ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE

BERNARD DACOROGNA

HIDEYUKI KOSHIGOE

On the different notions of convexity for rotationally invariant functions

Annales de la faculté des sciences de Toulouse 6e série, tome 2, nº 2 (1993), p. 163-184

<http://www.numdam.org/item?id=AFST_1993_6_2_2_163_0>

Annales de la Faculté des Sciences de Toulouse Vol. II, n° 2, 1993

On the different notions of convexity for rotationally invariant functions[*]

BERNARD DACOROGNA[1] and HIDEYUKI KOSHIGOE[2]

RÉSUMÉ. — Soit $R^{2\times 2}$ l'ensemble des matrices 2×2 et $f : R^{2\times 2} \to \mathbb{R}$. Sous l'hypothèse d'invariance par rotation de f, on montre que les conditions de convexité et de polyconvexité prennent une forme plus simple qu'usuellement ; alors que ce n'est pas le cas pour les notions de quasiconvexité et convexité de rang 1. On montre aussi que le calcul des différentes enveloppes convexes est aussi facilité sous cette hypothèse d'invariance.

ABSTRACT. — Let $R^{2\times 2}$ be the set of 2×2 matrices and $f : R^{2\times 2} \to \mathbb{R}$. We assume that f is rotationally invariant and we show that the notions of convexity and polyconvexity are simpler than for general f ; while this is not the case for quasiconvexity and rank one convexity. We finally show how this also simplifies the computations of the different convex envelopes.

0. Introduction

Let $R^{2\times 2}$ be the set of 2 by 2 real matrices and $R_d^{2\times 2}$ be the subset of all diagonal matrices. We denote By O the set of orthogonal matrices, i.e.

$$\mathrm{O} = \left\{ U \in R^{2\times 2} \mid UU^t = I \right\}$$

where U^t denotes the transpose of U and I is the identity. By O^+, we mean that

$$\mathrm{O}^+ = \left\{ U \in \mathrm{O} \mid \det U = 1 \right\}.$$

(*) Reçu le 27 janvier 1993

(1) Département de Mathématiques, E.P.F.L., CH-1015 Lausanne (Suisse)
(2) Institute of Applied Mathematics, Chiba University (Japan)

We will be interested in functions $f : R^{2\times 2} \to \mathbb{R}$ which are rotationally invariant, i.e.

$$f(U\xi V) = f(\xi) \quad \text{for every } \xi \in R^{2\times 2}\,,\ U,\, V \in \mathrm{O}^+\ . \tag{H}$$

For such functions, we will study the different notions of convexity used in the calculus of variations (namely: convexity, polyconvexity, quasiconvexity and rank one convexity, see below for a precise definition). Most examples of vectorial calculus of variations are functions f satisfying (H). In particular a very important case is the one where (H) is satisfied for every $U,\, V \in \mathrm{O}$ (instead of O^+). It has been intensively studied (see Ball [2], Buttazzo-Dacorogna-Gangbo [3], Ciarlet [4] or Dacorogna [5] for more references).

To illustrate more concretely the hypothesis (H), let $h : \mathbb{R}^2 \to \mathbb{R}$ and denote for $\xi \in R^{2\times 2}$, by $|\xi|^2 = \sum_{i,j=1}^{2} \xi_{ij}^2$ the Euclidean norm and by $\det \xi$ the determinant. If f is of the form $f(\xi) = h\big(|\xi|^2\,, \det \xi\big)$ then f satisfies (H).

Our main results will be that to test the convexity or the polyconvexity of f, it is enough to test them on diagonal matrices. This might be in some concrete examples an important computational simplification. It also reduces significantly the computations of the convex or the polyconvex envelopes of a given function.

Surprisingly we will show that these results do not extend to rank one convex functions. We will give two examples showing that it is not enough to infer the rank one convexity of a function f from its rank one convexity tested only on diagonal matrices. More precisely if either

$$f_{\alpha,b} = |\xi|^{2\alpha} - 2^{\alpha-1} b |\det \xi|^{\alpha}$$

or

$$f_{\alpha,b} = |\xi|^{2\alpha}\big(|\xi|^2 - 2\,b\det \xi\big)\,,$$

we will show that for a certain choice of the parameters α and b, $f_{\alpha,b}$ is rank one convex when restricted to diagonal matrices, while it is not rank one convex (on the whole of $R^{2\times 2}$).

Our article is inspired by those of Buttazzo-Dacorogna-Gangbo [3], Dacorogna-Douchet-Gangbo-Rappaz [6] and Iwaniec-Lutoborski [7, Prop. 10.2]. It is divided into five sections; the three first deal with the notions of convexity, polyconvexity and rank one convexity respectively. The fourth one is devoted to some results on the different convex envelopes of a given function f satisfying (H). The last one gives an application of the fourth section to a concrete example.

Most of our results can be extended to $R^{n\times n}$ $(n \geq 2)$, but we have prefered for the sake of clarity to restrict ourselves to the case $n = 2$.

1. Convexity on diagonal matrices

We first start with ordinary convexity. The main result of this section is Theorem 1.1.

THEOREM 1.1.— *Let* $f : R^{2\times 2} \to \mathbb{R}$ *satisfy (H). Then the following properties are equivalent:*

(i) f *is convex,*

(ii) $f\restriction_{R_d^{2\times 2}}$ *is convex.*

Remarks

(i) By $f\restriction_{R_d^{2\times 2}}$ is convex, we mean that for every ξ, η diagonal matrices and for every $\lambda \in [0, 1]$,

$$f\big(\lambda\xi + (1-\lambda)\eta\big) \leq \lambda f(\xi) + (1-\lambda) f(\eta).$$

(ii) This result might not be new, but we are unaware of any precise reference.

Before proceeding with the proof, we introduce some notations (following Alibert-Dacorogna [1]).

Notations.— Let

$$\xi = \begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix} \in R^{2\times 2}.$$

Define

$$\widetilde{\xi} = \begin{pmatrix} \xi_{22} & -\xi_{21} \\ -\xi_{12} & \xi_{11} \end{pmatrix}, \quad \xi^+ = \frac{1}{2}(\xi + \widetilde{\xi}), \quad \xi^- = \frac{1}{2}(\xi - \widetilde{\xi}).$$

Observe that if $\langle \cdot\, ; \cdot \rangle$ denotes the scalar product in $R^{2\times 2}$,

(i) $\det(\xi + \eta) = \det\xi + \langle \widetilde{\xi}\, ; \eta\rangle + \det\eta, \quad 2\det\xi = \langle\widetilde{\xi}\, ; \xi\rangle;$

(ii) $\xi = \xi^+ + \xi^-, \quad \widetilde{\xi} = \xi^+ - \xi^-;$

(iii) $2\det\xi^+ = |\xi^+|^2, \quad 2\det\xi^- = -|\xi^-|^2$;

(iv) $|\xi|^2 = |\xi^+|^2 + |\xi^-|^2, \quad 2\det\xi = |\xi^+|^2 - |\xi^-|^2 = 2\det\xi^+ + 2\det\xi^-$;

(v) $\langle\xi\,;\eta\rangle = \langle\xi^+\,;\eta^+\rangle + \langle\xi^-\,;\eta^-\rangle$ and $\langle\xi^+\,;\eta^-\rangle = \langle\xi^-\,;\eta^+\rangle = 0$;

(vi) $|\xi|^2 - 2\det\xi = 2|\xi^-|^2, \quad |\xi|^2 + 2\det\xi = 2|\xi^+|^2$;

(vii) if $t \in [0\,,1]$, then

$$\left|\left(t\xi + (1-t)\eta\right)^{\pm}\right| \le t|\xi^{\pm}| + (1-t)|\eta^{\pm}|\,.$$

Remark.— Note that if

$$\xi = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad (a,\ b \in \mathbb{R})\,,$$

then

$$|\xi^+| = \frac{|a+b|}{\sqrt{2}}\,, \quad |\xi^-| = \frac{|a-b|}{\sqrt{2}}\,.$$

Before proceeding with the proof of Theorem 1.1, we prove two intermediate results.

LEMMA 1.2.— *Let $g : \mathbb{R}^2 \to \mathbb{R}$ be convex and satisfy*

$$g(a,b) = g(b,a) = g(-a,-b) \quad \textit{for every } a,\ b \in \mathbb{R} \tag{1.1}$$

then

$$g(sx+ty\,,\ sx-ty) \le g(x+y\,,\ x-y) \tag{1.2}$$

for every $x,\ y \in \mathbb{R}$ and $s,\ t \in [0\,,1]$.

Proof.— We first prove that for every $a \in \mathbb{R}$, $x \in \mathbb{R}$, $b \in \mathbb{R}$ and $y \in R$, then

$$g(a,a) \le g(a+x\,,\ a-x)\,, \tag{1.3}$$

$$g(b,-b) \le g(b+y\,,\ -b+y)\,. \tag{1.4}$$

In fact, using the convexity of g and (1.1), we get

$$\begin{aligned} g(a,a) &= g\left(\frac{1}{2}(a+x\,,\ a-x) + \frac{1}{2}(a-x\,,\ a+x)\right) \\ &\le \frac{1}{2}g(a+x\,,\ a-x) + \frac{1}{2}g(a-x\,,\ a+x) = g(a+x\,,\ a-x)\,, \end{aligned}$$

i.e. (1.3).

Similarly to get (1.4) we use the convexity of g and (1.1), namely

$$\begin{aligned} g(b,-b) &= g\left(\frac{1}{2}(b+y\,,\,-b+y)+\frac{1}{2}(b-y\,,\,-b-y)\right) \\ &\le \frac{1}{2}g(b+y\,,\,-b+y)+\frac{1}{2}g(b-y\,,\,-b-y)=g(b+y\,,\,-b+y)\,. \end{aligned}$$

We now prove (1.2) using the convexity of g, (1.3) and (1.4):

$$\begin{aligned} g(sx+ty\,,\,sx-ty) &= g\big(s(x+ty\,,\,x-ty)+(1-s)(ty,-ty)\big) \\ &\le sg(x+ty\,,\,x-ty)+(1-s)g(ty,-ty) \\ &\le sg(x+ty\,,\,x-ty)+(1-s)g(x+ty\,,\,x-ty) \\ &= g(x+ty\,,\,x-ty) \\ &= g\big(t(x+y\,,\,x-y)+(1-t)(x,x)\big) \\ &\le tg(x+y\,,\,x-y)+(1-t)g(x,x) \\ &\le tg(x+y\,,\,x-y)+(1-t)g(x+y\,,\,x-y) \\ &= g(x+y\,,\,x-y)\,. \end{aligned}$$

Hence the result. □

LEMMA 1.3.— *Let g be as in lemma 1.2, then there exists a function $f : R^{2\times 2} \to \mathbb{R}$ such that*

$$f \text{ is convex,} \tag{1.5}$$

$$f \text{ satifies (H),} \tag{1.6}$$

$$f\restriction_{R_d^{2\times 2}} = g\,, \quad i.e., \quad f = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = g(a,b)\,. \tag{1.7}$$

Proof.— Let us define $f : R^{2\times 2} \to \mathbb{R}$ by

$$f(\xi) = g\left(\frac{|\xi^+|+|\xi^-|}{\sqrt{2}}\,,\,\frac{|\xi^+|-|\xi^-|}{\sqrt{2}}\right) \tag{1.8}$$

of every $\xi \in R^{2\times 2}$. Observe that (1.6) holds trivially. To prove (1.7) we let

$$\xi = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

We then get immediately that

$$f(\xi) = g\left(\frac{1}{2}\left(|a+b| + |a-b|\right), \frac{1}{2}\left(|a+b| - |a-b|\right)\right) = g(a,b)\,.$$

We finally prove (1.5). For any ξ, $\eta \in R^{2\times 2}$ and $\lambda \in [\,0\,,\,1\,]$, we set

$$x = \lambda\left|\xi^+\right| + (1-\lambda)\left|\eta^+\right|, \quad y = \lambda\left|\xi^-\right| + (1-\lambda)\left|\eta^-\right|, \tag{1.9}$$

$$sx = \left|\lambda\xi^+ + (1-\lambda)\eta^+\right|, \quad ty = \left|\lambda\xi^- + (1-\lambda)\eta^-\right|. \tag{1.10}$$

Note that

$$s, t \in [\,0\,,\,1\,]. \tag{1.11}$$

Hence using Lemma 1.2 and (1.8)-(1.11), we get

$$\begin{aligned} f\big(\lambda\xi + (1-\lambda)\eta\big) &= g\left(\frac{sx+ty}{\sqrt{2}}, \frac{sx-ty}{\sqrt{2}}\right) \\ &\leq g\left(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}\right) \\ &\leq \lambda g\left(\frac{\left|\xi^+\right| + \left|\xi^-\right|}{\sqrt{2}}, \frac{\left|\xi^+\right| - \left|\xi^-\right|}{\sqrt{2}}\right) + \\ &\quad + (1-\lambda) g\left(\frac{\left|\eta^+\right| + \left|\eta^-\right|}{\sqrt{2}}, \frac{\left|\eta^+\right| - \left|\eta^-\right|}{\sqrt{2}}\right) \\ &= \lambda f(\xi) + (1-\lambda) g(\eta)\,. \,\square \end{aligned}$$

We now turn to the proof of Theorem 1.1.

Proof of Theorem 1.1

(i)$\Rightarrow$(ii) is trivial.

(ii)$\Rightarrow$(i) For

$$\xi = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in R_d^{2\times 2},$$

let $g(a,b) = f(\xi)$, Then g satisfies the conditions of Lemma 1.2. Using then Lemma 1.3, we deduce that there exists $\tilde{f} : R^{2\times 2} \to \mathbb{R}$ such that

$$\tilde{f} \text{ is convex,} \tag{1.12}$$

$$\tilde{f} \text{ satisfies (H),} \tag{1.13}$$

$$\tilde{f}\restriction_{R_d^{2\times 2}} = g\,. \tag{1.14}$$

Since f and $\tilde{f}$ satisfy (H), we deduce that $f = \tilde{f}$ on the whole of $R^{2\times 2}$ and thus the result. $\square$

2. Polyconvexity on diagonal matrices

We now turn our attention to the notion of polyconvexity. Let us recall the definition (see Ball [2] or Dacorogna [5] for more details).

DEFINITIONS

(i) *A function* $f : R^{2\times 2} \to \mathbb{R}$ *is said to be polyconvex if there exists* $g : R^{2\times 2} \times \mathbb{R} \to \mathbb{R}$ *convex such that*

$$f(\xi) = g(\xi\,,\,\det\xi) \quad \textit{for every } \xi \in R^{2\times 2}\,.$$

(ii) *We say that* $f\restriction_{R_d^{2\times 2}}$ *is polyconvex if there exists* $g : \mathbb{R}^3 \to \mathbb{R}$ *convex such that for every*

$$\xi = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in R_d^{2\times 2}\,, \quad f(\xi) = g(a, b, ab)\,.$$

We then have the main result of this section.

THEOREM 2.1.— *Let* $f : R^{2\times 2} \to \mathbb{R}$ *satisfy (H). The following conditions are then equivalent:*

(i) f *is polyconvex,*

(ii) $f\restriction_{R_d^{2\times 2}}$ *is polyconvex,*

(iii) *the following holds*

$$\sum_{i=1}^{4} \lambda_i f(A_i) \geq f\left(\sum_{i=1}^{4} \lambda_i A_i\right)$$

for every $A_i \in R_d^{2\times 2}$, *every* $\lambda_i \geq 0$ *with* $\sum_{i=1}^{4} \lambda_i = 1$, *such that*

$$\sum_{i=1}^{4} \lambda_i \det(A_i) = \det\left(\sum_{i=1}^{4} \lambda_i A_i\right) \ ;$$

in particular if $g : \mathbb{R}^3 \to \mathbb{R}$ *is defined by*

$$g(a, b, \delta) \equiv \inf\left\{ \sum_{i=1}^{4} \lambda_i f\begin{pmatrix} a_i & 0 \\ 0 & b_i \end{pmatrix} \;\middle|\; \sum_{i=1}^{4} \lambda_i (a_i, b_i, a_i b_i) = (a, b, \delta) \right\}$$

then g is convex and

$$f\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = g(a, b, ab)\,,$$

$$f(\xi) = g\left(\frac{|\xi^+| + |\xi^-|}{\sqrt{2}}\,,\ \frac{|\xi^+| - |\xi^-|}{\sqrt{2}}\,,\ \det\xi\right)$$

for every $\xi \in R^{2\times 2}$;

(iv) *for every $\eta \in R^{2\times 2}_d$, there exist $\alpha(\eta)$, $\beta(\eta)$, $\gamma(\eta) \in R$ such that*

$$f(\xi) \geq f(\eta) + \left\langle \begin{pmatrix} \alpha(\eta) & 0 \\ 0 & \beta(\eta) \end{pmatrix} ; \xi - \eta \right\rangle + \gamma(\eta)(\det\xi - \det\eta)$$

for every $\xi \in R^{2\times 2}_d$ and $\langle \cdot\,;\cdot\rangle$ denotes the scalar product in $R^{2\times 2}$; in partucular if

$$h(x, y, \delta) \equiv \sup_{a,b\in \mathbb{R}} \Big\{ \alpha(a,b)(x-a) + \beta(a,b)(y-b) + \\ + \gamma(a,b)(\delta - ab) + f\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \Big\}\,,$$

then h is convex and

$$f\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = h(x, y, xy)\,,$$

$$f(\xi) = h\left(\frac{|\xi^+| + |\xi^-|}{\sqrt{2}}\,,\ \frac{|\xi^+| - |\xi^-|}{\sqrt{2}}\,,\ \det\xi\right)$$

for every $\xi \in R^{2\times 2}$.

Remarks

(i) One should compare this result with that of Ball [2] (see also Dacorogna [5]) for general polyconvex functions. For example from (iii) we see that to test polyconvexity it is enough to take 4 diagonal matrices instead of 6 general matrices. In R^n, $n \geq 2$, the gain will be obviously even bigger (for example if $n = 3$, with a theorem similar to the above one, we need only 8 diagonal matrices instead of 20 general one).

(ii) A similar observation can be made with (iv).

Proof of Theorem 2.1

(i)$\Rightarrow$(ii) is trivial.

(ii)$\Rightarrow$(i) Since $f\lceil_{R^{2\times 2}_d}$ is polyconvex, there exists $g:\mathbf{R}^3\to\mathrm{I\!R}$ convex such that

$$f\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = g(a,b,ab) \quad \text{for every } a,\, b\in\mathrm{I\!R}\,. \tag{2.2}$$

Moreover under the assumption (H) on f, we deduce that

$$g(a,b,ab) = g(-a,-b,ab) = g(b,a,ab) \quad \text{for every } a,\, b\in\mathrm{I\!R}\,. \tag{2.3}$$

Now set

$$G(a,b,\delta) \equiv \frac{1}{4}\left\{g(a,b,\delta)+g(b,a,\delta)+g(-a,-b,\delta)+g(-b,-a,\delta)\right\}, \tag{2.4}$$

from which we have

$$G:\mathbf{R}^3\to\mathrm{I\!R} \text{ is convex,} \tag{2.5}$$

$$G(a,b,\delta)=G(b,a,\delta)=G(-a,-b,\delta), \tag{2.6}$$

$$G(a,b,ab) = f\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}. \tag{2.7}$$

Applying Lemma 1.3 to $G(\,\cdot\,,\,\cdot\,,\delta)$, we find that there exists $\widetilde{G}: R^{2\times 2}\times\mathrm{I\!R}\to\mathrm{I\!R}$ such that

$$\widetilde{G} \text{ is convex,} \tag{2.8}$$

$$\widetilde{G}(R_\alpha\xi R_\beta,\delta)=\widetilde{G}(\xi,\delta) \text{ for every } R_\alpha,\ R_\beta\in \mathrm{O}^+ \text{ and } \xi\in R^{2\times 2}, \tag{2.9}$$

$$\widetilde{G}\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},\delta\right) = G(a,b,\delta)\,. \tag{2.10}$$

(The convexity of $\widetilde{G}$ in (2.8) is obtained exactly as in the proof of Lemma 1.3.)

In particular, it follows from (2.10), (2.7) and (2.2) that

$$\widetilde{G}\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},ab\right) = g(a,b,ab) = f\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \tag{2.11}$$

and

$$\widetilde{G}(\xi,\det\xi) = g\left(\frac{|\xi^+|+|\xi^-|}{\sqrt{2}}\,,\ \frac{|\xi^+|-|\xi^-|}{\sqrt{2}}\,,\ \det\xi\right) \tag{2.12}$$

for every a, $b \in \mathbb{R}$ and $\xi \in R^{2\times 2}$. Here we used the fact

$$\frac{|\xi^+| + |\xi^-|}{\sqrt{2}} \cdot \frac{|\xi^+| - |\xi^-|}{\sqrt{2}} = \det \xi \,.$$

Hence for $\xi \in R^{2\times 2}$ with

$$R_\alpha \xi R_\beta = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} ,$$

$$\begin{aligned} \tilde{G}(\xi, \det \xi) &= \tilde{G}\big(R_\alpha \xi R_\beta \,,\, \det(R_\alpha \xi R_\beta)\big) \\ &= \tilde{G}\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, ab\right) = f\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \\ &= f(R_\alpha \xi R_\beta) = f(\xi) \,. \end{aligned}$$

Therefore f is polyconvex.

(ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv) The proof of this is classical and in particular identical to that of Dacorogna [5, th. 1.3, p. 106]. □

3. Rank one convexity on diagonal matrices

As we mentioned in the introduction, a theorem as those of the preceding sections cannot be proved for rank one convex functions. Let us recall first the following definitions.

DEFINITIONS

(i) A function $f : R^{2\times 2} \to \mathbb{R}$ *is rank one convex if*

$$f\big(t\xi + (1-t)\eta\big) \le t f(\xi) + (1-t) f(\eta)$$

for every $t \in [\,0\,,\,1\,]$, ξ, $\eta \in R^{2\times 2}$ *with* $\det(\xi - \eta) = 0$.

(ii) Similarly $f\restriction_{R_d^{2\times 2}}$ *is said to be rank one convex if the above inequality holds for every diagonal matrix* ξ *and* η *with* $\det(\xi - \eta) = 0$.

Remark. — As well known (see the references), we always have

$$f \text{ convex } \underset{\not\Leftarrow}{\Longrightarrow} f \text{ polyconvex } \underset{\not\Leftarrow}{\Longrightarrow} f \text{ rank one convex.}$$

Before producing counterexamples to the implication

$$f\restriction_{R_d^{2\times 2}} \text{ rank one convex } \Longrightarrow f \text{ rank one convex}$$

(the converse implication being trivial) we give a very elementary characterisation of rank one convexity on $R_d^{2\times 2}$ for some C^2 functions.

PROPOSITION 3.1. — *Let*

$$f(\xi) = g\big(|\xi|^2\,,\, \det \xi\big)$$

with $g \in C^2(\mathbb{R}^2)$. The following is then equivalent

(i) $f\restriction_{R_d^{2\times 2}}$ is rank one convex;

(ii) g satisfies

$$\begin{aligned} 4g_{xx}(a^2+b^2\,,\, ab)a^2 + 4g_{xy}(a^2+b^2\,,\, ab)ab\, + \\ + g_{yy}(a^2+b^2\,,\, ab)b^2 + 2g_x(a^2+b^2\,,\, ab) \geq 0 \end{aligned}$$

for every a, $b \in \mathbb{R}$ and where

$$g_{xx} = \frac{\partial^2 g}{\partial x^2}\,, \quad g_{xy} = \frac{\partial^2 g}{\partial x \partial y}\,,$$

$$g_{yy} = \frac{\partial^2 g}{\partial y^2}\,, \quad g_x = \frac{\partial g}{\partial x}\,.$$

Proof

(i)$\Rightarrow$(ii) Let

$$\xi = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \in R_d^{2\times 2}, \quad \eta = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in R_d^{2\times 2}$$

with $\det \eta = xy = 0$ and $t \in \mathbb{R}$. Let

$$\begin{aligned} \varphi(t) = f(\xi + t\eta) &= f\begin{pmatrix} u+tx & 0 \\ 0 & v+ty \end{pmatrix} \\ &= g\big(u^2+v^2+2t(ux+vy)+t^2(x^2+y^2)\,,\, uv+t(vx+uy)\big)\,. \end{aligned}$$

Since $f\lceil_{R_d^{2\times 2}}$ is rank one convex, then necessarily $\varphi''(0) \geq 0$. A direct computation, bearing in mind that $xy = 0$, leads to

$$\begin{aligned}\varphi''(0) = 4g_{xx}(u^2+v^2\,,\,uv)(u^2x^2+v^2y^2) + 4g_{xy}(u^2+v^2\,,\,uv)uv(x^2+y^2)\\ + g_{yy}(u^2+v^2\,,\,uv)(u^2y^2+v^2x^2) + 2g_x(u^2+v^2\,,\,uv)(x^2+y^2) \geq 0\,.\end{aligned}$$

Dividing by x^2+y^2 and setting

$$a^2(x^2+y^2) = u^2x^2+v^2y^2\,, \quad b^2(x^2+y^2) = u^2y^2+v^2x^2$$

and bearing in mind that $xy = 0$, we obtain the claimed result.

(ii)$\Rightarrow$(i) follows elementarily from the above proof. □

Remark.— A similar computation is done in Dacorogna-Douchet-Gangbo-Rappaz [6, prop. 1.1]. It shows that rank one convexity on the whole of $R^{2\times 2}$ is equivalent to

$$4g_{xx}(x,y)a^2 + 4g_{xy}(x,y)ab + g_{yy}(x,y)b^2 + 2g_x(x,y) \geq 0$$

for every $(x,y,a,b) \in \mathbb{R}^4$ with

$$\begin{cases} a^2+b^2 \leq x\,, \\ (a+b)^2 - x \leq 2y \leq x - (a-b)^2\,. \end{cases}$$

One sees clearly that Proposition 3.1. follows from the above condition if we set $x = a^2+b^2$ ($\Rightarrow y = ab$).

We may now give the two following counterexamples which are implicitly contained in Dacorogna-Douchet-Gangbo-Rappaz [6, Prop. 1.6 and 1.8]. For the explicit computations of the following constants b_1 and b_2, we refer to the above paper.

Counterexample 3.1.— Let $\mathrm{b} \geq 0$, $\alpha > 2+\sqrt{2}$ and

$$f_{\alpha,\mathrm{b}}(\xi) = |\xi|^{2\alpha} - 2^{\alpha-1}\mathrm{b}\left|\det\xi\right|^{\alpha}\,.$$

Then

$$f_{\alpha,\mathrm{b}} \text{ is rank one convex } \iff \mathrm{b} \leq b_2 \tag{3.1}$$

$$f_{\alpha,\mathrm{b}}\lceil_{R_d^{2\times 2}} \text{ is rank one convex } \iff \mathrm{b} \leq b_1 \tag{3.2}$$

(for the precise value of b_1, b_2, see the above paper) and $b_2 < b_1$.

Counterexample 3.2.— Let $\mathrm{b} \geq 0$, $\alpha > (9+5\sqrt{5})/4$ and

$$f_{\alpha,\mathrm{b}}(\xi) = |\xi|^{2\alpha}\big(|\xi|^2 - 2\mathrm{b}\det\xi\big)\,.$$

Then

$$f_{\alpha,\mathrm{b}} \text{ is rank one convex } \iff \mathrm{b} \leq b_2 \tag{3.3}$$

$$f_{\alpha,\mathrm{b}}\restriction_{R_d^{2\times 2}} \text{ is rank one convex } \iff \mathrm{b} \leq b_1 \tag{3.4}$$

(for the precise value of b_1, b_2, see the above paper) and $b_2 < b_1$.

4. The different envelopes

We now turn our attention to some properties of the envelopes of a given function f satisfying (H).

Before being explicit, we need one more notion of convexity (introduced by Morrey [8]).

DEFINITION .— *Let $f : R^{2\times 2} \to \mathbb{R}$ be continuous. Then f is said to be quasiconvex if for every $\Omega \subset \mathbb{R}^2$ bounded domain and for every $u \in C_0^\infty(\Omega\,;\mathbb{R}^2)$ (i.e. $u = (u_1, u_2) \in \big(C^\infty(\Omega)\big)^2$ and has compact support) and for every $\xi \in R^{2\times 2}$,*

$$f(\xi)\cdot \mathrm{meas}(\Omega) \leq \int_\Omega f\big(\xi + \nabla u(x)\big)\,\mathrm{d}x\,.$$

Remarks

(i) In minimisation problems of the calculus of variations, this is the right notion. As seen by its definition, it is very hard, in practice, to check such a condition. In fact one always have

$$f \text{ convex } \Longrightarrow f \text{ polyconvex } \Longrightarrow f \text{ quasiconvex}$$
$$\Longrightarrow f \text{ rank one convex.}$$

The reverse of the last implication is still open (although Sverak [10] has produced a counterexample in higher dimension). The fact that f quasiconvex $\not\Rightarrow f$ polyconvex can be found in Sverak [9] and Alibert-Dacorogna [1].

(ii) Note immediately that if f satisfies (H) the quasiconvexity of f cannot be infered from the quasiconvexity on diagonal matrices (since every continuous function is quasiconvex when restricted to diagonal matrices).

For a given function $f : R^{2\times 2} \to \mathbb{R}$, we define the convex, polyconvex, quasiconvex and rank one convex envelope to be respectively

$$\begin{aligned} Cf &= \sup\{g \le f \mid g \text{ convex}\}, \\ Pf &= \sup\{g \le f \mid g \text{ polyconvex}\}, \\ Qf &= \sup\{g \le f \mid g \text{ quasiconvex}\}, \\ Rf &= \sup\{g \le f \mid g \text{ rank one convex}\}. \end{aligned}$$

Then the following is well known (*cf.* Dacorogna [5]);

$$Cf(A) = \inf\left\{ \sum_{i=1}^{5} \lambda_i f(A_i) \;\middle|\; \sum_{i=1}^{5} \lambda_i A_i = A \right\}$$

$$Pf(A) = \inf\left\{ \sum_{i=1}^{6} \lambda_i f(A_i) \;\middle|\; \sum_{i=1}^{6} \lambda_i (A_i, \det A_i) = (A, \det A) \right\}$$

for every $A \in R^{2\times 2}$.

The following result is elementary and we will not prove it (it is completely identical to that of Theorem 3.1 of Buttazzo-Dacorogna-Gangbo [3]).

THEOREM 4.1.— *If $f = R^{2\times 2} \to \mathbb{R}$ satisfy (H), then so do Cf, Pf, Qf and Rf.*

Less trivially we have a better characterisation of Cf and Pf when f satisfies (H).

THEOREM 4.2.— *Lef $f : R^{2\times 2} \to \mathbb{R}$ satisfy (H) and*

$$f(\xi) \ge \langle \alpha ; \xi \rangle + \beta$$

for some $\alpha \in R^{2\times 2}$, $\beta \in \mathbb{R}$ and for every $\xi \in R^{2\times 2}$.

(A) Let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$\begin{aligned} g(a,b) = \inf\Bigg\{ \sum_{i=1}^{3} \lambda_i f\begin{pmatrix} a_i & 0 \\ 0 & b_i \end{pmatrix} \;\Bigg|\; \lambda_i \ge 0\,,\ \sum_{i=1}^{3} \lambda_i = 1\,, \\ \sum_{i=1}^{3} \lambda_i a_i = a\,,\ \sum_{i=1}^{3} \lambda_i b_i = b \Bigg\}\,, \end{aligned} \tag{4.1}$$

then

$$Cf(\xi) = g\left(\frac{|\xi^+| + |\xi^-|}{\sqrt{2}}, \frac{|\xi^+| - |\xi^-|}{\sqrt{2}}\right) \text{ for every } \xi \in R^{2\times 2}. \tag{4.2}$$

(B) Let $h^* : \mathbb{R}^2 \to \mathbb{R}$ *be defined by*

$$h^*(x^*, y^*) = \sup_{x,y\in\mathbb{R}} \left\{ xx^* + yy^* - f\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \right\} \tag{4.3}$$

and $h^{**} : \mathbb{R}^2 \to \mathbb{R}$ *be defined by*

$$h^{**}(x, y) = \sup_{x^*,y^*} \left\{ xx^* + yy^* - h^*(x^*, y^*) \right\}, \tag{4.4}$$

then

$$Cf(\xi) = h^{**}\left(\frac{|\xi^+| + |\xi^-|}{\sqrt{2}}, \frac{|\xi^+| - |\xi^-|}{\sqrt{2}}\right). \tag{4.5}$$

Remarks

(i) As in Theorem 2.1, we see that (A) gives a much easier way to compute the convex envelope than the usual one. Indeed we need only 3 diagonal matrices, instead of 5 general matrices (as usually implied by Carathéodory's Theorem).

(ii) Theorem 4.2 might, as Theorem 2.1, be known by those working in convex analysis, but we are unaware of any place where it is explicity quoted.

(iii) (B) in Theorem 4.2 should be compared with Theorem 3.2 in Buttazzo-Dacorogna-Gangbo [3].

Proof of Theorem 4.2

(A) Let g be as stated. Observe that by Carathéodory's Theorem, g is convex (*cf.* Dacorogna [5, th. 1.1, p. 201]) and because of the invariance of f, one has $g(a, b) = g(b, a) = g(-a, -b)$. So setting for $\xi \in R^{2\times 2}$

$$\tilde{f}(\xi) = g\left(\frac{|\xi^+| + |\xi^-|}{\sqrt{2}}, \frac{|\xi^+| - |\xi^-|}{\sqrt{2}}\right),$$

we get from Lemma 1.3 that $\tilde{f}$ is convex and

$\tilde{f}(R_\alpha \xi R_\beta) = \tilde{f}(\xi)$ for every $\xi \in R^{2\times 2}$ and every $R_\alpha,\ R_\beta \in O^+$,
$\tilde{f} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = g(a,b)$.

We now show that in fact $\tilde{f} = Cf$ as claimed. To do this, we first prove that $\tilde{f} \leq Cf$ and then that $Cf \leq \tilde{f}$.

Step 1. $\tilde{f} \leq Cf$

Oberve that $\tilde{f} \leq f$. This is easy, since for every $\xi \in R^{2\times 2}$, we can find $R_\alpha,\ R_\beta \in O^+$ such that

$$R_\alpha \xi R_\beta = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},$$

then

$$\tilde{f}(\xi) = \tilde{f}(R_\alpha \xi R_\beta) = \tilde{f}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = g(a,b) \leq f\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = f(\xi).$$

Since $\tilde{f}$ is convex and less than f, we conclude that $\tilde{f} \leq Cf$.

Step 2. $\tilde{f} \geq Cf$

Let as above $\xi \in R^{2\times 2}$. Then there exist $R_\alpha,\ R_\beta \in O^+$ such that

$$R_\alpha \xi R_\beta = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

and thus using the definition of g and Theorem 4.1 we get

$$\begin{aligned}
\tilde{f}(\xi) &= \tilde{f}(R_\alpha \xi R_\beta) = g(a,b) \\
&= \inf\left\{ \sum_{i=1}^{3} \lambda_i f\begin{pmatrix} a_i & 0 \\ 0 & b_i \end{pmatrix} \,\middle|\, \sum_{i=1}^{3} \lambda_i a_i = a\,,\ \sum_{i=1}^{3} \lambda_i b_i = b \right\} \\
&\geq \inf\left\{ \sum_{i=1}^{3} \lambda_i Cf\begin{pmatrix} a_i & 0 \\ 0 & b_i \end{pmatrix} \,\middle|\, \sum_{i=1}^{3} \lambda_i a_i = a\,,\ \sum_{i=1}^{3} \lambda_i b_i = b \right\} \\
&= Cf\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = Cf(\xi).
\end{aligned}$$

Hence the result.

(B) This is proved similarly. Let h^{**} be as stated. Then it is clear that h^{**} is convex $h^{**}(a,b) = h^{**}(b,a) = h^{**}(-a,-b)$. Thus from Lemma 1.3, we get that if

$$\widetilde{f}(\xi) = h^{**}\left(\frac{|\xi^+| + |\xi^-|}{\sqrt{2}}, \frac{|\xi^+| - |\xi^-|}{\sqrt{2}}\right)$$

then $\widetilde{f}$ is convex, rotationally invariant and

$$\widetilde{f}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = h^{**}(a,b).$$

To show that $Cf = \widetilde{f}$, it is therefore sufficient to prove that $\widetilde{f} \leq Cf$ and $Cf \leq \widetilde{f}$ and this is done as in (A). □

We end up this section with the corresponding theorem for polyconvex functions.

THEOREM 4.3.— *Let $f : R^{2\times 2} \to \mathbb{R}$ satisfy (H) and*

$$f(\xi) \geq \langle \alpha\,;\xi\rangle + \beta \det \xi + \gamma$$

for some $\alpha \in R^{2\times 2}$, β, $\gamma \in \mathbb{R}$ and for every $\xi \in R^{2\times 2}$.

(A) Let $g : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$g(a,b,\delta) = \inf\left\{ \sum_{i=1}^{4} \lambda_i f\begin{pmatrix} a_i & 0 \\ 0 & b_i \end{pmatrix} \,\middle|\, \lambda_i \geq 0\,,\ \sum_{i=1}^{4} \lambda_i = 1\,,\right.$$
$$\left. \sum_{i=1}^{4} \lambda_i (a_i, b_i, a_i b_i) = (a,b,\delta) \right\} \tag{4.18}$$

then

$$Pf(\xi) = g\left(\frac{|\xi^+| + |\xi^-|}{\sqrt{2}}, \frac{|\xi^+| - |\xi^-|}{\sqrt{2}}, \det \xi\right). \tag{4.19}$$

(B) Let $h^p = \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$h^p(a,b,\delta) = \sup_{x,y}\left\{ ax + by + \delta xy - f\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\right\}. \tag{4.20}$$

Let $h^{pp} : \mathbf{R}^3 \to \mathbb{R}$ be defined by

$$h^{pp}(a,b,\delta) = \sup_{x,y,\varepsilon} \left\{ ax + by + \delta\varepsilon - h^{p}(x,y,\varepsilon) \right\} , \tag{4.21}$$

then

$$Pf(\xi) = h^{pp}\left(\frac{|\xi^{+}| + |\xi^{-}|}{\sqrt{2}} , \frac{|\xi^{+}| - |\xi^{-}|}{\sqrt{2}} , \det \xi \right) . \tag{4.22}$$

for every $\xi \in R^{2\times 2}$.

Remarks

(i) As in Theorem 3.1, we see that (A) gives a much easier way to compute Pf than the usual one (see Dacorogna [5]). Indeed we need only 4 diagonal matrices instead of 6 general ones.

(ii) (B) of the theorem should be compared to Proposition 3.3 of Buttazzo-Dacorogna-Gangbo [3].

Proof of Theorem 4.3

(A) Let g be as stated. It is clear by Carathéodory's Theorem that $g : \mathbb{R}^3 \to \mathbb{R}$ is convex. Furthermore since f satisfies (H), we immediately get that

$$g(a,b,\delta) = g(-a,-b,\delta) = g(b,a,\delta) .$$

We therefore use Theorem 2.1 to get $\widetilde{G} : R^{2\times 2} \times \mathbb{R} \to \mathbb{R}$ such that

(i) $\widetilde{G}$ is convex, $\widetilde{G}(\xi,\delta) = g\left(\frac{|\xi^{+}| + |\xi^{-}|}{\sqrt{2}} , \frac{|\xi^{+}| - |\xi^{-}|}{\sqrt{2}} , \delta \right)$;

(ii) $\widetilde{G}(R_\alpha \xi R_\beta \, , \delta) = \widetilde{G}(\xi,\delta)$ for every $\xi \in R^{2\times 2}$, R_α, $R_\beta \in O^{+}$;

(iii) $\widetilde{G}\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} , \delta \right) = g(a,b,\delta)$.

We now show that in fact

$$Pf(\xi) = \widetilde{G}(\xi \, , \det \xi) .$$

To do this, we first prove $\widetilde{G} \leq Pf$, then $\widetilde{G} \geq Pf$.

Step 1. $\widetilde{G} \le Pf$

Let $\xi \in R^{2\times 2}$, R_α, $R_\beta \in O^+$ be such that

$$R_\alpha \xi R_\beta = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} .$$

Then we have

$$\begin{aligned}\widetilde{G}(\xi\,,\,\det\xi) &= \widetilde{G}\big(R_\alpha\xi R_\beta\,,\,\det(R_\alpha\xi R_\beta)\big) = g(a,b,ab)\\ &\le f\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = f(\xi)\,.\end{aligned}$$

Since $\widetilde{G}$ is convex and in view of the above inequality we deduce that $\widetilde{G} \le Pf$.

Step 2. $\widetilde{G} \ge Pf$

Let ξ, R_α, R_β be as above. Using the definition of g, Pf and Theorem 4.1, we get

$$\begin{aligned}\widetilde{G}(\xi\,,\,\det\xi) &= g(a,b,ab)\\ &= \inf\left\{\sum_{i=1}^{4}\lambda_i f\begin{pmatrix} a_i & 0 \\ 0 & b_i \end{pmatrix} \;\middle|\; \sum_{i=1}^{4}\lambda_i(a_i,b_i,a_ib_i) = (a,b,ab)\right\}\\ &\ge \inf\left\{\sum_{i=1}^{4}\lambda_i Pf\begin{pmatrix} a_i & 0 \\ 0 & b_i \end{pmatrix} \;\middle|\; \sum_{i=1}^{4}\lambda_i(a_i,b_i,a_ib_i) = (a,b,ab)\right\}\\ &= Pf\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = Pf(\xi)\,.\end{aligned}$$

Thus **(A)** is established.

(B) This is proved similarly. Let h^{pp} be as stated. Observe that h^{pp} is trivially convex and has the required invariance to apply Theorem 2.1. We get $\widetilde{H} : R^{2\times 2} \times \mathbb{R} \to \mathbb{R}$ such that

(i) $\widetilde{H}$ is convex;

(ii) $\widetilde{H}(R_\alpha\xi R_\beta\,,\,\delta) = \widetilde{H}(\xi,\delta)$;

(iii) $\widetilde{H}\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\,,\,\delta\right) = h^{pp}(a,b,\delta)$.

Using the same argument as above, we deduce that

$$Pf(\xi) = \widetilde{H}(\xi,\det\xi)\,. \ \square$$

5. An example

In this section, we shall give an example of how to apply the above results.

Let $g : \mathbb{R} \to \mathbb{R}$ be such that $g(t) = g(-t)$ and $f : R^{2\times 2} \to \mathbb{R}$ be defined by

$$f(\xi) = g\big(\sqrt{|\xi|^2 - 2\det\xi}\big)\,. \tag{5.1}$$

We then have the following Proposition.

PROPOSITION 5.1

$$Cf(\xi) = Pf(\xi) = Qf(\xi) = Rf(\xi) = g^{**}\big(\sqrt{|\xi|^2 - 2\det\xi}\big)\,.$$

Remark.— A similar result holds for $g\big(\sqrt{|\xi|^2 + 2\det\xi}\big)$.

Proof of Proposition 5.1

We devide the proof into two steps.

Sept 1.— It follows trivially from Theorem 1.1 that

$$g^{**}\big(\sqrt{|\xi|^2 - 2\det\xi}\big) \leq Cf(\xi)\,.$$

Sept 2.— In order to conclude the proof, we only need to show that $Rf(\xi) \leq g^{**}\big(\sqrt{|\xi|^2 - 2\det\xi}\big)$. Using Corollary 2.2.9 in Dacorogna [5], we have that for any $\varepsilon > 0$, there exist $\lambda^\varepsilon \in [0, 1]$, α^ε and β^ε with $\lambda^\varepsilon \alpha^\varepsilon + (1 - \lambda^\varepsilon)\beta^\varepsilon = x - y$ such that

$$\varepsilon + g^{**}(x - y) \geq \lambda^\varepsilon g(\alpha^\varepsilon) + (1 - \lambda^\varepsilon) g(\beta^\varepsilon)\,. \tag{5.2}$$

Now set

$$a^\varepsilon = \alpha^\varepsilon + y\,, \quad b^\varepsilon = \beta^\varepsilon + y$$

and

$$\xi_1 = \begin{pmatrix} a^\varepsilon & 0 \\ 0 & y \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} b^\varepsilon & 0 \\ 0 & y \end{pmatrix}.$$

Then $\det(\xi_1 - \xi_2) = 0$ and

$$\lambda^\varepsilon \xi_1 + (1 - \lambda^\varepsilon)\xi_2 = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$

Hence using (5.2) and Theorem 5.1.1 in Dacorogna [5], we have

$$
\begin{aligned}
&Rf\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \leq \\
&\leq \inf\left\{ \sum_{i=1}^{2} \lambda_i f(A_i) \,\middle|\, \sum_{i=1}^{2} \lambda_i A_i = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \text{ with } \det(A_1 - A_2) = 0 \right\} \\
&\leq \lambda^{\varepsilon} f(\xi_1) + (1-\lambda^{\varepsilon}) f(\xi_2) = \lambda^{\varepsilon} g(\alpha^{\varepsilon}) + (1-\lambda^{\varepsilon}) g(\beta^{\varepsilon}) \\
&\leq g^{**}(x-y) + \varepsilon\,.
\end{aligned}
$$

Letting ε tend to 0, we get

$$
Rf\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \leq g^{**}(x-y)\,. \tag{5.3}
$$

Let $\xi \in R^{2\times 2}$, then there exist P_α, $P_\beta \in \mathrm{O}^+$ such that

$$
P_\alpha \xi P_\beta = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\,.
$$

Using Theorem 4.1 and (5.3), we find that

$$
Rf(\xi) = Rf\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \leq g^{**}(x-y) = g^{**}\left(\sqrt{|\xi|^2 - 2\det\xi}\right)
$$

and thus the conclusion. □

Acknowledgements

The second author wishes to thank the "Fonds National Suisse de la Recherche" for financial support while visiting E.P.F.L.

References

[1] ALIBERT (J.J.) and DACOROGNA (B.) .— *An example of a quasiconvex function that is not polyconvex in two dimensions*, Arch. Rational Mech. Anal. **117** (1992), pp. 155-166.

[2] BALL (J.M.) .— *Convexity conditions and existence theorems in nonlinear elasticity*, Arch. Rational Mech. Anal. **64** (1977), pp. 337-403.

[3] BUTTAZZO (G.), DACOROGNA (B.) and GANGBO (W.) .— *On the envelopes of functions depending on singular values of matrices*, to appear in Boll. Unione Mat. Italiana.

[4] CIARLET (P.) .— *Mathematical elasticity*, Vol. 1, North Holland, 1987.

[5] DACOROGNA (B.) .— *Direct Methods in the Calculus of Variations*, Springer-Verlag, Berlin, 1989.

[6] DACOROGNA (B.), DOUCHET (J.), GANGBO (W.) and RAPPAZ (J.) .— *Some examples of rank one convex functions in dimension two*, Proc. Roy. Soc. Edinburgh **114 A** (1990), pp. 135-150.

[7] IWANIEC (T.) and LUTOBORSKI (A.) .— *Integral estimates for null Lagrangians*, to appear in Arch. Rational Mech. Anal.

[8] MORREY (C.B.) .— *Multiple Integrals in the Calculus of Variations*, Springer-Verlag, Berlin, 1966.

[9] SVERAK (V.) .— *Quasiconvex functions with subquadratic growth*, Proc. Roy. Soc. London A **433** (1991), pp. 723-725.

[10] SVERAK (V.) .— *Rank-one convexity does not imply quasiconvexity*, Proc. Roy. Soc. Edinburgh **120 A** (1992), pp. 185-189.