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Completion by Gamma-convergence for optimal control problems[(*)]

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RÉSUMÉ. — On considère des problèmes de contrôle abstraits de la forme

$$\min\{J(u,y) : y \in \operatorname{argmin} G(u,\,\cdot\,)\}$$

et on identifie les problèmes relaxés par la complétion de l'espace des contrôles dans la topologie de la Γ-convergence pour les fonctionnelles $G(u,\,\cdot\,)$. On montre que plusieurs exemples rentrent dans ce cadre abstrait.

ABSTRACT. — Abstract optimal control problems of the form

$$\min\{J(u,y) : y \in \operatorname{argmin} G(u,\,\cdot\,)\}$$

are considered, and the associated relaxed problems are identified by completing the space of controls with respect to the Γ-convergence of the state functionals $G(u,\,\cdot\,)$. Several examples are shown to fall in this framework.

1. Introduction and preliminary results

Many optimal control problems can be written in the form

$$\min\{J(u,y) : y \in \operatorname{argmin} G(u,\,\cdot\,)\} \tag{1.1}$$

where y is the *state variable* varying in a space Y, u is the *control variable* varying in a set U, $J : U \times Y \to \overline{\mathbb{R}}$ is the *cost functional*, and $G : U \times Y \to \overline{\mathbb{R}}$ is a functional whose Euler-Lagrange equation (if any) is the state equation. We shall call G the *state functional.*

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Usually Y is a function space endowed with a metric structure, while U is just a set with no topological structure. The natural topology on U that takes into account the convergence of minimizers of G is the one related to Γ-convergence of the mappings $G(u,\,\cdot\,)$: in other words, if we endow U with the convergence defined by

$$u_h \to u \quad \Leftrightarrow \quad \Gamma\text{-}\lim_{h\to\infty} G(u_h,\,\cdot\,) = G(u,\,\cdot\,) \tag{1.2}$$

under some lower semicontinuity and coerciveness assumptions we get (see Corollary 2.3) the existence of solutions of (1.1) by the standard direct method of the calculus of variations.

When these assumptions are not fullfilled, we have to pass to the relaxed formulation of (1.1) which is here introduced by considering relaxed controls $\widehat{u} \in \widehat{U}$, where the set $\widehat{U}$ is constructed as the completion of U with respect to the convergence (1.2). The usual properties of relaxed problems are shown in Proposition 2.8.

Finally, three main examples are illustrated in the framework of this abstract theory. The first one is related to some shape optimization problems recently considered by Buttazzo and Dal Maso in [2], [3], [4] in which the control set U is the class of all domains contained in a given open subset Ω of $\mathbb{R}^n$; the second and the third ones are problems where the control is on the coefficients of the differential state equation.

In the following we consider a separable metric space of states; a functional $G : Y \to \overline{\mathbb{R}} =]-\infty, +\infty]$ will be called *coercive* if for every $t \in \mathbb{R}$ there exists a compact subset K_t of Y such that

$$\{y \in Y : G(y) \le t\} \subseteq K_t\,.$$

The set $\operatorname{argmin} G$ is defined as the set of all minimum points of G on Y where G is finite, and for every set E we define

$$\chi_E(x) = \begin{cases} 0 & \text{if } x \in E \\ +\infty & \text{otherwise.} \end{cases}$$

Given a sequence (G_h) of functionals from Y into $\overline{\mathbb{R}}$ we recall that (G_h) is said to Γ-converge to a functional G if for every $y \in Y$

$$\begin{cases} \text{i) } \forall\, y_h \to y \quad G(y) \le \liminf\limits_{h\to\infty} G_h(y_h) \\ \text{ii) } \exists\, y_h \to y \quad G(y) \ge \limsup\limits_{h\to\infty} G_h(y_h) \end{cases} \tag{1.3}$$

and the following results hold (see for instance De Giorgi and Franzoni [10] or Dal Maso [8]).

THEOREM 1.1. — *Properties of* Γ*-convergence:*

i) every Γ*-limit is lower semicontinuous on* Y*;*

ii) if (G_h) *is equi-coercive on* Y *and* Γ*-converges to* G*, then* G *is coercive too and so it admits a minimum on* Y*. Moreover, if* G *is not identically* $+\infty$ *and* $y_h \in \operatorname{argmin} G_h$ *then there exists a subsequence of* (y_h) *which converges to an element of* $\operatorname{argmin} G$*;*

iii) from every sequence (G_h) *of functionals on* Y *it is possible to extract a subsequence* Γ*-converging to a functional* G *on* Y*.*

Concerning the metrizability of the Γ-convergence we have the following result (see Dal Maso [8, Theorem 10.22]).

THEOREM 1.2. — *Let* $\Psi : Y \to \overline{\mathbb{R}}$ *be a coercive function and let* $S_\Psi(Y)$ *be the class of all lower semicontinuous functions from* Y *into* $\overline{\mathbb{R}}$ *which are greater or equal to* Ψ*. Then* $S_\Psi(Y)$ *endowed with the* Γ*-convergence is a compact metric space in the sense that there exists a compact distance* δ_Γ *on* $S_\Psi(Y)$ *such that*

$$\delta_\Gamma(G_h, G) \to 0 \quad \Leftrightarrow \quad \Gamma\text{-}\lim_{h\to\infty} G_h = G\,.$$

2. Existence of optimal pairs and relaxation

We start this section by finding conditions which, via Tonelli's direct method of the calculus of variations, ensure the existence of optimal pairs for the control problems (1.1). Setting

$$F(u,y) = J(u,y) + \chi_{\operatorname{argmin} G(u,\,\cdot\,)}(y) \tag{2.1}$$

the minimum problem (1.1) becomes the minimum problem for F on the whole product space $U \times Y$. We assume that the following conditions hold.

For every $u \in U$ the mapping $G(u,\,\cdot\,)$ is not identically $+\infty$; (2.2)

the mappings $G(u,\,\cdot\,)$ are lower semicontinuous and locally equi-coercive that is, for every compact set $K \subseteq U$ there exists a lower semicontinuous coercive function $\Psi_K : Y \to \overline{\mathbb{R}}$ such that $G(u,y) \geq \Psi_K(y)$ for every $(u,y) \in K \times Y$; (2.3)

a topology on the control set U is a priori given, stronger than Γ-convergence for $G(u,\,\cdot\,)$, that is
$$u_h \to u \quad \Rightarrow \quad \Gamma\text{-}\lim_{h\to\infty} G(u_h,\,\cdot\,) = G(u,\,\cdot\,)\,; \tag{2.4}$$

J is sequentially lower semicontinuous on $U \times Y$; (2.5)

$J(u,y)$ is sequentially coercive with respect to u, that is
$$J(u_h,y_h) \le C \in \mathbb{R} \quad \Rightarrow \quad (u_h) \text{ is relatively compact in } U\,. \tag{2.6}$$

THEOREM 2.1.— *Under assumptions (2.4), (2.5) the functional F in (2.1) is sequentially lower semicontinuous on $U \times Y$.*

Proof.— Let $(u_h,y_h) \to (u_0,y_0)$ in $U \times Y$; we have to prove that
$$F(u_0,y_0) \le \liminf_{h\to\infty} F(u_h,y_h)\,. \tag{2.7}$$

Passing to subsequences and avoiding the trivial case in which the right-hand side of (2.7) is $+\infty$, we may assume that $y_h \in \operatorname{argmin} G(u_h,\,\cdot\,)$. From (2.4) and Theorem 1.1 ii) we obtain that $y_0 \in \operatorname{argmin} G(u_0,\,\cdot\,)$ so that, by (2.5)
$$F(u_0,y_0) = J(u_0,y_0) \le \liminf_{h\to\infty} J(u_h,y_h) = \liminf_{h\to\infty} F(u_h,y_h)\,. \square$$

THEOREM 2.2.— *Under assumptions (2.2), (2.3), (2.4), (2.6) the functional F in (2.1) is sequentially coercive on $U \times Y$.*

Proof.— Let (u_h,y_h) be a sequence in $U \times Y$ and assume $F(u_h,y_h) \le C$. Then $J(u_h,y_h) \le C$ and $y_h \in \operatorname{argmin} G(u_h,\,\cdot\,)$ so that by (2.6) we have, possibly passing to subsequences, that (u_h) tends to some u_0 in U. Hence $G(u_h,\,\cdot\,)$ Γ-converges to $G(u_0,\,\cdot\,)$. Thanks to equi-coerciveness (2.3), Theorem 1.1 applies and (y_h) turns out to be relatively compact in Y. $\square$

COROLLARY 2.3.— *Under assumptions (2.2), ..., (2.6) problem (1.1) admits at least a solution.*

Remark 2.4.— The problems which we consider are quite general, indeed G doesn't need to be an integral functional but, for instance, it may be the indicator of a set Λ of admissible pairs, that is $G(u,y) = \chi_\Lambda(u,y)$ where Λ is a subset of $U \times Y$.

We consider now the case when the control set U is not a priori topologized, and condition (2.5) is not fullfilled; then it's easy to see that in this situation the existence of a solution for problem (1.1) may fail. We still assume (2.2) and (2.3), and

$$\text{the mapping } \mathcal{G} : U \to S_\Psi(Y) \text{ defined by } \mathcal{G}(u) = G(u, \cdot\,) \text{ is one-to-one.} \tag{2.8}$$

We endow U with the distance

$$d(u, v) = \delta_\Gamma\big(\mathcal{G}(u), \mathcal{G}(v)\big)$$

being δ_Γ the compact distance on $S_\Psi(Y)$ given by Theorem 1.2. In this way (U, d) is a metric space, $\mathcal{G}$ is an isometry, and

$$u_h \to u \text{ in } U \quad \Leftrightarrow \quad \Gamma\text{-}\lim_{h\to\infty} G(u_h, \cdot\,) = G(u, \cdot\,)\,.$$

Let now $(\widehat{U}, \widehat{d})$ be the completion of the metric space (U, d); we define the mapping $\widehat{\mathcal{G}} : \widehat{U} \to S_\Psi(Y)$ as the unique isometry which extends $\mathcal{G}$: more precisely

$$\widehat{\mathcal{G}}(\widehat{u}) = \Gamma\text{-}\lim_{h\to\infty} \mathcal{G}(u_h)$$

where (u_h) is any sequence $\widehat{d}$-converging to $\widehat{u}$. Therefore we may define $\widehat{G} : \widehat{U} \times Y \to \overline{\mathbb{R}}$ by $\widehat{G}(\widehat{u}, \cdot\,) = \widehat{\mathcal{G}}(\widehat{u})$, and we have

$$\widehat{u}_h \to \widehat{u} \text{ in } \widehat{U} \quad \Leftrightarrow \quad \Gamma\text{-}\lim_{h\to\infty} \widehat{G}(\widehat{u}_h, \cdot\,) = \widehat{G}(\widehat{u}, \cdot\,)\,.$$

PROPOSITION 2.5.— *The metric space* $(\widehat{U}, \widehat{d})$ *is compact.*

Proof.— Since $\widehat{\mathcal{G}}$ is an isometry and $\widehat{U}$ is complete, $\widehat{\mathcal{G}}(\widehat{U})$ is a complete subspace of the compact one $S_\Psi(Y)$, so that $\widehat{\mathcal{G}}(\widehat{U})$ is compact. Hence, using again the fact that $\widehat{\mathcal{G}}$ is an isometry, we get that $\widehat{U}$ is compact too. □

Define now the following functionals on $\widehat{U} \times Y$

$$J_\infty(\widehat{u}, y) = \begin{cases} J(\widehat{u}, y) & \text{if } \widehat{u} \in U \\ +\infty & \text{otherwise} \end{cases}$$

$$\widehat{J} = \mathrm{sc}^-(\widehat{U} \times Y) J_\infty \quad \text{the l.s.c. envelope of } J_\infty \text{ on } \widehat{U} \times Y$$

$$F_\infty(\widehat{u}, y) = J_\infty(\widehat{u}, y) + \chi_{\operatorname{argmin} \widehat{G}(\widehat{u}, \cdot\,)}(y)$$

$$\widehat{F}_\infty(\widehat{u}, y) = \widehat{J}(\widehat{u}, y) + \chi_{\operatorname{argmin} \widehat{G}(\widehat{u}, \cdot\,)}(y)$$

and consider the relaxed control problem associated to (1.1)

$$\min\{\widehat{J}(\widehat{u},y) : y \in \operatorname{argmin} \widehat{G}(\widehat{u},\,\cdot\,)\}. \tag{2.9}$$

THEOREM 2.6. — *Assume (2.3), (2.8) and*

for every $\widehat{u} \in \widehat{U}$ the minimum point of $\widehat{G}(\widehat{u},\,\cdot\,)$ on Y is unique; *(2.10)*

there exist a function $\Phi : U \to \mathrm{IR}$ bounded on the d-bounded sets and a function $\omega : Y \times Y \to \mathrm{IR}$ with $\lim_{z\to y} \omega(y,z) = 0, \quad \forall\, y \in Y,$ such that for every $u \in U$ and $y, z \in Y$, $\left|J(u,y) - J(u,z)\right| \le \Phi(u)\,\omega(y,z)$. *(2.11)*

Then problem (2.9) is actually the relaxed problem of (1.1) in the sense that $\widehat{F} = \mathrm{sc}^-(\widehat{U}\times Y)F_\infty$.

Proof. — Since $\widehat{F} \le F_\infty$ the inequality $\widehat{F} \le \mathrm{sc}^-(\widehat{U}\times Y)F_\infty$ follows immediately from the lower semicontinuity of $\widehat{F}$ on $\widehat{U}\times Y$. In order to prove the opposite inequality, let $(\widehat{u}_0, y_0) \in \widehat{U}\times Y$ be such that $\widehat{F}(\widehat{u}_0, y_0) < +\infty$; then $y_0 \in \operatorname{argmin} \widehat{G}(\widehat{u}_0,\,\cdot\,)$ and there exist two sequences (u_h) in U and (y_h) in Y such that

$$\widehat{F}(\widehat{u}_0, y_0) = \widehat{J}(\widehat{u}_0, y_0) = \lim_{h\to\infty} J(\widehat{u}_h, y_h). \tag{2.12}$$

Take z_h as the unique minimum point of $G(u_h,\,\cdot\,)$ in Y; by the assumption (2.10) and Theorem 1.1 ii) we get $z_h \to y_0$ in Y so that, by (2.11) and (2.12)

$$\begin{aligned}
&\mathrm{sc}^-(\widehat{U}\times Y)F_\infty(\widehat{u}_0, y_0) \le \\
&\quad\le \liminf_{h\to\infty} F_\infty(u_h, z_h) = \liminf_{h\to\infty} J_\infty(u_h, z_h) \le \\
&\quad\le \liminf_{h\to\infty}\big[J_\infty(u_h, y_h) + \Phi(u_h)\,\omega(y_0, y_h) + \Phi(u_h)\,\omega(y_0, z_h)\big] = \\
&\quad= \widehat{F}(\widehat{u}_0, y_0).\ \square
\end{aligned}$$

Remark 2.7. — Under the assumptions (2.11) it is easy to see that

$$\widehat{J}(\widehat{u},y) = \liminf_{u\to\widehat{u}} J(u,y), \quad \forall\,(\widehat{u},y) \in \widehat{U}\times Y.$$

Summarizing, for the relaxed problem (2.9) the following results hold.

PROPOSITION 2.8. — *Under the assumptions of Theorem 2.6, we have:*

i) *the relaxed problem (2.9) has always a solution;*

ii) $\inf\{J(u,y) : u \in U,\ y \in \operatorname{argmin} G(u,\cdot)\} =$
$= \min\{\widehat{J}(\widehat{u},y) : \widehat{u} \in \widehat{U},\ y \in \operatorname{argmin} \widehat{G}(\widehat{u},\cdot)\}$;

iii) *if (u_h, y_h) is a minimizing sequence for (1.1), then there exists a subsequence converging in $\widehat{U} \times Y$ to a solution $(\widehat{u}, y)$ of (2.9);*

iv) *if (2.5) holds and if $(u,y) \in U \times Y$ is a solution of the relaxed problem (2.9), then (u,y) is a solution of (1.1).*

Example 2.9. — The uniqueness hypothesis (2.10) cannot be dropped, as the following example shows. Take $U = \mathbb{Q}^+ \setminus \{0\}$, $Y = (\mathbb{R}, |\cdot|)$, $J(u,y) = y^2$, $G(u,y) = (y^2-1) \vee (y^2 u)$. Note that hypothesis (2.8) is satisfied. Then it is easy to check that $\widehat{U} = \big([\,0\,,+\infty\,], \widehat{d}\big)$ and

$$\widehat{G}(\widehat{u},y) = \begin{cases} (y^2-1) \vee (y^2 \widehat{u}) & \text{if } \widehat{u} \in [\,0\,,+\infty\,[\\ \chi_{\{0\}}(y) & \text{if } \widehat{u} = +\infty. \end{cases}$$

It is $\widehat{F}(0,1/2) = 1/4$ but $\mathrm{sc}^- F_\infty(0,1/2) = +\infty$. Indeed

$$\begin{aligned} \mathrm{sc}^- F_\infty(0,1/2) &= \\ &= \inf\left\{\liminf_{h\to\infty} F_\infty(\widehat{u}_h, y_h) : \widehat{u}_h \to 0\,,\ y_h \to \frac{1}{2}\right\} \\ &= \inf\Big\{\liminf_{h\to\infty}\big[y_h^2 + \chi_{\operatorname{argmin}(y_h^2-1)\vee y_h^2 u_h}(u_h, y_h)\big] : u_h \in U\,, \\ &\qquad\qquad u_h \to 0\,,\ y_h \to \frac{1}{2}\Big\} \\ &= \inf\left\{\liminf_{h\to\infty}\big[y_h^2 + \chi_{\{0\}}(y_h)\big] : u_h \in U\,,\ u_h \to 0\,,\ y_h \to \frac{1}{2}\right\} = +\infty\,. \end{aligned}$$

3. Some examples

In this section we show some applications of the abstract framework illustrated in the previous sections. The first example deals with a class of shape optimization problems recently studied by Buttazzo and Dal Maso ([2], [3], [4], and references therein) in which the set U is the class of all domains contained in a given open subset Ω of $\mathbb{R}^n$; this class has no linear or convex structure, and usual topologies are not suitable for the problems one would like to consider.

To set the problem more precisely, let Ω be a bounded open subset of $\mathbb{R}^n$ $(n \geq 2)$, let $f \in L^2(\Omega)$ and let $j : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Borel function. Consider the shape optimization problem

$$\min\left\{\int_\Omega j\big(x, y_A(x)\big)\,\mathrm{d}x : A \in \mathcal{A}(\Omega)\right\} \tag{3.1}$$

where $\mathcal{A}(\Omega)$ is the family of all open subsets of Ω and y_A is the solution of the Dirichlet problem

$$\begin{cases} -\Delta y = f \text{ in } A \\ y \in H_0^1(A) \end{cases}$$

extended by zero to $\Omega \setminus A$. Setting $U = A(\Omega)$, $Y = H_0^1(\Omega)$ with the strong topology of $L^2(\Omega)$, and

$$J(A, y) = \int_\Omega j\big(x, y(x)\big)\,\mathrm{d}x$$

$$G(A, y) = \int_\Omega \big[|Dy|^2 - 2fy\big]\,\mathrm{d}x + \chi_{\{y=0 \text{ on } \Omega\setminus A\}}(y)$$

the minimization problem (3.1) can be written in the form

$$\min\big\{J(A, y) : A \in U\,,\ y \in \operatorname{argmin} G(A, \cdot)\big\}\,. \tag{3.2}$$

Note that assumptions (2.2), (2.3) and (2.8) are fullfilled, with

$$\psi(y) = \alpha \int_\Omega |Dy|^2\,\mathrm{d}x - \beta$$

for suitable constants α, β. In order to identify the relaxed problem associated to (3.2) we have to characterize the completion $\widehat{U}$ of U with respect to the distance induced by the Γ-convergence on the functionals $G(A,\cdot)$. This has be done by Dal Maso and Mosco in [9] where it is shown that $\widehat{U}$ coincides with the space $\mathcal{M}_0(\Omega)$ of all nonnegative Borel measures, possibly $+\infty$ valued, which vanish on all sets of capacity zero. Moreover, for every $\mu \in \mathcal{M}_0(\Omega)$ and $y \in H_0^1(\Omega)$ we have

$$\widehat{G}(\mu, y) = \int_\Omega \left[|Dy|^2 - 2fy\right] \mathrm{d}x + \int_\Omega y^2 \,\mathrm{d}\mu\,.$$

The relation $y \in \operatorname{argmin} \widehat{G}(\mu,\cdot)$ can also be written, via Euler-Lagrange equation, in the form

$$\begin{cases} -\Delta y + \mu y = f \\ y \in H_0^1(\Omega) \end{cases}$$

which must be intended in the weak sense: $y \in H_0^1(\Omega) \cap L^2_\mu(\Omega)$ and

$$\int_\Omega DyD\varphi \,\mathrm{d}x + \int_\Omega y\varphi \,\mathrm{d}\mu = \int_\Omega f\varphi \,\mathrm{d}x\,, \quad \forall\ \varphi \in H_0^1(\Omega) \cap L^2_\mu(\Omega)\,.$$

The uniqueness property (2.10) follows straightforward. If we assume on the integrand j

$j(x,\cdot)$ is continuous for a.e. $x \in \Omega$; (3.3)

for suitable $a \in L^1(\Omega)$ and $b \in \mathbb{R}$ we have $|j(x,s)| \le a(x) + b|s|^2$ for a.e. $x \in \Omega$, for every $s \in \mathbb{R}$, (3.4)

the functional J turns out to be continuous in the strong topology of L^2 so that (2.11) is fulfilled, Theorem 2.6 and Proposition 2.8 apply, and the relaxed problem of (3.2) can be written in the form

$$\min\left\{ \int_\Omega j\big(x, y(x)\big)\,\mathrm{d}x : \mu \in \mathcal{M}_0(\Omega)\,,\ y \in H_0^1(\Omega)\,,\ -\Delta y + \mu y = f \right\}.$$

The second example we consider is the case of a control problem where the control occurs on the coefficient of the state equation. More precisely, given $\alpha > 0$ take

$$U = \left\{u \in L^1(0,1) : u \ge \alpha \text{ a.e. on } (0,1)\right\}$$
$$Y = H_0^1(0,1) \text{ with the strong topology of } L^1(0,1)$$

and consider the optimal control problem

$$\min\left\{\int_0^1 \big[g(x,u)+\varphi(x,y)\big]\,dx : u\in U\,,\ y\in Y\,,\ -(uy')'=f\right\}. \tag{3.5}$$

Here $f\in L^2(0,1)$, and g, φ are Borel functions from $(0,1)\times\mathbb{R}$ into $\overline{\mathbb{R}}$ with

$\varphi(x,\cdot)$ is continuous on $\mathbb{R}$ for a.e. $x\in(0,1)$, (3.6)

for a suitable function $\omega(x,t)$ integrable in x and increasing in t we have
$\big|\varphi(x,s)\big|\le\omega\big(x,|s|\big)\,,\quad\forall\ (x,s)\in(0,1)\times\mathbb{R}.$ (3.7)

Setting for any $(u,y)\in U\times Y$

$$J(u,y)=\int_0^1\big[g(x,u)+\varphi(x,y)\big]\,dx$$

$$G(u,y)=\int_0^1\big[uy'^2-2fy\big]\,dx$$

we obtain that problem (3.5) can be written in the form

$$\min\big\{J(u,y):u\in U\,,\ u\in U\,,\ y\in Y\,,\ y\in\operatorname{argmin}G(u,\cdot)\big\}.$$

It is well-known that

$$\Gamma\text{-}\lim_{h\to\infty}G(u_h,\cdot)=G(u,\cdot)\quad\Leftrightarrow\quad\frac{1}{u_h}\to\frac{1}{u}\ \text{in weakly}\ {}^*L^\infty(0,1)\,;$$

therefore, by applying Theorem 2.6, we obtain $\widehat{U}=U$, $\widehat{G}=G$, and

$$\widehat{J}(u,y)=\int_0^1\big[\gamma(x,u)+\varphi(x,y)\big]\,dx$$

where $\gamma(x,s)=\beta^{**}(x,1/s)$ being ** the convexification operator (with respect to the second variable) and

$$\beta(x,t)=\begin{cases}g(x,1/t) & \text{if } t\in\,]\,0\,,\,1/\alpha\,]\\ +\infty & \text{otherwise.}\end{cases}$$

For instance, if $\alpha<1$ and $g(x,s)=|s-1|$ we have

$$\gamma(x,s)=\begin{cases}s-1 & \text{if } s\ge1\\ \alpha(1-s)/s & \text{if } \alpha\le s<1.\end{cases}$$

An analogous computation can be done in the case

$$U=\left\{u\in L^1(0,1): u\geq 0\,,\ \int_0^1 \frac{1}{u}\,\mathrm{d}x\leq c\right\}\,,\qquad (c>0)\,.$$

In this case, in order to satisfy the coerciveness assumption (2.3), it is better to consider

$$Y=BV(0,1) \text{ with the strong topology of } L^1(0,1)$$

$$G(u,y)=\int_0^1\left[uy'^2-2fy\right]\mathrm{d}x+\chi_{\{y(0)=y_0\,,\,y(1)=y_1\}}(y)+\chi_{\{y'\ll \mathrm{d}x\}}(y)$$

where $y'\ll \mathrm{d}x$ denotes the constraint that y' is a measure absolutely continuous with respect to the Lebesgue measure. Following Buttazzo and Freddi [5] we obtain that $\widehat{U}$ coincides with the set of positive measures μ on $[\,0\,,\,1\,]$ such that $\mu([\,0\,,\,1\,])\leq c$ and

$$\begin{aligned}\widehat{G}(\mu,y)=\int_{]\,0\,,\,1\,[}\left|\frac{\mathrm{d}y'}{\mathrm{d}\mu}\right|^2\mathrm{d}\mu-2\int_{]\,0\,,\,1\,[}fy\,\mathrm{d}x\,+\\ +\frac{\left|y^+(0)-y_0\right|^2}{\mu(\{0\})}+\frac{\left|y^-(1)-y_1\right|^2}{\mu(\{1\})}+\chi_{\{y'\ll\mu\}}(y)\end{aligned}$$

where $\mathrm{d}y'/\mathrm{d}\mu$ is the Radon-Nikodym derivative of y' with respect to μ. It is not difficult to see that assumptions (2.3), (2.8), (2.10) are fulfilled, with

$$\psi(y)=a\|y\|_{BV}^2-b$$

for suitable positive constants a, b. It remains to compute the functional $\widehat{J}$. Assume for simplicity that $g(x,s)=g(s)$ and that (3.6) and (3.7) hold; then we obtain

$$\widehat{J}(\mu,y)=\int_0^1\beta^{**}\big(\mu^a(x)\big)\,\mathrm{d}x+\int_{[\,0\,,\,1\,]}(\beta^{**})^\infty(\mu^s)+\int_0^1\varphi(x,y)\,\mathrm{d}x$$

where $\beta(t)=g(1/t)$, $\mu=\mu^a\cdot\mathrm{d}x+\mu^s$ is the Lebesgue-Nikodym decomposition of μ, and $\left(\beta^{**}\right)^\infty$ is the recession function of β^{**}. For instance, if $g(s)=|s-1|^2$, the relaxed problem of (3.5) has the form

$$\min\left\{\int_{\{\mu^a\leq 1\}}\left|\frac{\mu^a-1}{\mu^a}\right|^2\mathrm{d}x+\int_0^1\varphi(x,y)\,\mathrm{d}x : \mu([\,0\,,\,1\,])\leq c\,,\right.$$
$$\left. y\in\operatorname{argmin}\widehat{G}(\mu,\,\cdot\,)\right\}\,.$$

In this last example we consider a class of optimal control problems for two-phases conductors which has been studied by Cabib and Dal Maso ([6], [7]). As in the second, also in this case the control occurs on the coefficients of the state equation which is actually of elliptic partial differential kind. Precisely, let Ω be a bounded open subset of $\mathbb{R}^n$ and α, β be two real positive numbers, $f \in L^2(\Omega)$, U is the set of all functions $u : \Omega \to \mathbb{R}$ with the property that there exists a Borel subset $A \subset \Omega$ such that

$$u = \alpha 1_A + \beta 1_{\Omega \setminus A} \tag{3.8}$$

and $Y = H_0^1(\Omega)$ with the strong topology of L^2.

Consider the optimal control problem

$$\min\big\{ J(u, y) : -\operatorname{div}(uDy) = f \text{ in } \Omega\,,\ u = 0 \text{ on } \partial\Omega \big\}$$

where the cost functional is still of the form

$$J(x, y) = \int_A g(x)\,\mathrm{d}x + \int_\Omega \varphi(x, y)\,\mathrm{d}x$$

where $g : \mathbb{R}^n \times U \to \mathbb{R}$ is a given function in $L^1(\Omega)$, and $\varphi : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Charathéodory integrand which satisfy the growth condition

$$\big|\varphi(x, z)\big| \le c_1(x) + c_2 z^2 \quad \text{for suitable } c_1 \in L^1(\Omega)\,,\ c_2 \in \mathbb{R}\,.$$

The energy functional G is now given by

$$G(u, y) = \int_\Omega \big(u|Dy|^2 - 2fy \big)\,\mathrm{d}x\,.$$

The completion $\widehat{U}$ of U with respect to the G-convergence of the state equation or, equivalently, the Γ-convergence of the functionals $G(u, \cdot)$ has been characterized by Lurie and Cherkaev ([11], [12]) for the two-dimensional case and more recently by Murat and Tartar ([13], [14]) for the general case. They proved that $\widehat{U}$ is the space of all symmetric $n \times n$ matrices $A(x) = \big(a_{ij}(x)\big)$ whose eigenvalues $\lambda_1(x) \le \lambda_2(x) \le \cdots \le \lambda_n(x)$ satisfy for a suitable $t \in [\,0\,,1\,]$ depending on x to the following $n+2$ inequalities

$$\sum_{i=1}^n \frac{1}{\lambda_i - \alpha} \le \frac{1}{\nu_t - \alpha} + \frac{n-1}{\mu_t - \alpha}$$

$$\sum_{i=1}^n \frac{1}{\beta - \lambda_i} \le \frac{1}{\beta - \nu_t} + \frac{n-1}{\beta - \mu_t}$$

$$\nu_t \le \lambda_i \le \mu_t\,,\ i = 1, \ldots, n$$

where μ_t and ν_t respectively denote the arithmetic and the harmonic mean of α and β, namely

$$\mu_t = t\alpha + (1-t)\beta$$

$$\nu_t = \left(\frac{t}{\alpha} + \frac{1-t}{\beta}\right)^{-1}.$$

For instance, when $n = 2$, then $\widehat{U}$ consists of all symmetric 2×2 matrices whose eigenvalues $\lambda_1(x)$, $\lambda_2(x)$ belong for every $x \in \Omega$ to the following convex domain D of $\mathbf{R}^2$

$$D = \left\{(\lambda_1, \lambda_2) \in [\alpha\,,\beta] \times [\alpha\,,\beta] : \frac{\alpha\beta}{\beta + \alpha - \lambda_1} \leq \lambda_2 \leq \alpha + \beta - \frac{\alpha\beta}{\lambda_1}\right\}.$$

The functional $\widehat{G}$ turns out to be

$$\widehat{G}(A, y) = \int_\Omega \left[\sum_{i,j=1}^{n} a_{ij}(x) D_j y D_i y - 2fy\right] \mathrm{d}x\,.$$

The computation of the functional $\widehat{J}$ can be found in Cabib [6]. We have

$$\widehat{J}(A, y) = \int_\Omega \left[\widehat{g}(x, A) + \varphi(x, y)\right] \mathrm{d}x$$

where

$$\widehat{g}(x, A) = \begin{cases} g(x)\dfrac{\beta - \underline{\mu}(A)}{\beta - \alpha} & \text{if } g(x) \leq 0 \\[2ex] g(x)\dfrac{\beta - \overline{\mu}(A)}{\beta - \alpha} & \text{if } g(x) \geq 0 \end{cases}$$

and

$$\underline{\mu}(A) = \max\left\{\lambda_n\,,\, \beta + \frac{(n-1)\beta + \alpha}{1 - \beta\sum_{i=1}^{n}(\beta - \lambda_i)^{-1}}\right\}$$

$$\overline{\mu}(A) = \alpha + \frac{(n-1)\alpha + \beta}{1 + \alpha\sum_{i=1}^{n}(\lambda_i - \alpha)^{-1}}\,.$$

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