary dichotomy and perturbations of the coefficient matrix

DEFINITION 2 ([8]). — *The subspace Y of the underlying vector space E induces an ordinary dichotomy of the solutions of equation (1) on the interval $[\tau_0, +\infty)$ if for some projector $P(P^2 = P)$ with range $R(P) = Y$ there exists a constant N such that*

$$\begin{cases} |X(t)X^{-1}(\tau_0)PX(\tau_0)X^{-1}(s)| \leq N & \text{for } t \geq s \geq \tau_0 \\ |X(t)X^{-1}(\tau_0)(I - P)X(\tau_0)X^{-1}(s)| \leq N & \text{for } s \geq t \geq \tau_0 \end{cases} \quad (4)$$

where I stands for the unit matrix.

DEFINITION 3. — *Let P be a projector ($P^2 = P$). The function*

$$G(t, s) = \begin{cases} X(t)PX^{-1}(s) & \text{for } t \geq s \geq \tau_0; \\ X(t)(P - I)X^{-1}(s) & \text{for } s \geq t \geq \tau_0, \end{cases}$$

will be called Green's function for equation (1).

We shall use the following properties of Green's function which are verified immediately :

$$\frac{\partial G(t, s)}{\partial t} = A(t)G(t, s) \quad \text{for } t \neq s,$$

$$G(t_i + 0, t) = B_i G(t_i, t)$$

$$\text{and } G(t, t + 0) - G(t, t - 0) = -I \quad \text{for } t \neq t_i, i = 1, 2, \dots$$

3. Main results

THEOREM 1. — *Let the impulse matrices $B_i, i = 1, 2, \dots$, of equation (1) be nonsingular and let the subspace Y induce an ordinary dichotomy of the solutions of equation (1) on the interval $[t_0, +\infty)$ with a projector P and a constant N . If*

$$\int_{t_0}^{\infty} |\tilde{A}(\theta)| d\theta = K < \frac{1}{N(2N + 1)} \quad (5)$$

then the perturbed equation (2) also has an ordinary dichotomy on interval $[t_0, +\infty)$.

Proof. — Let $X(t)$ be the fundamental matrix of equation (1) for which $X(t_0) = I$. The bounded solutions $y(t)$ of equation (2) are just the bounded solutions of the integral equation

$$y(t) = X(t)\eta + \int_{t_0}^{\infty} G(t, \theta)\tilde{A}(\theta)y(\theta) d\theta, \quad \eta \in Y, \quad (6)$$

since for $t \neq t_i$

$$\begin{aligned} \frac{dy(t)}{dt} &= X'(t)\eta + \frac{d}{dt} \left(\int_{t_0}^t G(t, \theta)\tilde{A}(\theta)y(\theta) d\theta + \int_t^{\infty} G(t, \theta)\tilde{A}(\theta)y(\theta) d\theta \right) = \\ &= A(t)X(t)\eta + \int_{t_0}^{\infty} A(t)G(t, \theta)\tilde{A}(\theta)y(\theta) d\theta + \\ &\quad + G(t, t-0)\tilde{A}(t)y(t) - G(t, t+0)\tilde{A}(t)y(t) = \\ &= (A(t) + \tilde{A}(t))y(t), \end{aligned}$$

and for $t = t_i$

$$\begin{aligned} y(t_i + 0) &= X(t_i + 0)\eta + \int_{t_0}^{\infty} G(t_i + 0, \theta)\tilde{A}(\theta)y(\theta) d\theta = \\ &= B_i X(t_i)\eta + \int_{t_0}^{\infty} B_i G(t_i, \theta)\tilde{A}(\theta)y(\theta) d\theta = B_i y(t_i). \end{aligned}$$

Denote by H the Banach space of all bounded piecewise continuous vector-valued functions $y(t)$ in the interval $[t_0, +\infty)$ with points of discontinuity of the first kind at $t = t_i, i = 1, 2, \dots$ and with a norm

$$\|y\| = \sup_{t \geq t_0} |y(t)|.$$

The linear operator $Ty(t) = \int_{t_0}^{\infty} G(t, \theta)\tilde{A}(\theta)y(\theta) d\theta$ maps H into itself since

$$|Ty(t)| \leq \int_{t_0}^{\infty} |G(t, \theta)| |\tilde{A}(\theta)| |y(\theta)| d\theta \leq NK \|y\|.$$

This implies that $|T| \leq NK < 1$ and by the contraction mapping principle the integral equation (6) for each $\eta \in Y$ has exactly one solution $y \in H$ which depends linearly on η , i.e. $y(t) = F(t)\eta$ where $F(t)$ is a bounded piecewise continuous matrix on the interval $[t_0, +\infty)$ with points of discontinuity of the first kind at $t = t_i, i = 1, 2, \dots$. Moreover, from $y = X(t)\eta + Ty$ we obtain

$$\|y\| \leq N\|\eta\| + |T|\|y\| \leq N\|\eta\| + NK\|y\|,$$

i.e.

$$\|y\| \leq \frac{N}{1 - NK} |\eta| \quad \text{and} \quad |F(t)| \leq \frac{N}{1 - NK}.$$

Let \tilde{Y} be the subspace of E consisting of the initial values $y(t_0)$ of the bounded solutions of the integral equation (6)

$$\begin{aligned} y(t_0) &= \eta + \int_{t_0}^{\infty} G(t_0, \theta) \tilde{A}(\theta) F(\theta) \eta \, d\theta = \\ &= \eta - (I - P) \left(\int_{t_0}^{\infty} X^{-1}(\theta) \tilde{A}(\theta) F(\theta) \, d\theta \right) \eta = \\ &= (I - (I - P)QP) \eta, \end{aligned}$$

where

$$Q = (I - P) \left(\int_{t_0}^{\infty} X^{-1}(\theta) \tilde{A}(\theta) F(\theta) \, d\theta \right) P.$$

The operator Q is bounded

$$|Q| \leq N \cdot K \cdot \frac{N}{1 - NK} \cdot |P|.$$

The operator $I - (I - P)QP$ maps the subspace Y onto \tilde{Y} . It has a bounded inverse one $I + (I - P)QP$. The operator

$$\tilde{P} = (I - (I - P)QP)P(I - (I - P)QP)^{-1} = P - (I - P)QP$$

is a projector with range $R(\tilde{P}) = \tilde{Y}$. The supplementary projector $I - \tilde{P} = (I - P)(I + QP)$ has a range $R(I - \tilde{P}) = Z$.

First we shall estimate the solutions issuing from \tilde{Y} . By (6)

$$\begin{aligned} \eta &= X^{-1}(s)y(s) - X^{-1}(s) \int_{t_0}^{\infty} G(s, \theta) \tilde{A}(\theta) y(\theta) \, d\theta = \\ &= X^{-1}(s)y(s) - \int_{t_0}^s P X^{-1}(\theta) \tilde{A}(\theta) y(\theta) \, d\theta \\ &\quad - \int_s^{\infty} (P - I) X^{-1}(\theta) \tilde{A}(\theta) y(\theta) \, d\theta = \\ &= P X^{-1}(s)y(s) - \int_{t_0}^s P X^{-1}(\theta) \tilde{A}(\theta) y(\theta) \, d\theta, \end{aligned}$$

i.e.

$$\begin{aligned} y(t) &= X(t)\eta + \int_{t_0}^{\infty} G(t, \theta)\tilde{A}(\theta)y(\theta) d\theta = \\ &= X(t)PX^{-1}(s)y(s) - X(t) \int_{t_0}^s PX^{-1}(\theta)\tilde{A}(\theta)y(\theta) d\theta + \\ &\quad + \int_{t_0}^{\infty} G(t, \theta)\tilde{A}(\theta)y(\theta) d\theta = \\ &= X(t)PX^{-1}(s)y(s) + \int_s^{\infty} G(t, \theta)\tilde{A}(\theta)y(\theta) d\theta. \end{aligned}$$

Hence for $t \geq s$

$$|y(t)| \leq N|y(s)| + N \int_s^{\infty} |\tilde{A}(\theta)||y(\theta)| d\theta.$$

Let us fix s and set $N|y(s)| = \alpha$. The cone of the nonnegative piecewise continuous functions $\varphi(t) = |y(t)|$ is invariant with respect to the linear operator $L\varphi(t) = N \int_s^{\infty} |\tilde{A}(\theta)|\varphi(\theta) d\theta$. Hence

$$L|y(\theta)| \leq L\alpha + L^2|y(t)| = NK\alpha + NKL|y(t)|,$$

i.e.

$$|y(t)| \leq \alpha + L|y(t)| \leq \alpha + \frac{NK\alpha}{1 - NK} = \frac{\alpha}{1 - NK}.$$

Hence for $t \geq s$

$$|y(t)| \leq \frac{N|y(s)|}{1 - NK}. \quad (7)$$

Let $z(t)$ be a solution with initial condition $z(t_0) \in Z$. From the formula

$$z(s) = X(s)z(t_0) + \int_{t_0}^s X(s)X^{-1}(\theta)\tilde{A}(\theta)z(\theta) d\theta$$

we express $z(t_0)$ and in view of $(I - P)z(t_0) = z(t_0)$ for $t < s$ we obtain that

$$\begin{aligned} z(t) &= X(t)(I - P)X^{-1}(s)z(s) - \int_{t_0}^s X(t)(I - P)X^{-1}(\theta)\tilde{A}(\theta)z(\theta) d\theta + \\ &\quad + \int_{t_0}^t X(t)X^{-1}(\theta)\tilde{A}(\theta)z(\theta) d\theta = \\ &= X(t)(I - P)X^{-1}(s)z(s) + \int_{t_0}^t X(t)PX^{-1}(\theta)\tilde{A}(\theta)z(\theta) d\theta - \\ &\quad - \int_t^s X(t)(I - P)X^{-1}(\theta)\tilde{A}(\theta)z(\theta) d\theta \end{aligned}$$

and get to the integral inequality

$$|z(t)| \leq N|z(s)| + N \int_{t_0}^s |\tilde{A}(\theta)| |z(\theta)| d\theta.$$

The linear operator $L_1\varphi(t) = N \int_{t_0}^s |\tilde{A}(\theta)| \varphi(\theta) d\theta$ is monotone and, as for (7), we obtain that for $t \leq s$

$$|z(t)| \leq \frac{N}{1 - NK} |z(s)|. \quad (8)$$

Now let $x(t) = y(t) + z(t)$ be an arbitrary solution of the differential equation (2). From the formula

$$x(s) = X(s)x(t_0) + \int_{t_0}^s X(s)X^{-1}(\theta)\tilde{A}(\theta)x(\theta) d\theta$$

we express $x(t_0)$ and in view of (6) we obtain that

$$\begin{aligned} y(s) &= X(s)Px(t_0) + \int_{t_0}^{\infty} G(s, \theta)\tilde{A}(\theta)y(\theta) d\theta = \\ &= X(s)PX^{-1}(s)x(s) - \int_{t_0}^s X(s)PX^{-1}(\theta)\tilde{A}(\theta)x(\theta) d\theta + \\ &\quad + \int_{t_0}^s X(s)PX^{-1}(\theta)\tilde{A}(\theta)y(\theta) d\theta + \\ &\quad + \int_s^{\infty} X(s)(P - I)X^{-1}(\theta)\tilde{A}(\theta)y(\theta) d\theta = \\ &= X(s)PX^{-1}(s)x(s) - \int_{t_0}^s X(s)PX^{-1}(\theta)\tilde{A}(\theta)z(\theta) d\theta + \\ &\quad + \int_s^{\infty} X(s)(P - I)X^{-1}(\theta)\tilde{A}(\theta)y(\theta) d\theta. \end{aligned}$$

In view of (7) and (8) we get to the inequality

$$\begin{aligned} |y(s)| &\leq N|x(s)| + N \int_{t_0}^s |\tilde{A}(\theta)| |z(\theta)| d\theta + N \int_s^{\infty} |\tilde{A}(\theta)| |y(\theta)| d\theta \leq \\ &\leq N|x(s)| + \frac{N^2K}{1 - NK} |z(s)| + \frac{N^2K}{1 - NK} |y(s)| \leq \\ &\leq \frac{N}{1 - NK} |x(s)| + \frac{2N^2K}{1 - NK} |y(s)|, \end{aligned}$$

i.e.

$$|y(s)| \leq \frac{N}{1 - NK - 2N^2K} |x(s)|. \quad (9)$$

Moreover,

$$|z(s)| \leq |x(s)| + |y(s)| \leq \frac{1 + N - NK - 2N^2K}{1 - NK - 2N^2K} |x(s)|. \quad (10)$$

From (7), (8), (9) and (10) we obtain that for $t \geq s \geq t_0$, $|y(t)| \leq N_1|x(s)|$ and for $t \geq s \geq t_0$, $|z(t)| \leq N_1|x(s)|$ where

$$N_1 = \frac{N(1 + N - NK - 2N^2K)}{(1 - NK)(1 - NK - 2N^2K)} > 0,$$

i.e. the perturbed equation (2) has an ordinary dichotomy as well.

LEMMA 1 ([8]).— Let τ_0 and τ be fixed real numbers in the interval $[t_0, +\infty)$ and let the impulse matrices B_i , $i = 1, 2, \dots$, of equation (1) be nonsingular. If equation (1) has an ordinary dichotomy on the interval $[\tau_0, +\infty)$, then it has an ordinary dichotomy on the interval $[\tau, +\infty)$ as well.

COROLLARY 1.— Let the impulse matrices B_i , $i = 1, 2, \dots$, be nonsingular and let equation (1) have an ordinary dichotomy on the interval $[t_0, +\infty)$. If

$$\int_{t_0}^{\infty} |\tilde{A}(\theta)| d\theta < \infty$$

then the perturbed equation (2) also has an ordinary dichotomy on the interval $[t_0, +\infty)$.

Proof.— Since the integral $\int_{t_0}^{\infty} |\tilde{A}(\theta)| d\theta$ is convergent, then a sufficiently large number τ_0 can be found such that condition (5) should hold. Since the impulse matrices B_i , $i = 1, 2, \dots$, are nonsingular, then equation (1) has an ordinary dichotomy with the same constant N on the interval $[\tau_0, +\infty)$ as well. Then by theorem 1 the perturbed equation (2) also has an ordinary dichotomy on the interval $[\tau_0, +\infty)$ and by lemma 1 it has an ordinary dichotomy on each interval $[\tau, +\infty)$, $\tau \geq t_0$ as well.

References

- [1] COPPEL (W.A.) . — *Dichotomies in stability theory*,
Lecture Notes in Math., Springer-Verlag, **629** (1978).
- [2] DALECKII (JU.L.) and KREIN (M.G.) . — *Stability of solutions of differential equations in Banach spaces*,
Amer. Math. Soc. Transl., Providence, R.I. (1974).
- [3] DISHLIEV (A.B) and BAINOV (D.D.) . — *Continuous dependence of the solution of a system of differential equations with impulses on the impulse hypersurfaces*,
J. Math. Anal. Appl., **135**, n° 2 (1988), pp. 369-382.
- [4] HEKIMOVA (M.A.) and BAINOV (D.D.) . — *Periodic solutions of singularly perturbed systems of differential equations with impulse effect*,
ZAMP, **36** (1985), pp. 520-537.
- [5] LAKSHMIKANTHAM (V.) and XINZHI LIU . — *Stability for impulsive differential systems in terms of two measures*,
Appl. Math. Comp. (to appear).
- [6] LAKSHMIKANTHAM (V.) and XINZHI LIU . — *On quasi stability for impulsive differential systems*,
Nonlinear Analysis (to appear).
- [7] MASSERA (J.L.) and SCHÄFFER (J.J.) . — *Linear differential equations and functional analysis*,
I, Ann. of Math., **67** (1958), pp. 517-573.
- [8] MILEV (N.V.) and BAINOV (D.D.) . — *Dichotomies for linear differential equations with variable structure and impulse effect*,
(to appear).
- [9] PALMER (K.J.) . — *A perturbation theorem for exponential dichotomies*,
Proc. Roy. Soc. Edinburgh, **106A** (1987), pp. 25-37.
- [10] SIMONOV (P.S.) and BAINOV (D.D.) . — *Stability with respect to part of the variables in systems with impulse effect*,
J. Math. Appl., **117**, n° 1 (1986), pp. 247-263.