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# Non-uniformly hyperbolic billiards[(*)]

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RÉSUMÉ. — On donne une caractérisation de l'hyperbolicité (non uniforme) ou *comportement chaotique* de fonctions différentiables, avec singularités, en termes de formes quadratiques de Liapounov. Une analyse générale des courbes convexes désirables comme composantes régulières de la frontière d'un billard plan chaotique est aussi obtenue. On prouve que tout arc convexe suffisament petit peut faire partie d'un tel billard. On donne des descriptions de classes très amples de billards plans avec ce type de propriétés ergodiques. On montre qu'un arc de circonférence, plus petit qu'une demi-circonférence, peut être $C^4$-perturbé sans perdre le comportement chaotique des billards de Bunimovich.

ABSTRACT. — We give a characterization of (non-uniform) hyperbolicity or *chaotic behavior* of smooth maps with singularities in terms of Lyapunov quadratic forms. A general analysis of focusing curves that are suitable as regular components of a chaotic plane billiard is obtained. It is proved that any sufficiently small focusing arc can be part of the boundary of such billiards. We provide descriptions of very large classes of plane billiards with these ergodic properties. We show that less than a half circumference can be $C^4$-perturbed keeping the chaotic behavior of the Bunimovich-type billiards.

## 0. Introduction

Let $M$ be a smooth compact $d$-manifold, $\nu$ a probability measure absolutely continuous with respect to the Riemannian volume measure, $N$ a subset such that $\nu(N) = 0$, $H = M \setminus N$, and let $f : H \to H$ be the restriction to $H$ of a $\nu$-invariant $C^r$ diffeomorphism defined on an open subset of $M$, $r \geq 2$.

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Motivated by billiard problems, one imposes certain conditions on the derivatives of $f$ (see Section 1, and Katok & Strelcyn [10, Part I and V]) and call such a $f$ a discontinuous dynamical system or a smooth map with singularities. In $N$ the map is not defined.

It is well known that the existence of non zero Lyapunov exponents for $f$ allows one to construct locally invariant submanifolds. If all Lyapunov exponents are different from zero in $\Lambda$, a non-uniform hyperbolic decomposition is obtained at each point of $\Lambda$ and (if $\Lambda$ has positive measure) there is a countable number of invariants sets $\Lambda_i$ of positive measure $\left(\sum \nu(\Lambda_i) = \nu(\Lambda)\right)$ such that $f\restriction_{\Lambda_i}$ is ergodic. See, for exemple, Pesin & Sinai [20] for a survey of such results.

So, the study of Lyapunov exponents in sets of measure one for a given smooth map with singularities is a reasonable approximation to the knowledge of its ergodic properties. We will say that such dynamical systems have Pesin region of measure one, or are non-uniformly hyperbolic or that they have chaotic behavior, if the set of points where all the Lyapunov exponents are different from zero has measure one.

In the words of Pesin himself [19] "*there is a deep informal connection*" between the results on dispersing billiards of Sinai [22] and his own abstract theory. Supporting Pesin's remark, there is the fact that the works of Sinai (and Bunimovich) were based on the study of ergodic properties of the geodesic flow on manifolds of negative curvature. See the introduction of Markarian [18].

In our previous paper we combine the methods of Lewowicz to study hyperbolic properties considering the asymptotic behavior of some quadratic forms, with the point of view given by geometrical considerations concerning the way in which geodesics get apart on a manifold (norm of Jacobi fields).

In this work we give a characterization of non-uniform hyperbolicity in terms of quadratic forms. In particular, the chaotic behavior of a smooth dynamical system with singularities implies the existence of an increasing non degenerate quadratic form. So it becomes natural to look for these quadratic forms in order to try to find all the focusing curves that can be part of the boundary of a chaotic billiard.

Then, aside from the geometrical motivations, we analyze directly, in terms of basic elements of the boundary curves, what the coefficient of adequate quadratic forms should be in order to have chaotic behavior. The main result we obtain in this direction is that any short focusing arc can be a component of the boundary of a chaotic billiard. This allows us to describe very large classes of plane billiards with these ergodic properties.

The present paper is organized as follows. Section 1 consists of statements and proofs of theorems that characterize non-uniform hyperbolicity in terms of quadratic forms.

In Section 2, we give the elements necessary to study billiards in $\mathbb{R}^2$, and the geometrical motivations that allow us to understand better why we use certain quadratic forms in our previous paper.

Section 3 is devoted to a general study of focusing curves that are suitable as regular components of the boundary of a billiard with non-uniform hyperbolicity. It is proved that any sufficiently small focusing arc can be part of the boundary of such billiards.

Finally in Section 4, we show that an arc of circumference with length less than half that of the circumference may be $C^4$ perturbed keeping chaotic behavior of Bunimovich-type billiards, which best known example is the stadium.

It seems that a more detailed study of the structure and length of the local manifolds whose existence derive from Pesin theory is the natural way to treat the ergodicity of billiard systems with chaotic behavior. And it would be very interesting to apply our methods to study non-uniform hyperbolicity of other *physical systems: billiards* with non ellastic hits, motion of particles on potential fields, etc. See Sinai [21], Kubo [12] and Baldwin [2].

## 1. Characterization of non-uniform hyperbolicity of smooth maps with singularities in terms of quadratic forms

In this section let $M$, $d$, $\nu$, $N$, $H$, $f$ be as in the Introduction, $d \geq 2$, $r \geq 1$. We will assume also that

$$\log^+ \left\| \left(f^{\pm 1}\right)'_x \right\| \in L^1(H, \nu)\,, \quad \log^+ s = \max\{\log s\,,\, 0\}\,,$$

a condition that is needed to apply the ergodic multiplicative theorem of Osedelets.

We say that $x$ is a regular (Osedelets) point of $f$ if there exist numbers $\lambda_1(x) > \lambda_2(x) > \cdots > \lambda_m(x)$ and a decomposition $T_xM = E_1(x) \oplus E_2(x) \oplus \cdots \oplus E_m(x)$ such that

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \left\| (f^n)'_x \omega \right\| = \lambda_i(x)$$

for every $0 \neq \omega \in E_i(x)$ and every $1 \leq i \leq m(x)$. $E_j(x)$ is the proper subspace of the Lyapunov exponent $\lambda_j(x)$. The theorem of Oseledets establishes that the set of regular points has measure one.

$\Sigma(f)$ will denote the Pesin region, that is the set of regular points that have only non zero exponents. If $\nu(\Sigma(f)) = 1$ we will say that the map $f$ (or the dynamical system defined by it) is non-uniform hyperbolic or has chaotic behavior.

$B : TM \to \mathbb{R}$ is a quadratic form on $M$ if $B_x : T_xM \to \mathbb{R}$ is a quadratic form in the usual sense. If $f : M \to M$ is a diffeomorphism we denote by $f^\sharp B$ (pull back of $B$ by $f$) the quadratic form $(f^\sharp B)_x u = B_{f(x)}(f'_x u)$. $B$ is non degenerate on a subset $N \subset M$ if $B_x$ is non degenerate for every $x \in N$, that is, the associate matrix of $B_x$ in any base has non zero eigenvalues. $B$ is positive in $N$ if $B_x u > 0$ for every $x \in N$ and every non zero $u \in T_xM$.

For any quadratic form $B$ on the orbit of $x \in M$ we define

$$S_x = \{u \in T_xM : B((f^n)'u) < 0\,,\ n \geq 0\}$$
$$U_x = \{u \in T_xM : B((f^n)'u) > 0\,,\ n \leq 0\}\,.$$

For $x \in \Sigma(f)$, let be

$$E_x^s = \bigoplus_{\lambda_i(x)<0} E_i(x)\,, \quad E_x^u = \bigoplus_{\lambda_i(x)>0} E_i(x)\,.$$

THEOREM 1.— *Let $B : TM \to \mathbb{R}$ a quadratic form such that:*

*(i) for $y \in H$, $T_yM = L_y \bigoplus K_y$ with $B(v) < 0$ (resp. $> 0$) for every non zero $v \in L_y$ (resp. $K_y$) and $0 < \dim L_y < d$ with $\dim L_{f^n(x)} = \dim L_x$ for every $n \in \mathbb{Z}$ and every $x \in H$;*

*(ii) $B_x$ depends measurably on $x$, and is non degenerate in $H$;*

*(iii) $P_x = (f^\sharp B - B)_x$ is positive for every $x \in H$.*

*Then $\nu(\Sigma(f)) = 1$ and for every $x \in \Sigma(f)$*

$$S_x = E_x^s\,, \quad U_x = E_x^u\,.$$

*Remarks*

• A slightly stronger version of the theorem can be proved in the same way: instead of condition *(iii)* we assume that $P$ is positive eventually: $P \geq 0$; and for almost every $x \in M$ there exists $k = k(x) \in \mathbb{N}$ such that

$$B\big((f^{k+1})'u\big) - B\big((f^k)'u\big) > 0$$

for every non zero $u \in T_xM$. This fact is pointed out in part (b) of the proof.

• Hypotesis *(i)* is verified if, either:

(1) $M$ is a 2-manifold, since $S_x$ and $U_x$ are non void subspaces and then $\dim S_x = \dim U_x = 1$, see Markarian [18]; or

(2) $B$ is continuous and non degenerate in $M$, and $M$ is connected. In this case the non degeneracy of $B$ ensures the decomposition $T_yM = L_y \bigoplus K_y$ and the dimension of $L_y$ depends neither on the special choice of the decomposition nor on the point $y \in H$.

THEOREM 2.— *If $r \geq 2$ and $\nu\big(\Sigma(f)\big) = 1$, there exists a quadratic form $B : TM \to \mathbb{R}$ such that conditions (i)-(iii) of Theorem 1 are verified.*

These theorems are ergodic versions of Theorem 2.1 in Lewowicz [14] where it is proved that the existence of a continuous quadratic form $B$ verifying *(iii)* is equivalent to $f$ being Anosov. Also a simple version of Theorem 1, was proved in Markarian [18].

*Proof of Theorem 1*

(a) For $n > 0$, $x \in M$, let $w_n \in T_{f^n(x)}M$ be such that $B_{f^n(x)}w_n < 0$. Then $B_x\big((f^{-n})'w_n\big) < 0$ and $\big\{(f^{-n})'w_n\big\} \subset T_xM$ has a convergent subsequence, say, to $w_\infty$.

If $B\big((f^N)'w_\infty\big) > 0$ for some $N \geq 0$. $B\big((f^{n_j})'w_\infty\big) > 0$ for $n_j > N$ and the same inequality is valid in some neighbourdhood of $(f^{n_j})'w_\infty$ to which some $w_{n_j}$ must belong, $n_j$ being large enough.

(We have used repeatedly that $P > 0$.)

Then $B\big((f^n)'w_\infty\big) < 0$ and $w_\infty \in S_x$. So $S_x$ contains a one dimensional subspace. Similarly for $U_x$.

**(b)** Define, for each real $a > 0$,

$$M_a = \left\{ x \in H : |B_x v| < \frac{1}{a} \|v\|^2 \text{ for all } 0 \neq v \in T_x M \right\}$$

$$E_a = \left\{ y \in H : P_y w \geq a \|w\|^2 \text{ for all } w \in T_y M \right\}$$

$$F_a = \{ z \in H : P_z u > a B_z u \text{ for all } 0 \neq u \in U_z \} .$$

Then, if $0 < b \leq c$, $M_c \subset M_b$, $E_c \subset E_b$ and $F_c \subset F_b$. Also

$$\bigcup_{a>0} M_a = \bigcup_{a>0} E_a = H = \bigcup_{a>0} F_a .$$

The only non trivial fact here is the last equality and this is a consequence of the following observation: if $z \in H \setminus \bigcup_{a>0} F_a$ we can select a sequence $u_n \in U_z$. $\|u_n\| = 1$, $n \geq N$ with $P_z(u_n) \leq (1/n) B_z(u_n)$ and for an accumulation vector $u$ of $\{u_n\}$, $P_z(u) \leq 0$. If $P$ is positive eventually we can work instead of $E_a$ and $F_a$ with the sets

$$\left\{ y \in H : |B\big((f^{k+1})'w\big) - Bw \geq \frac{1}{k} \|w\|^2 \text{ for all } w \in T_y M \right\}$$

$$\left\{ z \in H : |B\big((f^{k+1})'u\big) - Bu > \frac{1}{k} Bu \text{ for all } 0 \neq u \in U_z \right\}$$

respectively, and study the points which have infinite iterates in these sets, as it is done in (c) and (e).

**(c)** As each $x \in H$ is in some $E_c$, applying the Poincaré Recurrence theorem we obtain an increasing sequence $\{n_k\}_{k \in \mathbb{N}}$ such that $f^{n_k}(x) \in E_c$, and for $u \in S_x$ we have

$$\begin{aligned} B\big((f^m)'u\big) - B(u) = \sum_{i=0}^{m-1} P\big((f^i)'u\big) \geq \\ \geq c \sum_{j=0}^{N} \big\|(f^{n_j})'u\big\|^2 , \quad n_N \leq m < n_{N+1} . \end{aligned}$$

So, if $\big\|(f^{n_j})'u\big\| \not\to 0$ is concluded that $B\big((f^m)'u\big) \to +\infty$ which is a contradiction ($B < 0$ on $S$). Then, if $u$, $v \in S_x$, $\lambda$, $\mu \in \mathbb{R}$, we have

$$\lim_{k \to +\infty} \big\|(f^{n_k})'(\lambda u + \mu v)\big\| = 0 .$$

This implies $B\big((f^n)'(\lambda u+\mu v)\big)<0$ since otherwise

$$B\big((f^{n_k})'(\lambda u+\mu v)\big) > B\big((f^N)'(\lambda u+\mu v)\big) \geq 0$$

which contradicts the previous statement. Then $S_x$ (and $U_x$) are subspaces of $T_xM$.

(d) Consider now the sequence $\big\{(f^{-n})'L_{f^n(x)}\big\}_{x\in\mathbb{N}} \subset T_xM$ and a limit subspace $L_\infty$ obtained in the following way: if $\{v_1^n, \ldots, v_k^n\}$ is an orthonormal basis of $(f^{-n})'L_{f^n(x)}$, by compacity, there exists a sequence $n_p \to +\infty$ such that $\lim_{p\to+\infty} v_j^{n_p} = v_j$. We have that $\{v_1, \ldots, v_k\}$ is linearly independent and $L_\infty$ is spanned by its vectors. Now if $1 \leq i \leq k$, for all $n \geq 0$, we have

$$\begin{aligned} B_{f^n(x)}\big((f^n)'_x v_i\big) &= B_{f^n(x)}\left((f^n)'_x\left(\lim_{p\to+\infty} v_i^{n_p}\right)\right)\\ &= \lim_{p\to+\infty} B_{f^n(x)}\big((f^n)'_x v_i^{n_p}\big) <\\ &< \lim_{p\to+\infty} B_{f^{n_p}(x)}\big((f^{n_p})' v_i^{n_p}\big) \leq 0\,. \end{aligned}$$

(It was used repeatedly that $B - f^\sharp B < 0$, and that $(f^{n_p})'v_i^{n_p} \in L_{f^{n_p}(x)}$.) So, $L_\infty \subset S_x$ (and $K_\infty \subset U_x$) and $\dim S_x \geq k(\dim U_x \geq n-k)$. Sinces $S_x \cap U_x = O_x$, it is concluded that $S_x \oplus U_x = T_xM$ for each $x \in H$.

(e) Let $x \in D = F_a \cap M_a$ for some $a > 0$. The measure of $D$ is as large as we need. For $u \in U_x$ we have

$$B(f'u) = B(u) + P(u) > B(u)(1+a)\,,$$

and if $f^{n_k}(x) \in D$ for an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$, $n_0 = 0$, then

$$\begin{aligned} B\big((f^{n_{k+1}})'u\big) &\geq B\big((f^{n_k+1})'u\big) > (1+a)B\big((f^{n_k})'u\big) \geq\\ &\geq (1+a)B\big((f^{n_{k-1}+1})'u\big) \geq\\ &\geq (1+a)^2 B\big((f^{n_{k-1}})'u\big) \geq \cdots > (1+a)^{k+1}B(u)\,. \end{aligned}$$

Since $f^{n_k}(x) \in M_a$, finally we have

$$\big\|(f^{n_{k+1}})'u\big\|^2 > a(1+a)^{k+1}B(u)\,.$$

Now, if $x$ is a regular point of $f$ and

$$N_n(x) = \#\{j : f^j(x) \in D\,,\ 0 \le j < n\} \le n\,,$$

it follows that

$$\begin{aligned}\lim_{m\to+\infty} \frac{1}{m} \log\left\|(f^m)'u\right\| &\ge \frac{1}{2} \liminf_{k\to+\infty} \frac{k+1}{n_{k+1}} \log(1+a) = \\ &= \frac{1}{2} \log(1+a) \lim_{n\to+\infty} \frac{N_n(x)}{n} > 0\,,\end{aligned}$$

for $x \in \overline{D}$ with $\nu(\overline{D}) > \nu(D)$. The existence and positiveness of the last limit derives from a standard application of the theorem of Birkhoff to $\chi_D$, the characteristic function of $D$: if

$$\overline{\chi}_D(x) = \lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_D\big(f^j(x)\big) = \lim_{n\to+\infty} \frac{N_{n(x)}}{n}$$

$\nu$-almost every $x \in H$, then

$$\nu(D) = \int_H \chi_D \,\mathrm{d}\nu = \int_M \overline{\chi}_D \,\mathrm{d}\nu$$

and as $\overline{\chi}_D(y) \le 1$, $y \in D$ it follows that $\overline{\chi}_D(x) > 0$ for $x \in D$ with $\nu(\overline{D}) \ge \nu(D)$.

(f) Now, if $u = \sum_{i=1}^m v_i \in U_x$, $v_i \in E_i(x)$, from (e) it follows that

$$0 < \lim_{n\to\infty} \frac{1}{n} \log\left\|(f^n)'u\right\| = \lambda_j$$

where $j$ is the smallest index $i$ such that $v_i \ne 0$. Also, if $s = \sum_{i=1}^m w_i \in S_x$, $w_i \in E_i(x)$,

$$0 > \lim_{n\to+\infty} \frac{1}{n} \left\|(f^n)'s\right\| = \lambda_r(x)$$

where $r$ is the largest index $i$ such that $w_i \ne 0$. So, if $v_\ell \in E_\ell(x)$, $v_\ell = u+s$, the definition of Lyapunov exponents and some elementary properties of $\log z$ permit to deduce that

$$\lambda_\ell(x) = \lim_{n\to+\infty} \frac{1}{n} \log\left\|(f^n)'v_\ell\right\| = \lim_{k\to+\infty} \frac{1}{n_k} \log\left\|(f^{n_k})'u\right\| = \lambda_j(x) > 0$$

if $u \ne 0$ because $(f^{n_k})'s \to 0$ if $k \to +\infty$, $f^{n_k}x \in D$.

Also

$$\lambda_\ell(x) = \lim_{n\to-\infty} \frac{1}{n} \log\left\|(f^n)' v_\ell\right\| = \lim_{k\to+\infty} \frac{1}{m_k} \log\left\|(f^{m_k})' s\right\| = \lambda_r(x) < 0$$

if $s \neq 0$.

So either $u$ or $s$ must be zero and $\lambda_\ell(x)$ coincides either with $\lambda_j(x)$ or with $\lambda_r(x)$, and all the Lyapunov exponents are different of zero.

(g) This argument also proves that

$$\bigoplus_{\lambda_i(x)<0} E_i \subset S_x \,, \qquad \bigoplus_{\lambda_i(x)>0} E_i \subset U_x$$

The statement of the theorem then follows. □

*Proof of theorem 2*

(a) We will use Lyapunov charts (Pesin [19]) in the version of Ledrappier & Young [13, Section 2 and Appendix]. we define

$$\lambda^+(x) = \min\{\lambda_i(x)\,,\, \lambda_i > 0\}\,, \quad \lambda^-(x) = \max\{\lambda_i(x)\,,\, \lambda_i < 0\}$$
$$u(x) = \dim E^u(x)\,, \quad s(x) = \dim E^s_x\,.$$

These four numbers are invariant along orbits and $u(x) + s(x) = d$ if $x \in \Sigma(f)$. For each $\ell \in \mathbb{N}$, let be

$$\Gamma_\ell = \left\{x \in \Sigma(f) : \lambda^-(x) \leq -\frac{1}{\ell} < \frac{1}{\ell} \leq \lambda^+(x)\right\} \quad \text{and} \quad \varepsilon = \frac{1}{100\,\ell}\,,\ \Gamma_0 = \emptyset\,,$$

$\Gamma_\ell$ is a measurable and invariant set. For $(x,z) \in \mathbb{R}^u \times \mathbb{R}^s$, let be

$$\left\|(x,z)\right\| = \max\{\left\|x\right\|_u\,,\, \left\|z\right\|_s\}$$

where $\|\cdot\|_u$ and $\|\cdot\|_s$ are the euclidean norms on $\mathbb{R}^u$ and $\mathbb{R}^s$ respectively. We will omit the subscripts in these norms. The closed disk in $\mathbb{R}^u$ of radius $a$ centered at 0 is denoted by $\mathsf{R}^u(a)$, $\mathsf{R}(a) = \mathsf{R}^u(a) \times \mathbb{R}^s(a)$. There exist a measurable function $\Lambda : \Sigma(f) \to [1\,,\infty)$, $\Lambda(f^{\pm1}x) \leq e^\varepsilon \Lambda(x)$ and an embedding $\phi_x : \mathsf{R}(\Lambda(x)^{-1}) \to M$ such that if $\widetilde{f}_x = \phi^{-1}_{f(x)} \circ f \circ \phi_x : U_x \to U_{f(x)}$ (connecting map), $U_x \subset \mathbb{R}^d$, then for each $x \in \Gamma_\ell$ we have

$$\left\|D\widetilde{f}_x(0)v\right\| \geq e^{(1/\ell)-\varepsilon}\left\|v\right\| \text{ for every } v \in \mathbb{R}^u$$
$$\left\|D\widetilde{f}_x(0)v\right\| \leq e^{-(1/\ell)+\varepsilon}\left\|v\right\| \text{ for every } v \in \mathbb{R}^s\,.$$

From here on we will omit the zero in $D\widetilde{f}_x(0)$.

**(b)** Then, if $L = e^{(1/\ell)-\varepsilon} > 1$, we have

$$\begin{cases} \|D\widetilde{f}_x^n v\| \geq L^n \|v\| \\ \|D\widetilde{f}_x^{-n} v\| \leq L^{-n} \|v\| \end{cases} \quad \text{for every } v \in \mathbb{R}^u$$

$$\begin{cases} \|D\widetilde{f}_x^n v\| \leq L^{-n} \|v\| \\ \|D\widetilde{f}_x^{-n} v\| \geq L^n \|v\| \end{cases} \quad \text{for every } v \in \mathbb{R}^s ,$$

and every $n \in \mathbb{N}$. So, if $w = v_1 + v_2$, $v_1 \in \mathbb{R}^u$, $v_2 \in \mathbb{R}^s$, let for example, be $\|w\| = \|v_1\| \geq \|v_2\|$, for consider $n > 0$ (if $\|w\| = \|v_2\| > \|v_1\|$ we must take $n < 0$). Then

$$\begin{aligned} \|D\widetilde{f}_x^n w\| &\geq \|D\widetilde{f}_x^n v_1\| - \|D\widetilde{f}_x^n v_2\| \geq \\ &\geq L^n \|v_1\| - L^{-n} \|v_2\| \geq (L^n - L^{-n}) \|v_1\| = \\ &= (L^n - L^{-n}) \|w\| . \end{aligned}$$

If $N_\ell$ is such that $L^{N_\ell} - L^{-N_\ell} \geq \sqrt{2}$, we have proved that *there exists a measurable and invariant function* $N : \Sigma(f) \to \mathbb{N}$ *defined by* $N(x) = N_\ell$ *if* $x \in \Gamma_\ell \setminus \Gamma_{\ell-1}$ *such that*

$$\|D\widetilde{f}_x^n w\|^2 \geq 2\|w\|^2 \quad \textit{for every} \quad w \in \mathbb{R}^d ,$$

*either for* $n \geq N(x)$ *or* $n \leq -N(x)$.

**(c)** *if* $\nu(\Sigma(f)) = 1$ *and* $B$ *is a measurable quadratic form on* $M$ *such that* $P > 0$ *on* $\Sigma(f)$, *then*

(c$_1$) $B_x < 0$ (resp. $> 0$) *on* $E_x^s$ *(resp.* $E_x^u$*)*;

(c$_2$) $B$ *is non degenerate.*

Actually, for every $v \in E_x^s \setminus \{0\}$, we have $\|(f^n)'_x v\| \to 0$. So $B_x(v) < 0$ since otherwise

$$B\big((f^n)'v\big) = P\big((f^{n-1})'v\big) + B\big((f^{n-1})'v\big) > B\big((f^{n-1})'v\big) = \cdots > B(v) \geq 0$$

which is a contradiction because as $f^{n_k}(x) \in M_a$ for some $a > 0$ and an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$, $n_0 = 0$ (Poincaré Recurrence Theorem), we have $\|(f^{n_i})'v\|^2 > aB\big((f^{n_i})'v\big)$. Now, if for some $x \in \Sigma(f)$ and some

$0 \neq w \in T_x M$ we have $B_x(v+w) = B_x(x)$ (in this case $B$ is degenerate), then for any $v \in E^s_x$ and every $0 \neq \lambda \in \mathbb{R}$,

$$B(v+\lambda w) = \lambda^2 B\Big(\frac{v}{\lambda} + w\Big) = \lambda^2 B\left(\frac{v}{\lambda}\right) < 0$$

and so the subspace generate by $E^s_x$ and $w$ intersects trivially $E^u_x$. Thus $w \in E^s_x$ and as $B(w) = 0$, it follows that $w = 0$.

(d) *If* $g_n : H \to \mathbb{R}$ *and* $N : H \to \mathbb{N}$ *are measurable functions, then* $\sum_{i=0}^{N} g_i$ *is a measurable function.*

The proof of this fact is standard.

(e) Obviously, $\Sigma(f) = \bigcup_{\ell=1}^{\infty} \Gamma_\ell \setminus \Gamma_{\ell-1}$. If $x \notin \Sigma(f)$ let $B_x w = 0$ for every $w \in T_x M$, and for each $x \in \Gamma_\ell \setminus \Gamma_{\ell-1}$ define

$$B_x w = \sum_{i=0}^{N_\ell - 1} \big\|\mathrm{D}\widetilde{f}_x^{N_\ell+i} \mathrm{D}\phi_x^{-1} w\big\|^2 - \big\|\mathrm{D}\widetilde{f}_x^{i} \mathrm{D}\phi_x^{-1} w\big\|^2$$

whose measurability follows from (b) and (d).

$$\begin{aligned} P_x w &= \big(F^\sharp B - B\big)_x w = B_{f(x)}(f'_x w) - B_x w \\ &= \sum_{i=0}^{N_\ell-1} \big\|\mathrm{D}\widetilde{f}_{f(x)}^{N_\ell+i}\, \mathrm{D}\phi_{f(x)}^{-1} f'_x w\big\|^2 - \big\|\mathrm{D}\widetilde{f}_{f(x)}^{i}\, \mathrm{D}\phi_{f(x)}^{-1} f'_x w\big\|^2 + \\ &\quad + \big\|\mathrm{D}\widetilde{f}_x^{N_\ell+i}\, \mathrm{D}\phi_x^{-1} w\big\|^2 + \big\|\mathrm{D}\widetilde{f}_x^{i}\, \mathrm{D}\phi_x^{-1} w\big\|^2 . \end{aligned}$$

Differentiating the condition of $\widetilde{f}_x$ it follows that $\mathrm{D}\phi_{f(x)}^{-1} f'_x = \mathrm{D}\widetilde{f}_x\, \mathrm{D}\phi_x^{-1}$, then

$$\begin{aligned} P_x w &= \sum_{\ell=0}^{N_\ell-1} \big\|\mathrm{D}\widetilde{f}_x^{N_\ell+i+1}\, \mathrm{D}\phi_x^{-1} w\big\|^2 - \big\|\mathrm{D}\widetilde{f}_x^{N_\ell+i}\, \mathrm{D}\phi_x^{-1} w\big\|^2 + \\ &\qquad - \big\|\mathrm{D}\widetilde{f}_x^{i+1}\, \mathrm{D}\phi_x^{-1} w\big\|^2 + \big\|\mathrm{D}\widetilde{f}_x^{i}\, \mathrm{D}\phi_x^{-1} w\big\|^2 \\ &= \big\|\mathrm{D}\widetilde{f}_x^{2N_\ell}\, \mathrm{D}\phi_x^{-1} w\big\|^2 - 2\big\|\mathrm{D}\widetilde{f}_x^{N_\ell}\, \mathrm{D}\phi_x^{-1} w\big\|^2 + \big\|\mathrm{D}\phi_x^{-1} w\big\|^2 . \end{aligned}$$

This number is bigger than either $\big\|\mathrm{D}\phi_x^{-1} w\big\|^2$ or $\big\|\mathrm{D}\widetilde{f}_x^{2N_\ell}\, \mathrm{D}\phi_x^{-1} w\big\|^2$ depending on the cases of the result in (b).

(f) So we can apply (c), and *(ii)* and *(iii)* in the statement of the theorem are proved. Finally, *(i)* follows from ($c_1$) and the invariance by $f'$ of the dimension of the proper subspaces in Oseledets decomposition. □

## 2. Plane billiards. Some known results

A plane billiard is the dynamical system describing the free motion of a point mass inside a bounded, connected region of the plane, with elastic reflections at the boundary. This consists of a finite set of curves $\partial Q_i$, $C^{r+1}$, $r \geq 1$, with curvature $\big(|K|\big)$ bounded. The regular components of the boundary, $\partial \widetilde{Q}_i = \partial Q_i \setminus \bigcup_{j \neq i} \partial Q_j$ can have positive (focusing components), negative (dispersing) or zero curvature (neutral). If $n(q)$ is the unit inward normal in $q \in \partial \widetilde{Q}_i$ then the parametrization $q(s)$ and the curvature $K(s)$, where $s$ is the arc length of the component $\partial Q_i$, are defined by

$$q''(s) = \frac{\mathrm{d}\vec{t}}{\mathrm{d}s} = K\vec{n} = Kiq'(s)\,.$$

Let $\pi$ be the natural projection from the tangent bundle to $\mathbb{R}^2$. Following (Cornfeld *et al.* [8]), we define the set

$$M_1 = \Big\{(q,v) \mid q \in \partial \widetilde{Q}_i\,,\ \|v\| = 1\,,\ \big\langle v\,,\, n(q)\big\rangle > 0\Big\}\,.$$

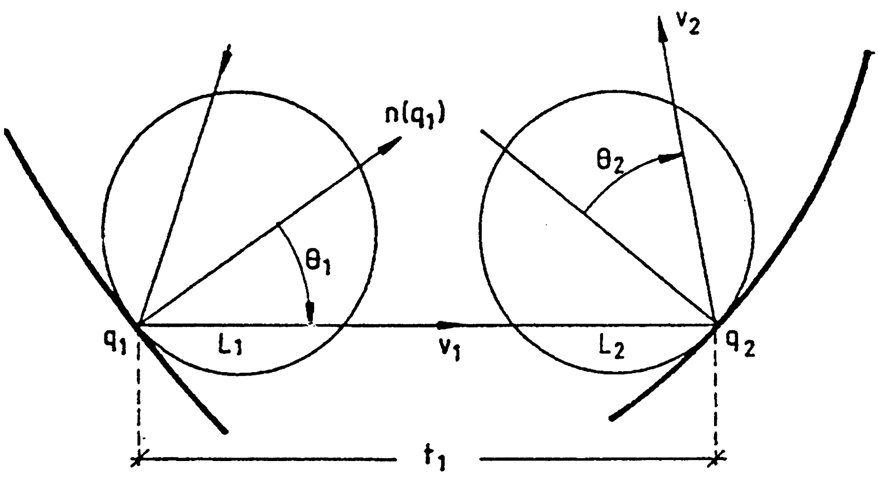


Fig. 1

Given $x_i = (q_1, v_1) \in M_1$, $T_{x_1}$ (if defined) is obtained moving forward in the billiard surface, in the direction $v_1$, a distance (time) $t_1$ till the intersection with $\partial \widetilde{Q}_j$ in $q_2$. Formally $Tx_1 = (q_2, v_2)$ where

$$v_2 = v_1 - 2 < n(q_2)\,, \quad v_1 > n(q_2) \quad \text{(see fig. 1).}$$

Let $N \subset M_1$ be the set of points where $T^k$ is not defined or not continuous for some $k \in \mathbb{N}$. See figure 2, where $x, y \in N$, $H = M_1 \setminus N$.

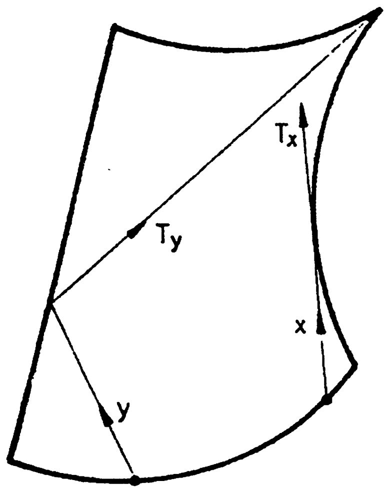


**Fig. 2**

The billiard transformation $T : H \to H$ is measurable, bijective, $C^r$ and $\nu$-measure preserving. As usual $\mathrm{d}\nu = \mathrm{d}s\,\mathrm{d}\theta \cos\theta$, normalized. The proof that $T$ verifies the conditions on the map $f$ of Section 1 is given in details in Part V (*Plane Billiards as smooth dynamical systems*, by J. M. Strelcyn) of Katok & Strelcyn [10]. In particular, to apply Corollary 4.1 and Theorem 5.1, it is needed for $\partial \widetilde{Q}_i$ to be $C^2$ and for $|K|$ to be uniformly bounded.

$T$ is a discontinuous dynamical system (or a smooth map with singularities) if, for example, either $\partial Q_i$ is real analytic or $C^3$ with non vanishing curvatures. These conditions are sufficient to apply Pesin's theory to the billiard transformation $T$.

A wave front is given by the curve $\big(q(s)\,,\, v(s)\big)$ in $M_1$ and so an element in the tangent space in $x \in M_1$, $x = \big(q(0)\,,\, v(0)\big)$ is

$$\big(q'(0)\,,\, v'(0)\big) = (q', v')\,.$$

In higher dimensions $v'$ must be replaced by the covariant derivative along $q(s)$.

Let be:

$s_1$, $s_2$ the arc length of $\partial Q_i$, $\partial Q_j$, in neighbourhoods of $q_1$, $q_2$, respectively (fig. 1);

$q_2(s_2(s_1)) = q_1(s_1) + t_1(s_1)v(s_1)$, the point evolution of a wave front;

$K_j(s_j)$, the curvatures in $q_j(s_j)$;

$\theta_j(s_j)$ the angles of $v_j(s_j)$ with $n(q_j(s_j))$;

$j = 1, 2$.

If the variable $(\cdot)$ is not indicated, it means that we are working in the base point; for example $K_2 = K_2(s_2(0))$.

It is simple to prove the following formulas where

$$\frac{\mathrm{d}s_2}{\mathrm{d}s_1} = \sigma', \quad \frac{\mathrm{d}\theta_2}{\mathrm{d}s_2} = \theta_2', \quad \frac{\mathrm{d}\theta_1}{\mathrm{d}s_1} = \theta_1' :$$

$$K_1 + \theta_1' = (K_2 - \theta_2')\sigma', \tag{1}$$

$$\cos\theta_1 = \left[t_1(K_2 - \theta_2') - \cos\theta_2\right]\sigma'. \tag{2}$$

These expressions correspond to Lemma 2.3 of Sinai [22] adjusted in Lemma 3 of Bunimovich [3], and are important for their construction of locally contracting and expanding fibres.

If $M_1$ is parametrized by $(s,\beta)$ instead of $(s,\theta)$, where $\beta$ is the angle formed by a fixed axis with $v$, we have

$$K_1 + \theta_1' = \frac{\mathrm{d}\beta_1}{\mathrm{d}s_1}, \quad -\theta_2' + K_2 = \frac{\mathrm{d}\beta_1}{\mathrm{d}s_2}. \tag{3}$$

If $q'$ and $v'$ are projected on $iv$, the positive $\pi/2$-rotated of $v$, we obtain a natural parametrization of the tangent space (remember transversal foliations in the study of geodesic flows). Let be

$$V_j = \langle v_j', iv_j\rangle, \quad \alpha_j = \langle q_j', iv_j\rangle, \quad j = 1, 2.$$

Then

$$\begin{aligned}
\alpha_2 &= -\sigma'\cos\theta_2 \\
\alpha_1 &= -\cos\theta_1 = \sigma'\left[\cos\theta_2 - t_1(K_2 - \theta_2')\right] \\
V_2 &= (K_2 + \theta_2')\sigma' \\
V_1 &= (K_1 + \theta_1') = (K_2 - \theta_2')\sigma'.
\end{aligned} \tag{4}$$

Sinai [22] proved that billiards whose components of the boundaries are all dispersing, are ergodic. Bunimovich [4] indicated that billiards whose focusing pieces of the boundaries have constant curvature and do not contain dispersing components are Bernoulli (the stadium, for example). Wojtkowski [25] proved that a large class of billiards with focusing arcs at the boundary have Pesin region of measure one. In Markarian [18], it is proved that the condition obtained by Wojtkowski for focusing curves ($\mathrm{d}^2R/\mathrm{d}s^2 < 0$, $R = 1/K$) is not typical. More precisely, open conditions for focusing curves, different that those in Wojtkowski [25] were obtained for the regular components of the boundary of chaotic billiards.

As we have applied a simpler version of Theorem 1 to study when Lyapunov exponents are non vanishing for several billiards, we will now explain briefly the motivations we had to use certain quadratic forms in such a study.

Consider first the geodesic flow in a compact manifold $M$ of negative curvature. The most natural "distance function" to study the evolution of geodesic flows is the norm of Jacobi fields, perpendicular to the trajectory. Let be $\eta = (p,v) \in SM$, an element of the unitary tangent bundle; $g^t : SM \to SM$ is the geodesic flow defined by

$$g^t\eta = \big(\gamma_\eta(t)\,,\ \gamma'_\eta(t)\big)$$

where $\gamma_\eta$ is the geodesic defined by $\eta$. $D_\eta g^t : T_\eta(SM) \to T_\eta(SM)$ can be consider restricted to $\eta^\perp \oplus \eta^\perp$ where $\eta^\perp = \big\{u \in T_P M : \langle v, u\rangle = 0\big\}$. With this restriction is not difficult to prove that $D_\eta g^t(x,y) = \big(J(t)\,,\ J'(t)\big)$ where $J(t)$ is the Jacobi field, perpendicular to $v$, such that $J(0) = x$, $J'(0) = y$ (we use the symbol $'$ to designate different types of derivatives).

Lewowicz [15] gave a proof, using the quadratic form

$$\left(\frac{1}{2}\,\|J\|^2\right)' = \langle J\,,\ J'\rangle\,,$$

of the fact that geodesic flows in manifolds of negative curvature are Anosov. Observe that the derivative of this quadratic form along the geodesics (equivalent to the $P$ of Theorem 1), is positive:

$$\big\langle J\,,\ J''\big\rangle + \big\langle J'\,,\ J'\big\rangle = -K(\gamma', J) + (J', J')\,.$$

In the case of plane billiards is very natural to take $\big\langle q'\,,\ iv\big\rangle = \alpha$ as the norm of the perpendicular Jacobi field, since in this expression it is reflected the geometry of the boundary, which determines the dynamics of the billiard. The derivative of $\alpha$ along the trajectory of the billiard is

$$\lim_{h\to 0} \frac{\langle q' + hv' , iv \rangle - \langle q', iv \rangle}{h} = \langle v' , iv \rangle = V .$$

Thus, continuing the analogy, we consider the quadratic form $B_x(q', v') = \alpha V$. Then

$$P_x(q', v') = \sigma'^2 \left[ -2K_2 \cos\theta_2 + t_1 (K_2 - \theta_2')^2 \right]$$

which is positive (eventually) if the regular components of the boundary are dispersing or neutral and the trajectories pass by dispersing components in finite time (i.e., for almost every $x$, there exist $k \in \mathbb{N}$ such that $\pi(T^k x)$ is in a dispersing component). This allows us to apply theorem 1 to Sinai's billiards (negative curvature) and to obtain a result somewhat weaker than that of Sinai [22].

For positive curvature the previous expressions do not work neither in geodesic flows nor in billiards. Therefore, turning to the original motivation, we consider again the quadratic form $\alpha^2$ and instead of taking the derivative directly to obtain $B$, we substract its values between one collision before the reflection and the previous one. So, we obtained

$$\alpha_2^2 - \alpha_1^2 = \langle q_1' + t_1 v_1' , iv_1 \rangle^2 - \langle q_1' , iv_1 \rangle^2 = t_1 \left[ t_1 V^2 + 2\alpha V \right] .$$

This, eliminating the first $t_1$ for simplification, was the second form studied in Markarian [18]. The change of $t_1$ by $L_1$ (time that the trajectory – or its continuation – spends inside the osculating circle of radius $R = 1/K$, before or after colliding with the boundary at $q_1$, see figure 1) is quite natural taking into account the geometric study of the question and the results of Wojtkowski [25]. This was our third quadratic form.

Now, it is difficult to continue the study of curves that are allowded in billiards with chaotic behavior by this *case by case method*. This is a main reason why we began our general study of "good" quadratic forms.

## 3. Plane billiards. General analysis. Focusing components

As it was observed in Section 2 a natural parametrization of the phase space of plane billiards is

$$\alpha = \langle q' , iv \rangle , \quad V = \langle v' , iv \rangle .$$

Then the general expression of a quadratic form $B_x$, $x = (q, v)$ is

$$B_x(q', v') = a\alpha^2 + 2b\alpha V + cV^2 ,$$

where $a$, $b$, $c$ are functions of $s, \theta$ (using the natural parametrization of $M_1$). In order to verify condition (ii) of theorem 1, $a$, $b$, $c$ must be measurable functions of $x$ and $ac - b^2 \neq 0$.

We study now which conditions must be verified by $a$, $b$, $c$ if (iii) of theorem 1 is required. We will use formulas (4) in Section 2 ($\theta$ is used instead of $\theta_2$).

$$\begin{aligned}
&P_{x_1}(q', v') = \\
&\quad = \left(T^\sharp B - B\right)_{x_1}(q', v') \\
&\quad = a_2\alpha_2^2 + 2b_2\alpha_2 V_2 + c_2 V_2^2 - (a_1\alpha_1^2 + 2b_1\alpha_1 V_1 + c_1 V_1^2) \\
&\quad = (a_2 - a_1)\alpha_2^2 + 2(b_2 - b_1)\alpha_2 V_2 + (c_2 - c_1)V_2^2 + a_1(\alpha_2^2 - \alpha_1^2) + \\
&\qquad + 2b_1(\alpha_2 V_2 - \alpha_1 V_1) + c_1(V_2^2 - V_1^2) \\
&\quad = \sigma'^2 \Big[(a_2 - a_1)\cos^2\theta + 2(b_2 - b_1)(K_2 + \theta')(-\cos^\theta) + \\
&\qquad + (c_2 - c_1)\left(K_2 + \theta'\right)^2 + a_1\left(2t_1\cos\theta(K_2 - \theta') - t_1^2(K_2 - \theta')^2\right) + \\
&\qquad + 2b_1\left(-2K_2\cos\theta + t_1\left(K_2 - \theta'\right)^2\right) + c_1 4K_2\theta'\Big] .
\end{aligned}$$

Then the sign of $P$ depends on

$$\begin{aligned}
&(c_2 - c_1 - a_1 t_1^2 + 2b_1 t_1)\theta'^2 + 2\theta'\left[-(b_2 - b_1)\cos\theta + K_2(c_2 - c_1) + \right. \\
&\qquad\qquad \left. - a_1 t_1\cos\theta + a_1 t_1^2 K_2 - 2b_1 t_1 K_2 + 2c_1 K_2\right] + \\
&\quad + (a_2 - a_1)\cos^2\theta - 2(b_2 - b_1)K_2\cos\theta + (c_2 - c_1)K_2^2 + \\
&\quad + 2a_1 t_1 K_2\cos\theta - a_1 t_1^2 K_2^2 - 4b_1 K_2\cos\theta + 2b_1 t_1 K_2^2 .
\end{aligned} \tag{5}$$

This expression is positive for every $\theta'$ iff

$$c_2 - c_1 - a_1 t_1^2 + 2b_1 t_1 > 0$$

and

$$\begin{aligned}
&\Big[-(b_2 - b_1)\cos\theta + (c_2 + c_1)K_2 - a_1 t_1\cos\theta + a_1 t_1^2 K_2 - 2b_1 t_1 K_2\Big]^2 + \\
&\quad - (c_2 - c_1 - a_1 t_1^2 + 2b_1 t_1)\Big[(a_2 - a_1)\cos^2\theta - 2(b_2 - b_1)K_2\cos\theta + \\
&\qquad +(c_2 - c_1)K_2^2 + 2a_1 t_1 K_2\cos\theta + 2b_1 t_1 K_2^2 - a_1 t_1^2 K_2^2\Big] < 0 .
\end{aligned}$$

We take $b_i = 1$ in order to have a control of the non degeneration of $B$ and to simplify the calculus. Hence, $P > 0$ iff

$$c_2 - c_1 - a_1 t_1^2 + 2t_1 > 0 \tag{6}$$

and

$$\begin{aligned}(c_1 - 2t_1 + a_1 t_1^2)\left[4c_2 - 2E + (a_2 - a_1)\frac{E^2}{4}\right] + \\ + c_2 E\left[2 - 2a_1 t_1 - (a_2 - a_1)\frac{E}{4}\right] + \frac{a_1^z t_1^2 E^2}{4} < 0\,, \quad \text{if } K_2 \neq 0\,,\end{aligned} \tag{7}$$

where $E = (2\cos\theta)/K_2$. If the component of $q_2$ is focusing ($K_2 > 0$), then $E = L_2$. In the case $K_2 = 0$, (7) becomes

$$(a_2 - a_1)(c_1 - c_2 - 2t_1) + a_1 a_2 t_1^2 < 0\,. \tag{7'}$$

The quadratic forms studied in Markarian [18] correspond to the following values of the parameters:

(a) $a_i = c_i = 0$ allows one to get ergodic properties in Sinai's billiards ($K < 0$);

(b) $a_i = 0$, $c_i = L_i$ gives the condition $\mathrm{d}^2 R/\mathrm{d}s^2 < 0$ for focusing curves (Wojtkowski [25]);

(c) $a_i = 0$, $c_i = t_i$ gives the condition $\mathrm{d}^2 R^{1/3}/\mathrm{d}s^2 < 0$ for focusing curves.

In the last two cases the conditions on the radius of curvature appear when we make a study of focusing curves compatible with the inequalities (6) and (7). Some more conditions on the distances between two different regular components of the boundary are needed. See Theorem A and B, in this section.

We study now the local conditions that a focusing curve must verify in order to be suitable as a part of the boundary of a chaotic billiard. We must look at the behavior of expression (6) and (7) when we have successive reflexions in the same component, with $\theta \cong \pm\pi/2$.

The first simple observation is this one: as $L_i \cong t_i \cong 0$ (see Appendix), if we suppose $a_i$, $c_i$ continuous in a neighbourhood of $(q_2\,, \pm\pi/2)$, the discriminant $\Delta$ of (5) verifies:

$$\Delta \cong 4c_1 c_2 \cong 4c_1^2\,.$$

As we must have $\Delta < 0$. We deduce $\lim_{\theta \to \pm\pi/2} c_i = 0$. If $c_i \ll t_i$ we have

$$\Delta \cong 4t_1 E \cong 4t_1^2 ,$$

and if $c_i \ll t_i$ we have

$$\Delta \cong 4c_1 c_2 \cong 4c_1^2 .$$

So, we must have $c_i \cong t_i$: this justify *a posteriori* our election of $c_i = t_i$ or $L_i$ in our previous paper.

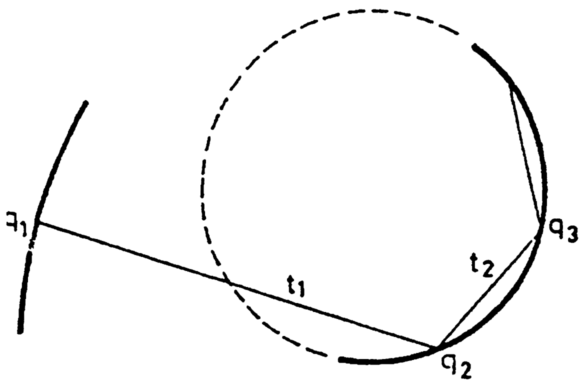


**Fig. 3**

Let $P(A) = q_2 + r(A)e^{iA}q_2'$ (fig. 3) be the polar parametrization of a $C^k$ curve ($k \geq 4$) where (see Appendix):

$$r(A) \doteq \dot{r}A + \frac{\ddot{r}}{2}A^2 + \frac{\dddot{r}}{6}A^3 + \cdots , \quad \dot{r} = \dot{r}(0) = \frac{2}{K_2} > 0 .$$

If the coefficients $a$, $c$ depend differentiably on $s$, $B$ (angle between the oriented tangent and the inward trajectory), in $q_2$ ($B = A$), then they may be developed as

$$c(s, B) = C_1 s + C_2 B + C_3 s^2 + C_4 sB + C_5 B^2 + \\ + C_6 s^3 + C_7 s^2 B + C_8 sB^2 + C_9 B^3 + \cdots$$

$$a(s, B) = A_0 + A_1 s + A_2 B + A_3 s^2 + \cdots$$

Then in $q_2$ we have $s = 0$, $B = A$, and

$$c_2 = c(0, A) = C_2 A + C_5 A^2 + C_9 A^3 + \cdots$$

As $c_2 \cong t_2$ for small $A$, then $C_2 = \dot{r}$ (for $A > 0$; if $A < 0$, $\theta$ close to $-\pi/2$, we must take $C_2 = -\dot{r}$. We will make all the calculus for positive $A$; there are equivalent expressions for $A < 0$). In $q_1$, we have

$$s = s(-A)$$

and

$$c_1 = (-C_1 + 1)\dot{r}A + \left(C_1 \frac{\ddot{r}}{2} + \frac{\ddot{r}}{2} + \dot{r}^2 C_3 - \dot{r}C_4 + C_5\right) A^2 + \cdots$$

As $c_1 \cong t_1$ in first approximation, we have $C_1 = 0$, and finally

$$\begin{aligned} c_1 &= \dot{r}A + \left[\frac{\ddot{r}}{2} + \dot{r}^2 C_3 - \dot{r}C_4 + C_5\right] A^2 + \\ &\quad + \left[\frac{\ddot{r}^2}{2\dot{r}} - \frac{\dddot{r}}{3} - \frac{\dot{r}}{3} - \dot{r}\ddot{r}C_3 + \right. \\ &\quad \left. + \frac{\ddot{r}}{\dot{r}} C_5 - \dot{r}^3 C_6 + \dot{r}^2 C_7 - \dot{r}C_8 + C_9\right] A^3 + \cdots \\ c_2 &= \dot{r}A + C_5 A^2 + C_9 A^3 + \cdots \end{aligned}$$

$$a_1 = A_0 + (-\dot{r}A_1 + A_2)A + \left[\frac{\ddot{r}}{2} A_1 + \frac{\ddot{r}}{2\dot{r}} A_2 + \dot{r}^2 A_3 - \dot{r}A_4 + A_5\right] A^2 + \cdots$$

$$a_2 = A_0 + A_2 A + A_5 A^2 + \cdots$$

Then (6) is immediately verified because

$$c_2 - c_1 - a_1 t_1^2 + 2t_1 = 2\dot{r}A + \left[-\frac{\ddot{r}}{2} + \dot{r}^2 C_3 - \dot{r}C_4 - \dot{r}^2 A_0 - \ddot{r}\right] A^2 + \cdots$$

which is positive ($\dot{r} > 0$) for small angles $A$.

The first member of (7) is now,

$$\begin{aligned} UA^3 + &\left[\frac{2C_5}{\dot{r}} U + \ddot{r}^2 - \frac{4}{3}\dot{r}(\dot{r} + \dddot{r}) + 2\ddot{r}C_5 + 4C_5^2 + \right. \\ &+ 2\dot{r}^2(-\ddot{r}C_3 - \dot{r}^2 C_6 + \dot{r}C_7 - C_8) + 2\dot{r}^2 A_0 C_5 - \dot{r}^2\ddot{r}A_0^2 + \\ &\left. - \frac{\dot{r}^4}{2} A_1 + \frac{\dot{r}^4}{4} A_0^2\right] A^4 + \cdots \end{aligned}$$

where $U = 3\dot{r}\ddot{r} + 2\dot{r}^3 C_3 - 2\dot{r}^2 C_4$.

Because of symmetry considerations it is natural to take $U = 0$, and then, for small $A$, we must have

$$\ddot{r}^2 - \frac{4}{3}\,\dot{r}(\dot{r} + \dddot{r}) + 2\ddot{r}C_5 + 4C_5^2 + 2\dot{r}^2(-\ddot{r}C_3 - \dot{r}^2C_6 + \dot{r}C_7 - C_8) + $$
$$+ 2\dot{r}^2A_0C_5 - \dot{r}^2\ddot{r}A_0 + \frac{\dot{r}^4}{4}\,(A_0^2 - 2A_1) < 0\,. \tag{8}$$

It is interesting to observe that cases (b) and (c) previously analized correspond to:

(b) $A_i = 0$: comparing the expressions of $c_2$ and $L_2$ we obtain $C_5 = 0$, $C_9 = -\dot{r}/6$ and comparing $L_1$ with $c_1$, it is deduced that $U = 0$ and

$$\dot{r}(-\ddot{r}C_3 - \dot{r}^2C_6 + \dot{r}C_7 - C_8) = \dot{r} - \frac{3\ddot{r}^2}{4\dot{r}} + \dddot{r}$$

so (8) becomes

$$-\frac{\ddot{r}^2}{2} + \frac{2\dot{r}}{3}\,(\dot{r} + \dddot{r}) < 0$$

which is the condition of Wojtkowski [25] written in a different way.

(c) $A_i = 0$, comparing the expressions of $c_2$ and $t_2$ we obtain $C_5 = \ddot{r}/2$, $C_9 = \dddot{r}\,/6$; and comparing $t_1$ with $c_1$, it is deduced that

$$\frac{\ddot{r}}{2} + \dot{r}^2C_3 - \dot{r}C_4 + C_5 = -\frac{\ddot{r}}{2} \rightarrow U = 0$$

and

$$\dot{r}(-\ddot{r}C_3 - \dot{r}^2C_6 + \dot{r}C_7 - C_8) = -\frac{\ddot{r}^2}{\dot{r}} - \frac{\dddot{r}}{3} - \frac{\dot{r}}{3}\,;$$

so (8) becomes

$$\ddot{r}^2 - \frac{2}{3}\,\dot{r}(\dot{r} + \dddot{r}) < 0\,,$$

which is condition (8) in Markarian [18].

Also, it is easy to observe that given any $C^4$ small focusing arc (that is, given $\dot{r} > 0$, $\ddot{r}$, $\dddot{r}$, in our polar coordinates), we can choose $C_i$, $A_i$ conveniently, in such a way that $U = 0$ and (8) are verified. For example, we can take $C_6$ in such a way that $-2\dot{r}^4C_6$ dominates all the other terms. *So, we have that any sufficiently small $C^4$ focusing arc can be a regular component of the boundary of a choatic billiard.*

A more general problem is how to construct all possible boundary pieces in order to get good ergodic properties for billiard: angles between consecutive components, distance between components, etc.

We show now how to construct plane billiards with chaotic behavior in which some of the regular components of the boundary are short focusing arcs. We take $a = 0$ in any case and for succesive hits in short focusing arcs $\mathcal{C}$, we define quadratic forms that verify condition (8).

*1(i).*— If $q_1, q_2 \in \mathcal{C}$ and $q_3 \notin \mathcal{C}$ we define $c_2 = L_2$. Then (6) is immediately verified since the first member, in first approximation, is equal to $2\dot{r}A$. (7) is verified if (8) is true with $C_5 = 0 = A_0 = A_1$: it is sufficient to take $C_6$ big enough.

*1(ii).*— If $q_1 \in \mathcal{C}$ and $q_2 \notin \mathcal{C}$ we consider initially only the condition $c_2 \geq 0$. In the first member of (6) we have $c_2 - L_1 + 2t_1 > 2t_1 - L_1 > 0$ if $2t_1 > L_1$; this means that the component of $q_2$ must be outside the semicurvature circle of any point of $\mathcal{C}$. If $K_2 = 0$, the first member of $(7')$ is zero. So, in order to maintain increasing $B$ along the trajectories, they must go eventually to the non neutral components. If $K_2 \neq 0$, the first member of (7) is $(L_1 - 2t_1)(4c_2 - 2E) + 2c_2E$. If $K_2 < 0$, then $E < 0$ and it is enough again to consider $2t_1 > L_1$. If $K_2 > 0$, we define $c_2 = L_2$ and then the previous expression is $2L_2(L_1 + L_2 - 2t_1)$ which is negative if $L_1 + L_2 < 2t_1$; this means that semicurvature circles of focusing components must not intersect themselves.

*1(iii).*— If $q_1, q_3 \notin \mathcal{C}$, $q_2 \in \mathcal{C}$, let $c_1 \geq 0$, $c_2 = L_2$. The first member of (6) is $L_2 - c_1 + 2t_1 > 2t_1 - c_1$; so (6) is verified if $2t_1 \geq c_1$. The first member of (7) is $(c_1 - 2t_1)2L_2 + 2L_2^2 = 2L_2(c_1 + L_2 - 2t_1)$. If the component of $q_1$ is dispersing or neutral we define $c_1 = 0$; if it is focusing, we define $c_1 = L_1$; then (7) is true if the arcs verify the conditions that appeared in 1(ii).

*1(iv).*— If $q_1 \notin \mathcal{C}$ and $q_2, q_3 \in \mathcal{C}$, the verification of (6) do not generate new conditions and, since $c_2 \simeq L_2$, the first member of (7) is

$$(c_1 - 2t_1)(4c_2 - 2L_2) + 2c_2L_2 \simeq 2c_2(c_1 + L_2 - 2t_1)$$

which is negative in the conditions that were found in 1(iii).

*1(v).*— If the segment of trajectory is between two components that are not short focusing arcs, the quadratic forms is defined with $c_i = L_i$ if $q_i$ is in a focusing component, and $c_i = 0$ in any other case. The results proved in

Markarian [18, case C], allows to assert that the focusing components must verify the condition of Wojtkowski: $\mathrm{d}^2R/\mathrm{d}s^2 < 0$.

Thus, we have proved the following theorem.

THEOREM A.— *Chaotic billiards can be constructed in the following way; the $C^3$ components of the boundary can be of any type, except that the focusing ones must be $C^4$ and verify $\mathrm{d}^2R/\mathrm{d}s^2 < 0$ or must be short. The semicurvature circles of any focusing arc do not contain parts of other components and the semicurvature circles of not adjacent focusing components do not intersect themselves; adjacent focusing components form interior angles bigger than $\pi$; focusing and dispering adjacent components form interior angles not less than $\pi$; focusing and neutral adjacent components have interior angles bigger than $\pi/2$.*

We will study now what happens if $c = t$ in any case that one of the extremes of the segment of trajectory is not in $\mathcal{C}$: $c_1 = t_1$ if $q_1$ or $q_2$ is not in $\mathcal{C}$.

*2(i).*— If $q_1$, $q_2 \in \mathcal{C}$, $q_3 \notin \mathcal{C}$ the analysis is like in 1(i) with

$$C_5 = \frac{\ddot{r}}{2}, \quad A_0 = A_1 = 0\,.$$

*2(ii).*— If $q_1 \in \mathcal{C}$, $q_2 \notin \mathcal{C}$, (6) is always true $(t_2 + t_1 > 0)$. If $K_2 = 0$ we obtain conditions as in 1(iii) on the trajectories that hit neutral components. If $K_2 \neq 0$, the first member of (7) is $-t_1(4t_2 - 2E) + 2t_2E$. If $K_2 < 0$, then $E \leq 0$ and (7) is verified. If $K_2 > 0$, suppose first that $t_1 \geq t_2 > L_2$; then

$$-t_1(2t_2 - L_2) + t_2L_2 < -2t_1t_2 + t_1t_2 + t_2^2 = t_2(-t_1 + t_2) \leq 0\,.$$

If $t_2 > t_1 > L_2$, then

$$-t_1(2t_2 - L_2) + t_2L_2 < t_1(-t_2 + t_1) < 0\,.$$

In both cases (7) is verified if the components of $q_1$ and $q_3$ are outside of the circle of curvature in $q_2$.

*2(iii).*— If $q_1$, $q_3 \notin \mathcal{C}$, $q_2 \in \mathcal{C}$, everything is like in 2(ii) because $c_i = t_i$, $i = 1, 2$, in both cases.

*2(iv).*— If $q_1 \notin \mathcal{C}$, $q_2, q_3 \in \mathcal{C}$, (6) is trivially verified $(t_2 + t_1 > 0)$. As in first approximation $c_2 \simeq L_2$, the first member of (7) is

$$-t_1(4c_2 - 2L_2) + 2c_2L_2 \simeq 2c_2(-t_1 + L_2)$$

which is negative if $t_1 > L_2$, i.e. all the other components are outside the circles of curvature of the points of $\mathcal{C}$.

*2(v).*— If the segment of trajectory is between to components of the boundary that are not short focusing arcs, the behavior of the quadratic form (and the resulting geometrical conditions) were studied in Markarian [18, case B].

So, we have proved the following theorem.

THEOREM B.— *Billiards with chaotic behavior can also be constructed taking a finite number of dispersing and neutral arcs, short $C^4$ focusing components and $C^4$ focusing arcs that verify $L_2(t_1 + t_2) < 2t_1t_2$. The curvature circles of focusing components must not intersect other not adjacent components of the boundary; the conditions of adjacent components are as in Theorem A.*

*Remarks*

(a) The meaning of short focusing component is that expression (8) is valid on it and $A$ is small enough so that this expression dominates the remainder of the Taylor development of the first member of (7).

(b) The focusing arcs mentioned in Theorem B are those that verify $\mathrm{d}^2R^{1/3}/\mathrm{d}s^2 > 0$, and some other conditions for long trajectories between points of the same component.

(c) The conditions on adjacent components are deduced from simple geometrical considerations.

(d) Billiards studied in Theorems A and B maintain chaotic behavior if their non neutral components are $C^4$-perturbed.

If we want to obtain simultaneously local and global conditions on $a_i$, $c_i$ we can write them as Taylor's developments of the natural parameters $L_i$, $t_i$:

$$c = \overline{C}_1L + \overline{C}_2L^2 + \overline{C}_4Lt + \overline{C}_5t^2 + \overline{C}_6L^3 + \cdots$$

$$a = \overline{A}_0 + \overline{A}_1L + \overline{A}_2t + \cdots$$

It is very interesting to observe that if in the succesive hits in a circumference ($L_1 = t_1 = L_2 = t_2$) we take such kind of $a_i$, $c_i$, then the first member of (7) becomes

$$\left(2c_1 - 2t_1 + \frac{a_1 t_1^2}{2}\right)^2 \geq 0 .$$

This implies that we can not use this type of coefficients in order to keep chaotic behavior in billiards that contain perturbed curves of an arc of circumference as part of the boundary. So we must define a sort of "strange" quadratic form to study this perturbations (Section 5).

If the last developments of $c$ and $a$ are substitued in (7), and we add the condition $\overline{C}_1 + \overline{C}_2 = 1$ ($c_i = L_i = t_i$ in first approximation), after a lot of routine calculus, we obtain

$$A^4 \left[\frac{2\dot r}{3}(\dot r + \dddot r)(2\overline{C}_1 - 1) + \ddot r^2\left(\overline{C}_1^2 - \frac{5\overline{C}_1}{2} + 1\right) - 2(1 + \overline{C}_1 - \overline{A}_0) \times \right.$$
$$\left. \times\ (\overline{C}_3 + \overline{C}_4 + \overline{C}_5)\dot r^2 \ddot r + (1 - \overline{C}_1)\overline{A}_0 \dot r^2 \ddot r + \overline{A}_0 \dot r^2 \ddot r + \frac{\overline{A}_0^2}{4}\dot r^4\right] + \cdots < 0 .$$

This type of study of the evolution of the quadratic form is in some sense more general because as the negativity of the expression in brackets gives a condition for the short trajectories, the global character of $L$ and $t$ admits the study of long trajectories. In order to ilustrate this assumption, we observe that if $\overline{C}_1 = 1/2$, $\overline{C}_i = 0$, $i > 2$; $\overline{A}_0 = a$, $\overline{A}_i = 0$, $i > 0$; we obtain the conditions

$$6a\ddot r + a^2 \dot r^2 < 0 ,$$
$$L_2(L_2 + t_2) + t_2(-3t_1 + L_1) + a t_1(2t_1 t_2 - L_2^2 - t_2 L_2) < 0 ,$$
$$L_2 + t_2 - L_1 + 3t_1 - 2a t_1^2 > 0 ,$$

which are consequences, respectively, of the local study on focusing components, and the global analysis of (7) and (6).

## 4. Perturbation of the billiards of Bunimovich

The proof provided by Bunimovich that of what we now call Bunimovich-type billiards are ergodic is very rigid: the focusing parts of the boundary must be circumferences and in particular no perturbations of them are allowed. In this section we prove that $C^4$-perturbations can be done in less than a half circumference keeping chaotic behavior in the resulting billiard, if the other components are conveniently design. As a consequence all the billiards of Bunimowich that contain less than half circumference (and only such kind of regular focusing components) are $C^4$-perturbable. In the case of the stadium, we can $C^4$ perturbe its half circumferences, maintaining its straight lines tangent to the perturbed curves at the contact points: the resulting billiard has Lyapunov exponents different from zero. In Bunimovich [5] there are some heuristic ideas referring to this problem. Victor Donnay [9] proved a similar result (only $C^6$-perturbations are allowed) using Lazutkin coordinates.

It is simple to see that a half circumference (or a bigger arc of it) can not be perturbed in general keeping chaotic behavior. Such arcs can be $C^\infty$-perturbed obtaining more than a half ellipse including the vertex of the short half-axis. But all the trajectories in elliptical billiards have caustics (curve such that all the intervals of a trajectory are tangent to it); in the case we are now interested, they are hyperbolas. See Cornfeld *et al.* [8, Chap. 6, Theorem 1]. The existence of caustics for sets of trajectories with positive measure is incompatible with Pesin region of measure one.

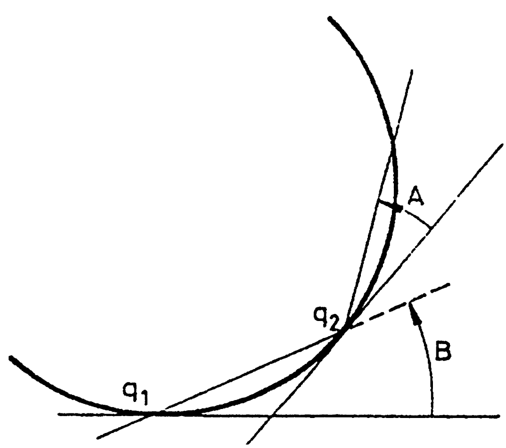


**Fig. 4**

Consider a circumference parametrized as in figure 4. We prove first that for $0 \leq \beta \leq \pi - \varepsilon$ (any $\varepsilon > 0$) the curve $\mathcal{C}$ can be $C^4$-perturbed keeping chaotic behavior in the resulting billiard, if the other components of the boundary are conveniently fitted.

If $q_1, q_2 \in \mathcal{C}$, $\beta_2 > \beta_1$ (there is a clear analogous for $\beta_1 > \beta_2$) we define the quadratic form by

$$a_1 = a\beta_1\,, \quad b_1 = 1\,, \quad c_1 = t_1\,;$$

$a > 0$ is a real constant to be estimated. If $q_1 \in \mathcal{C}$, $q_2 \notin \mathcal{C}$ then

$$a_1 = 0\,, \quad b_1 = 1\,, \quad c_1 = t_1\,.$$

We will analize three possible cases

*Case 1.*— $q_1, q_2, q_3 \in \mathcal{C}$. Then $t_1 = t_2 = L_1 = L_2 < R$ and the first member of (6) becomes

$$t_1 - a\beta_1 t_1^2 = t_1(1 - a\beta_1 t_1) > t_1(1 - a\pi R)\,.$$

So (6) is verified if $1 - a\pi R > 0$, or $a < 1/R\pi$.

The first member of (7) is now

$$\begin{aligned} t_1^2(-1 + a\beta_1 t_1)\left(2 + \frac{a(\beta_2 - \beta_1)t_1}{4}\right) &+ \\ + t_1^2\left[2 - 2a\beta_1 t_1 - \frac{a(\beta_2 - \beta_1)t_1}{4}\right] &+ \frac{a^2\beta_1^2 t_1^4}{4} = \\ = t_1^2\left[-\frac{a(\beta_2 - \beta_1)t_1}{2} + \frac{a^2\beta_1(\beta_2 - \beta_1)t_1^2}{4} + \frac{a^2\beta_1^2 t_1^2}{4}\right] & \\ = \frac{at_1^3}{4}\left[-2(\beta_2 - \beta_1) + a\beta_1\beta_2 t_1\right]. & \end{aligned}$$

So (7) is verified if

$$-2(\beta_2 - \beta_1) + a\beta_1\beta_2 t_1 < 0\,. \tag{9}$$

We distinguish now two situations.

**1(i)** $\beta_2 - \beta_1 = 2A \simeq 0$. It results

$$-2(\beta_2 - \beta_1) + a\beta_1\beta_2 t_1 < -4A + a\pi^2\left(\dot{r}A - \frac{\ddot{r}}{2}A^2 + \cdots\right).$$

In the circumference $\ddot{r} = \dddot{r} = 0$, $\dot{r} = -\dddot{r} = 2R$. And so in first approximation ($A \simeq 0$), (7) is verified if $-2 + a\pi^2 R < 0$, that is $a < 2/R\pi^2$. It is important to observe that (6) and (7) are verified for $C^4$-perturbed curves of $C$ as a consequece of the following facts: $U = 0$ independently of the values of $a_i$ (and it is only in this parameters where the variation of $\beta_i$, as a consequence of the perturbation, can influence; see (b) in Section 4; and (8) and the restrictions on $a$, are open conditions.

It is also useful to remark that if in the first member of (7) we substitute $c_i$, $a_i$ by the values indicated above, and do the local analysis, we obtain

$$\left[-\frac{2}{3}\dot{r}(\dot{r} + \dddot{r}) + \ddot{r}^2 + \frac{a^2\beta_1^2}{4}\dot{r}^4 - a\dot{r}^3\right] A^4 + \cdots$$

For small $A$ this expression is less than $(\pi^2\dot{r}^4/4)a^2 - \dot{r}^3 a + D$ where $D = -(2/3)\dot{r}(\dot{r} + \dddot{r}) + \ddot{r}^2$, so $P > 0$ if

$$\frac{2}{\pi^2\dot{r}^2}\left(\dot{r} - \sqrt{\dot{r}^2 - D\pi^2}\right) < a < \frac{2}{\pi^2\dot{r}^2}\left(\dot{r} + \sqrt{\dot{r}^2 - D\pi^2}\right).$$

In the case we are interested ($\dot{r} = 2R$, $\ddot{r} = 0$, $\dddot{r} = -2R$, $D = 0$), if $0 < a < 2/R\pi^2$ for $C^4$-perturbations of arcs of circumference, $0 \le \beta < \pi$, the resulting quadratic forms are increasing along short hits.

**1(ii)** $\beta_2 - \beta_1 \gg 0$. First we observe that

$$t_1 = t_2 = L_1 = L_2 = 2R \operatorname{sen}\frac{\beta_2 - \beta_1}{2}$$

and if $\beta_2 = y$, $\beta_1 = x$, the first member of (9) is

$$H(x, y) = 2aRxy \operatorname{sen}\frac{y - x}{2} - 2(y - x)$$

which sign must be studied in the triangle $\Delta : x \ge 0$, $y \le \pi$, $y \ge x$. We have

$$H_x(x, y) = 2aRy \operatorname{sen}\frac{y - x}{2} - aRxy\cos\frac{y - x}{2} + 2$$

$$H_{xx}(x, y) = aRy\left(-2\cos\frac{y - x}{2} - \frac{1}{2}x \operatorname{sen}\frac{y - x}{2}\right) < 0$$

since

$$0 < \frac{y - x}{2} < \frac{\pi}{2}.$$

So for each fixed $y_0$, the graphic of $z = H(x, y_0)$ is as in figure 5 with $H_x(y_0, y_0) = 2 - aRy_0^2$ which is positive for a $a < 2/R\pi^2$.

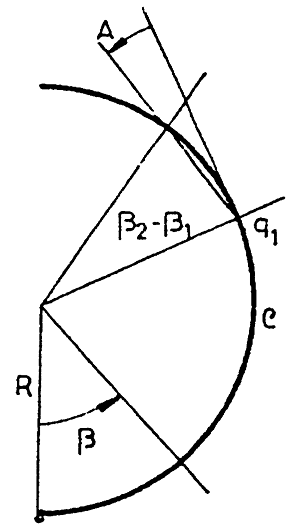


**Fig. 5**

This graphic is valid for the circumference, but we know (**1(i)**) that for its $C^4$-perturbation it is negative if $y \simeq x$. It is clear that if this is not the case, the graphic is far away from zero and negative. So, we have proved that the quadratic form is increasing for any trajectory that has there successive reflections on the circumference or its $C^4$-perturbations.

*Case 2.*— $q_1 \notin \mathcal{C}$, $q_2 \in \mathcal{C}$. We define $c_1 = t_1$, $a_1 = 0$ and must distinguish two cases depending where is $q_3$.

**2(i)** $q_3 \in \mathcal{C}, \beta_3 > \beta_2$. In this case (6) becomes $t_1 + t_2 > 0$ which is obviously true. And the first member of (7) is

$$- t_1 \left(4t_2 - 2L_2 + a\beta_2 \frac{L_2^2}{4}\right) + t_2 L_2 \left(2 - a\beta_2 \frac{L_2}{4}\right) =$$
$$= -a\beta_2 \frac{L_2^2}{4}(t_1 + t_2) + 2L_2(t_1 + t_2) - 4t_1 t_2 \,.$$

It is less than zero if $t_1 > L_2$ since in the arcs of circumference and its perturbations $t_2 \simeq L_2$.

**2(ii)** $q_2 \notin \mathcal{C}$. As it was indicated above, in this case $c_2 = t_2$, $a_2 = 0$. (6) is verified immediately and (7) becomes

$$2L_2(t_1 + t_2) - 4t_1t_2 < 0$$

which is true if $t_1,\, t_2 > L_2$.

So in both posibilities of case 2, the other components of the boundary must be outside the osculating circles of each point of the perturbed curves.

*Case 3.*— $q_1,\, q_2 \in \mathcal{C}$, $q_3 \notin \mathcal{C}$. (6) is verified if $t_2 + t_1 - a\beta_1 t_1^2 > 0$ and since we assume $t_2 \geq t_1$ (Case 2), this gives the same condition of Case 1:

$$0 < a < \frac{1}{R\pi}.$$

The first member of (7) is now

$$\begin{aligned} &(-t_1 + a\beta_1 t_1^2)\left(4t_2 - 2L_2 - a\beta_1 \frac{L_2^2}{4}\right) + \\ &\qquad + L_2 t_2 \left(2 - 2a\beta_1 t_1 + a\beta_1 \frac{L_2}{4}\right) + \frac{a^2\beta_1^2 t_1^2 L_2^2}{4} = \\ &= t_1(2L_2 - 4t_2) + 2t_2L_2 + \\ &\qquad + a\beta_1 \left[(t_1 + t_2)\frac{L_2^2}{4} + 4t_1^2 t_2 - 2t_1t_2L_2 - 2t_1^2 L_2\right]. \end{aligned}$$

In the circumference (and essentially the same in its perturbed), $t_1 = L_2$, and the last expression is

$$t_1\left[2t_1 - 2t_2 + a\beta_1 \frac{t_1}{4}(9t_2 - 7t_1)\right] < t_1\left[2t_1 - 2t_2 + a\pi\frac{R}{2}(9t_2 - 7t_1)\right],$$

which is negative if

$$a < \frac{4(t_2 - t_1)}{\pi R(9t_2 - 7t_1)}.$$

This bound for $a$ is away from zero if all the components of a Bunimovich-type billiard are circumferences (see the final remark in Case 2), or the neutral components are not tangent to the focusing ones. If the neutral components are tangent the result is also true, but we must add a general remark about the behavior of quadratic forms on such components.

If we define (as it was indicated at the beginning of Case 2) $c_2 = t_2$, $a_2 = 0$ on the neutral components ($K_2 = 0$), formulas (6) and (7′) indicate that the quadratic form is increasing (not strictly) when the trajectory hits on it. So, we can assume that $t_2$ is the time the trajectory spends between two non neutral components.

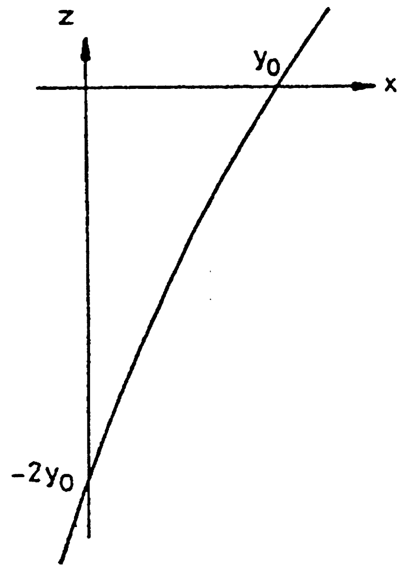


**Fig. 6**

For example, in the stadium of figure 6, it results $t_2 - t_1 > d' = d/10$, and $9t_2 - 7t_1 < 9t_2 < R'$, where $R'$ depends on $R$ and $d$ (observe that as $q_1, q_2 \in \mathcal{C}$, $q_3 \notin \mathcal{C}$, then $\theta_3 \geq \pi/4$). Finally, if we take

$$a = \min \left\{ \frac{4d}{RR'd}, \frac{2}{R\pi^2} \right\}$$

we have proved that chaotic behavior is kept if in the stadium the half circumferences are $C^4$-perturbed and the tangents at the end of the perturbed curves coincide with the neutral components.

## 5. Appendix
## Formulas for focusing components of a billiard

Let $P(A) = q_2 + r(A)\, e^{iA} q_2'$ be the polar parametrization of a $C^k$ focusing curve, $k \geq 4$, $r(0) = 0$. See figure 3. $\dot{r}(A)$ will indicate the first derivative with respect to $A$, $\dot{r} = \dot{r}(0)$, $\ddot{r} = \ddot{r}(0)$, etc.

Then

$$r(A) = \dot r A + \frac{\ddot r}{2} A^2 + \frac{\dddot r}{6} A^3 + \cdots ,$$

$$\begin{aligned} s(A) &= \int_0^A \sqrt{r^2(A) + \dot r^2(A)}\, \mathrm{d}A \\ &= \dot r A + \frac{\ddot r}{2} A^2 + \frac{1}{6}(\dot r + \dddot r)A^3 + \frac{1}{24}\ddddot r A^4 . \end{aligned}$$

For $A > 0$ we have $q_1 = q_2 + r(-A)\, e^{-iA} q_2'$. Then

$$\begin{aligned} t_2 &= r(A) = \dot r A + \frac{\ddot r}{2} A^2 + \frac{\dddot r}{6} A^3 + \frac{\ddddot r}{24} A^4 + \cdots \\ t_1 &= -r(-A) = \dot r A - \frac{\ddot r}{2} A^2 + \frac{\dddot r}{6} A^3 - \frac{\ddddot r}{24} A^4 + \cdots \end{aligned}$$

$$K(A) = \frac{r^2(A) + 2\dot r^2(A) - r(A)\ddot r(A)}{\big(r^2(A) + \dot r^2(A)\big)^{3/2}}$$

$$\begin{aligned} K_2 &= K(0) = \frac{2}{\dot r} \quad \longrightarrow \quad \dot r = \frac{2}{K_2} > 0 \\ K_1 &= K(-A) = \frac{2}{\dot r} + \frac{3\ddot r}{\dot r^2} A + \left( -\frac{2}{\dot r} + \frac{9}{2}\frac{\ddot r^2}{\dot r^3} - \frac{2\,\dddot r}{\dot r^2} \right) A^2 + \\ &\quad + \left( -\frac{11}{2}\frac{\ddot r}{\dot r^2} - \frac{35}{6}\frac{\ddot r\,\dddot r}{\dot r^3} + \frac{5}{6}\frac{\ddddot r}{\dot r^2} + \frac{13}{2}\frac{\ddot r^3}{\dot r^4} \right) A^3 + \cdots \end{aligned}$$

$$\begin{aligned} \cos\theta_2 &= \operatorname{sen} A = A - \frac{1}{6} A^3 + \cdots \\ \cos\theta_1 &= \operatorname{sen} B = \left( \frac{\tan^2 B}{1 + \tan^2 B} \right)^{1/2} = \left( \frac{r^2(-A)}{r^2(-A) + \dot r^2(A)} \right)^{1/2} \\ &= \frac{|r(-A)|}{\big(r^2(-A) + \dot r(-A)\big)^{1/2}} \\ &= A + \frac{\ddot r}{2\dot r} A^2 + \left( \frac{\ddot r^2}{2\dot r^2} - \frac{\dddot r}{3\dot r} - \frac{1}{2} \right) A^3 + \\ &\quad + \left( \frac{\ddot r^3}{2\dot r^3} - \frac{3\ddot r}{4\dot r} - \frac{7\ddot r\,\dddot r}{12\,\dot r^2} - \frac{\ddddot r}{8\dot r} \right) A^4 + \cdots \end{aligned}$$

$$
\begin{aligned}
L_2 &= \frac{2\cos\theta_2}{K_2} = \dot{r}A - \frac{\dot{r}}{6}\,A^3 + \cdots \\
L_1 &= \frac{2\cos\theta_1}{K_1} \\
&= \dot{r}A - \ddot{r}A^2 + \left(\frac{\dot{r}}{2} - \frac{\ddot{r}^2}{4\dot{r}} + \frac{2\,\dddot{r}}{3}\right)A^3 + \\
&\qquad + \left(\frac{\ddot{r}}{2} + \frac{2\ddot{r}\,\dddot{r}}{3\dot{r}} - \frac{\ddot{r}^3}{4\dot{r}^2} - \frac{7\,\ddddot{r}}{12}\right)A^4 + \cdots
\end{aligned}
$$

From the formula for $\cos\theta_1$ we obtain

$$
\begin{aligned}
B &= \mathrm{Arcsen}(\cos\theta_1) \\
&= A + \frac{\ddot{r}}{2\dot{r}}\,A^2 + \left(\frac{\ddot{r}^2}{2\dot{r}^2} - \frac{\dddot{r}}{3\dot{r}} - \frac{1}{3}\right)A^3 + \\
&\qquad + \left(\frac{\ddot{r}^3}{2\dot{r}^3} - \frac{\ddot{r}}{2\dot{r}} - \frac{7\ddot{r}\,\dddot{r}}{12\,\dot{r}^2} + \frac{\ddddot{r}}{8\dot{r}}\right)A^4 + \cdots
\end{aligned}
$$

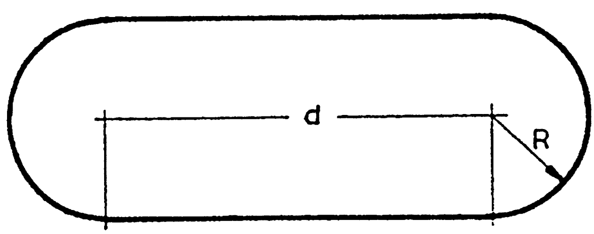


Fig. 7

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