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Closed hypersurfaces of S^4 with two constant symmetric curvatures(*)

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RÉSUMÉ. — Dans cet article comme dans ([AB1], [AB2]) nous étudions les hypersurfaces closes dans la 4-sphère unité, dont deux fonctions symétriques de courbure sont constantes. Si l'une est la courbure scalaire et est non négative, alors l'hypersurface est isoparamétrique. Il y a un résultat analogue pour les hypersurfaces dont la courbure moyenne et celle de Gauss-Kronecker sont constantes.

ABSTRACT. — In this paper as in ([AB1], [AB2]) we are concerned with the study of closed hypersurfaces in the unit 4-sphere with two constant symmetric curvature functions. If one of the assumptions refers to the scalar curvature of the hypersurface M as a non-negative constant then M must be isoparametric. There is a similar result for hypersurfaces with constant mean curvature and constant Gauss-Kronecker curvature.

1. Introduction

Let M be a closed hypersurface of the 4-dimensional Euclidean sphere S^4 with scalar curvature κ_M. In a recent paper [AB2], we considered the class $\mathcal{F}$ of closed oriented 3-dimensional hypersurfaces with constant mean curvature and constant scalar curvature immersed into the standard sphere S^4. It was shown that if $M \in \mathcal{F}$ and $\kappa_M \geq 0$ then M is isoparametric. For future reference, we state this result explicitly.

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THEOREM 1.1 [AB2].— *Let M be a closed hypersurface of the 4-sphere S^4. Suppose in addition that M has constant mean curvature H and constant scalar curvature $\kappa \geq 0$. Then M is isoparametric*

Recently, S. Chang [C] exhibited the following result.

THEOREM 1.2 [C].— *Let M^3 be a closed hypersurface of constant scalar curvature κ in S^4. If M has constant mean curvature then $\kappa \geq 0$.*

Remark 1.— If one combines Theorem 1.1 together with the non-existence result in [C] one obtains a complete classification of the family $\mathcal{F}$.

This paper is a continuation of our work in ([AB1], [AB2]) where we try to characterize closed hypersurfaces of S^4 with given geometric properties. We recall that the second fundamental form of a hypersurface $M^3 \subset S^4$ is locally represented by a symmetric 3×3 matrix A. In this particular case the second fundamental form is completely determined by the three symmetric curvatures

$$\begin{aligned} H_1 &= \operatorname{trace} A \\ H_2 &= \frac{H_1^2 - \operatorname{trace} A^2}{2} \\ H_3 &= \det A \,. \end{aligned} \tag{1.1}$$

The functions H_1 and H_3 are called the mean curvature and the Gauss–Kronecker curvature respectively. The function H_2 is, up to a constant, the scalar curvature of M. The hypersurface M is said to be isoparametric if H_1, H_2, H_3 are constant. If we suppose that just one of these functions is a constant then it is possible, in general, to find non isoparametric examples. One way of getting those examples is to use equivariant geometry methods [H]. The natural question at this point is what can be said about the family $\mathcal{F}_{r,s}$, $r < s$, of closed hypersurfaces $M \subset S^4$ that satisfy $\mathrm{d}H_r = \mathrm{d}H_s = 0$. The case where H_1 and H_2 are constant functions was completely solved in ([AB2], [C]). This leaves open the following.

PROBLEM 1.— *Determine $\mathcal{F}_{r,s}$ for $(r, s) \neq (1, 2)$.*

In this direction we have established the following results.

THEOREM 1.3.— *Let M be a closed oriented 3-dimensional hypersurface immersed in the standard 4-sphere. Suppose in addition that M has constant Gauss-Kronecker K and constant scalar curvature $\kappa \geq 0$. Then M is isoparametric.*

THEOREM 1.4.— *Let M be a closed oriented 3-dimensional hypersurface immersed in the standard 4-sphere with constant mean curvature H and constant Gauss–Kronecker curvature $K \neq 0$. Suppose in addition that $-3 \leq H/K$. Then M is isoparametric.*

In Theorem 1.4 we have assumed that H_1 and H_3 were constant functions. This case was partially treated in ([AB1], [R]). Using Theorem 1.4 we retrieved the following result.

THEOREM 1.5 [AB1].— *Let $M^3 \subset S^4$ be a closed minimally immersed hypersurface of S^4 with constant Gauss–Kronecker $K \neq 0$. Then M is the minimal Clifford torus*

$$S^2(\sqrt{2/3}) \times S^1(\sqrt{1/3}) \subset S^4 .$$

2. Preliminary results

The basic object of study in this note is a symmetric quadratic differential form which satisfies an identity of Coddazi–Mainardi type. In this section we will introduce this quadratic form, prove an algebraic lemma and fix the notation used throughout the paper.

2.1 Notations

Let M be a compact 3-dimensional Riemannian manifold with metric g, volume form vol and scalar curvature κ. Suppose a is a smooth symmetric tensor field on M of type $(0,2)$ and let A be the tensor field of type $(1,1)$ corresponding to a via g. In this section we will always assume that:

(I) the field ∇a of type $(0,3)$ is symmetric;

(II) $\mathrm{d}\sigma_r = 0$, $r = 2, 3$.

Here, as usual $\sigma_i : M \to \mathbb{R}$, denote the elementary symmetric functions of the eigenvalues $\lambda_1 \leq \lambda_2 \leq \lambda_3$ of the tensor field A. In particular $\sigma_1 = \operatorname{trace} A$, $\sigma_2 = \sum_{i<j} \lambda_i \lambda_j$ and $\sigma_3 = \det A$.

Let σ be the permutation given by $\sigma(1) = 2$, $\sigma(2) = 3$, $\sigma(3) = 1$. We now define

$$c_i = \lambda_{i'} - \lambda_{i''}\,, \quad \gamma = c_1 c_2 c_3 \tag{2.1}$$

where $i' = \sigma(i)$ and $i'' = \sigma(i')$. Obviously

$$\sum_{i\in I} c_i = 0\,. \tag{2.2}$$

The following identity can be found in [W]:

$$\gamma^2 = -4\sigma_3 f^3 + \sigma_2^2 f^2 + 18\,\sigma_2\sigma_3 f - 27\,\sigma_3^2 - 4\sigma_2^3\,. \tag{2.3}$$

In (2.3), σ_2, σ_3 are constants and $f : M \to \mathbb{R}$ is the C^∞ function given by

$$f = \operatorname{trace} A\,.$$

2.2 The structure equations

Now we will take a look at the structure equations on the open subset

$$Y = \{p \in M : \lambda_1(p) < \lambda_2(p) < \lambda_3(p)\}$$

of M. From now on we will assume that $Y \neq \emptyset$. Note that λ_i is smooth on Y for each $i \in Y = \{1, 2, 3\}$.

DEFINITION 2.1.— *We say that (U,ω) is admissible if:*

(i) U *is an open subset of* Y*;*

(ii) $\omega = (\omega_1, \omega_2, \omega_3)$ *is a smooth orthonormal coframe field on* U*;*

(iii) $\omega_1 \wedge \omega_2 \wedge \omega_3 = \mathrm{vol}$ *on* U*;*

(iv) $a = \sum_{i\in I} \lambda_i \omega_i \otimes \omega_i$.

Suppose (U,ω) is admissible. As is well known, there are smooth 1-forms ω_{ij} on U uniquely determined by the equations

$$\mathrm{d}\omega i = -\sum_{j=1}^{3} \omega_{ij} \wedge \omega_j\,, \quad \omega_{ij} + \omega_{ji} = 0\,, \quad i\,,\, j \in I\,. \tag{2.4}$$

To simplify the notation we let

$$\theta_i = \omega_{i''i'}\,.$$

Therefore

$$\mathrm{d}\omega_i = \theta_{i''} \wedge \omega_{i'} - \theta_{i'} \wedge \omega_{i''}\,, \quad i \in I\,. \tag{2.5}$$

Let the functions ρ_{ij}, i, $j \in I$, be determined by

$$d\theta_i = \theta_{i''} \wedge \theta_{i'} + \sum_{j \in I} \rho_{ij} \omega_{j''} \wedge \omega_{j'} . \tag{2.6}$$

Note that

$$\kappa = 2 \sum_{i \in I} \rho_{ii} . \tag{2.7}$$

The covariant derivative of the tensor field $a = \sum_{i,j \in I} a_{ij} \omega_i \otimes \omega_j$ is given by

$$\nabla a = \sum_{i,j,k} a_{ijk} \omega_i \otimes \omega_j \otimes \omega_k$$

where

$$\sum_k a_{ijk} \omega_k = da_{ij} - \sum_m a_{im} \omega_{mj} - \sum_m a_{mj} \omega_{mi} . \tag{2.8}$$

We are assuming that (U, ω) is admissible. Therefore $a_{ij} = \lambda_i \delta_{ij}$. We obtain from 2.8 that

$$d\lambda_i = \sum_{k \in I} a_{iik} \omega_k , \quad i \in I , \tag{2.9}$$

$$c_i \theta_i = \sum_{k \in I} a_{i'i''k} \omega_k , \quad i \in I . \tag{2.10}$$

Let the functions t_{ij}, i, $j \in I$ be determined by

$$\theta_i = \sum_{j \in I} t_{ij} \omega_j . \tag{2.11}$$

We obtain from (2.9), (2.10), (2.11) and the symmetry of ∇a that

$$c_1 t_{11} = c_2 t_{22} = c_3 t_{33} \tag{2.12}$$

$$c_i t_{ii'} = \lambda_{i'i''} \tag{2.13}$$

$$c_i t_{ii''} = \lambda_{i''i'} \tag{2.14}$$

where the functions λ_{ij}, $j \in I$, are determined by

$$d\lambda_i = \sum_{j \in I} \lambda_{ij} \omega_j \quad \text{on } U . \tag{2.15}$$

Now (2.12) gives

$$\gamma \sum_{i \in I} t_{ii} t_{i'i'} = (c_1 t_{11})^2 \sum_{i \in I} c_{i''} .$$

It follows that

$$\sum_{i \in I} t_{ii} t_{i'i'} = 0 . \tag{2.16}$$

On the other hand, we know that each pair $\big(x, \lambda_i(x)\big) \in M \times \mathbb{R}$ satisfies the polynomial equation $P(x, \lambda) = 0$ where

$$P(x, \lambda) = \prod_j (\lambda - \lambda_j) .$$

Differentiating the equation $P(x, \lambda_i) = 0$, $i = 1, 2, 3$, we obtain

$$\frac{\partial P}{\partial \lambda}(x, \lambda_i)\, \mathrm{d}\lambda_i = \lambda_i^2\, \mathrm{d}f .$$

This gives the following identities

$$\gamma\, \mathrm{d}\lambda_i = -c_i \lambda_i^2\, \mathrm{d}f , \quad i \in I . \tag{2.17}$$

Therefore

$$\gamma \lambda_{ij} = -c_i \lambda_i^2 f_j \tag{2.18}$$

where $\mathrm{d}f = \sum_{i \in j} f_i w_i$. From (2.13), (2.14) and (2.18), we obtain

$$c_{i'} t_{i'i''} = -\frac{c_{i''} \lambda_{i''}^2 f_i}{\gamma} , \quad c_{i''} t_{i''i'} = -\frac{c_{i'} \lambda_{i'}^2 f_i}{\gamma} .$$

Therefore

$$t_{i'i''} t_{i''i'} = \frac{\lambda_{i'}^2 \lambda_{i''}^2 f_i^2}{\gamma^2} \tag{2.19}$$

and

$$t_{i'i''} - t_{i''i'} = \left(\frac{c_{i'} \lambda_{i'}^2}{c_{i''}} - \frac{c_{i''} \lambda_{i''}^2}{c_{i'}} \right) \frac{f_i}{\gamma} .$$

An elementary computation gives

$$t_{i'i''} - t_{i''i'} = \frac{c_i^2 (\lambda_i \sigma_2 - 3\sigma_3) f_i}{\gamma^2} . \tag{2.20}$$

2.3 The 2-form ψ

As in [AB2] there is one and only one 2-form ψ on Y such that if (U, ω) is admissible then

$$\psi = \sum_{i \in I} \omega_i \wedge \theta_i \quad \text{on } U .$$

Suppose (U, ω) is admissible. Using equations (2.6) and (2.7) we obtain

$$\mathrm{d}(\omega_i \wedge \theta_i) = \omega_i \wedge \theta_{i'} \wedge \theta_{i''} - \omega_{i'} \wedge \theta_{i''} \wedge \theta_i - \omega_{i''} \wedge \theta_i \wedge \theta_{i'} + \rho_{ii} \,\mathrm{vol} .$$

Therefore

$$\mathrm{d}\psi = \frac{\kappa}{2} \,\mathrm{vol} - \sum_{i \in I} \omega_i \wedge \theta_{i'} \wedge \theta_{i''} .$$

Using (2.11) we get

$$\mathrm{d}\psi = \frac{\kappa}{2} \,\mathrm{vol} + \sum_{i \in I} (t_{i'i''} t_{i''i'} - t_{i'i'} t_{i''i''}) \,\mathrm{vol} .$$

We also obtain from (2.11) that

$$\mathrm{d}f \wedge \psi = \sum_{i \in I} f_i (t_{i'i''} - t_{i''i'}) \,\mathrm{vol}$$

so (2.16), (2.19), (2.20) gives

$$\mathrm{d}\psi = \frac{\kappa}{2} \,\mathrm{vol} + \sum_{i \in I} \frac{\lambda_{i'}^2 \lambda_{i''}^2 f_i^2}{\gamma^2} \,\mathrm{vol} \tag{2.21}$$

$$\mathrm{d}f \wedge \psi = \sum_{i \in I} c_i^2 f_i^2 \frac{\lambda_i \sigma_2 - 3\sigma_3}{\gamma^2} \,\mathrm{vol} . \tag{2.22}$$

2.4 An algebraic lemma

The following result is the key point in our proof of Theorem 1.3

LEMMA 2.2.— *Let M be a compact 3-dimensional Riemannian manifold with metric g, volume form* vol *and scalar curvature $\kappa \geq 0$. Suppose a is a smooth symmetric tensor field on M of type $(0, 2)$ and let A be the tensor field of type $(1, 1)$ corresponding to a via g. Suppose in addition that:*

(I) *the field* ∇a *of type* $(0,3)$ *is symmetric;*

(II) $\mathrm{d}\sigma_2 = \mathrm{d}\sigma_3 = 0$*;*

(III) $\sigma_3 \neq 0$.

Then $\mathrm{d}\sigma_1 = 0$.

Proof of Lemma 2.2

Without loss of generality we will assume that $\sigma_3 = -1$. From equation (2.3) we may write

$$\gamma^2 = Q \circ f \geq 0 \tag{2.23}$$

where Q is the polynomial given by

$$Q(x) = 4x^3 + \sigma_2^2 x^2 - 18\,\sigma_2 x - 4\sigma_2^3 - 27\,.$$

If $\gamma \equiv 0$, then $f(M) = Q^{-1}(0)$ and Lemma 2.2 will follow from the continuity of f. We will therefore assume that $\gamma \not\equiv 0$. We shall give the proof of Lemma 2.2 in two different steps.

Step 1.— $\gamma^{-1}(0) = \emptyset$.

Using equation (2.21) and Stokes' theorem we get

$$\int_M \left(\frac{\kappa}{2} + \left| \frac{A^{-1}(\nabla f)}{\gamma} \right|^2 \right) \mathrm{vol} = 0\,.$$

Since $\kappa \geq 0$ it follows that $\mathrm{d}f = 0$ on M and $f = \sigma_1 : M \to \mathbb{R}$ is a constant function.

Step 2.— $\gamma^{-1}(0) \neq \emptyset$.

The discriminant of the polynomial $Q(x)$ is given by

$$D = 4^2 (\sigma_2^3 - 27)^3\,.$$

We begin by assuming that $\sigma_2 < 3$. With this hypothesis the polynomial $Q(x)$ has only one real root β. Since $Q \circ f \geq 0$, it follows that $f \geq \beta$ on M. An easy computation shows that

$$\lambda_1 < 0 < \lambda_2 \leq \lambda_3\,. \tag{2.24}$$

For each $\varepsilon \geq 0$ sufficiently small we set

$$X_\varepsilon = f^{-1}[\beta\,,\, \beta + \varepsilon]\,, \quad Y_\varepsilon = M - X_\varepsilon\,. \tag{2.25}$$

Obviously, $M = X_\varepsilon \cup Y_\varepsilon$.

Note that $Y_0 \neq \emptyset$. For each $\varepsilon > 0$, we will choose a C^∞ function $\eta_\varepsilon : \mathbb{R} \to \mathbb{R}$ such that:

(a) $\eta'_\varepsilon \geq 0$, $0 \leq \eta_\varepsilon \leq 1$,

(b) $\eta_\varepsilon(t) = 0$ if $t \leq \beta + \varepsilon/3$,

(c) $\eta_\varepsilon(t) = 1$ if $t \geq \beta + \varepsilon$,

and then we apply Stokes'theorem to

$$\mathrm{d}\big((\eta_\varepsilon \circ f)\psi\big) = (\eta_\varepsilon \circ f)\,\mathrm{d}\psi + (\eta'_\varepsilon \circ f)\,\mathrm{d}f \wedge \psi$$

to obtain

$$\int_{Y_0} (\eta_\varepsilon \circ f)\,\mathrm{d}\psi + \int_{Y_0} (\eta'_\varepsilon \circ f)\,\mathrm{d}f \wedge \psi = 0\,. \tag{2.26}$$

From (2.21), (2.22) we get

$$\mathrm{d}\psi = \left(\frac{\kappa}{2} + \left|\frac{A^{-1}(\nabla f)}{\gamma}\right|^2\right) \mathrm{vol} \tag{2.27}$$

$$\mathrm{d}f \wedge \psi = \sum_{i \in I} c_i^2 f_i^2 \frac{\lambda_i \sigma_2 + 3}{\gamma^2}\ \mathrm{vol}\ . \tag{2.28}$$

It is easy to see that for $i = 2, 3$ and $\varepsilon \geq 0$ sufficiently small

$$\lambda_i \sigma_2 + 3 \geq 1 \quad \text{on } X_\varepsilon\,.$$

It follows from (2.26), (2.27) and (2.28) that

$$0 \leq \int_{Y_0} (\eta_\varepsilon \circ f)\,\mathrm{d}\psi \leq -\int_{Y_0} (\eta'_\varepsilon \circ f) \frac{c_1^2 \lambda_1 \sigma_2 f_1^2}{\gamma^2}\ \mathrm{vol}\ . \tag{2.29}$$

An argument analogous to one given in [AB2] shows that

$$\lim_{\varepsilon \to 0} \int_{Y_0} (\eta_\varepsilon \circ f)\,\mathrm{d}\psi = 0\,. \tag{2.30}$$

From this it follows that $f : M \to \mathbb{R}$ is a constant function which is a contradiction.

We next assume that $\sigma_2 = 3$. In this case the polynomial $Q(x)$ is given by $Q(x) = 4(x+3)^2(x-\beta)$, where $\beta = 15/4$. Since $\gamma^2 = Q \circ f \not\equiv 0$ it follows that $f \geq \beta$ on M. The proof proceeds exactly as before and also leads to a contradiction.

The remaining case $\sigma_2 > 3$ is analogous and will not be done here.

3. Proof of Theorem 1.3

Let $x : M \to S^4$ be an isometric immersion of M into the standard 4-sphere. We suppose in addition that M has constant scalar curvature $\kappa \geq 0$ and constant Gauss–Kronecker curvature K. Choose a unit normal vector field ν along x, and denote by h the second fundamental form associated to ν. If the Gauss–Kronecker curvature $K \neq 0$, Theorem 1.3 will follow directly from Lemma 2.2. Suppose now that $K \equiv 0$. With the notation of section 2, we have

$$\gamma^2 = \sigma_2^2(f^2 - 4\sigma_2) \geq 0 .$$

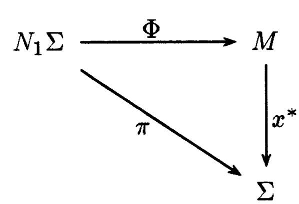
The equality is reached only at points $p \notin Y$, where as in section 2

$$Y = \{p \in M \mid \lambda_1 < \lambda_2 < \lambda_3\} .$$

We will assume first that $0 \leq \kappa < 6$. In this case $\gamma^2 \neq 0$ and $M = Y$. Using (2.21) and Stokes's theorem we get

$$\kappa = 0 = f_2 . \tag{3.1}$$

Let $x^* : M \to S^4$ be the associated Gauss map of x. It is defined pointwise as the image of the unit normal translated to the origin of $\mathbb{R}^5$. Since $\kappa = 0$ we have that $\lambda_2 = 0$ and $\lambda_1\lambda_3 = -3$. Therfore x^* is a map of rank two. It follows that the image $x^*(M)$ is a regular surface $\Sigma^2 \subset S^4$. Using (2.8) and the fact that $\lambda_2 = 0$ it is not difficult to see that M is foliated by great circles $S^1 \subset S^4$. These great circles are the lines of curvatures associated to the principal curvature $\lambda_2 = 0$. One can see easily that M^3 is the tube of geodesic radius $\pi/2$ over Σ^2 in S^4. We have the following commuting diagram



Here π is the natural projection, $N_1\Sigma$ is the unit normal bundle of Σ and Φ is the polar mapping given by $\Phi(p,\nu) = \nu$ (see [L]). Let $(p,\nu) \in N_1\Sigma$ and denote by $\kappa_i(p,\nu)$, $i = 1, 2$, the principal curvatures of Σ^2 in the normal direction ν. Since $\Phi(N_1\Sigma) = M^3$ it follows that

$$\kappa_1(p,\nu) = \frac{1}{\lambda_1}(\nu)\,, \quad \kappa_2(p,\nu) = \frac{1}{\lambda_3}(\nu)\,.$$

By a standard argument we can find a normal direction $\nu \in N_1\Sigma$ such that $\kappa_1 + \kappa_2 = 0$. Therefore $\lambda_1 = -\sqrt{3}$ and $\lambda_3 = \sqrt{3}$. To finish this part of the proof of the theorem we note that from (3.1) it follows that $f_2 = 0$. This implies that λ_1 and λ_3 are constant in each leaf. It follows that λ_1 and λ_3 are constant on M. Therefore M is Cartan's minimal hypersurface of S^4.

We will assume now that $\kappa > 6$. With this assumption it is easily seen that M is not an isoparametric hypersurface and $\sigma_2 > 0$. Therefore

$$f^2 - \beta^2 \geq 0$$

where $0 < \beta = 2\sqrt{\sigma_2}$. Without loss of generality we will assume that $f \geq \beta$. This gives in particular that $0 = \lambda_1 < \lambda_2 \leq \lambda_3$. Using (2.22) we see that for any C^∞ function $\eta : \mathbb{R} \to \mathbb{R}$:

$$d\big((\eta \circ f)\psi\big) = (\eta \circ f)\, d\psi + \sum_{i \in I} (\eta' \circ f) \frac{c_i^2 \lambda_i \sigma_2 f_i^2}{\gamma^2}\ \mathrm{vol}\,. \tag{3.2}$$

Since $f(M) \neq \{\beta\}$ we may choose a sufficiently small $\varepsilon > 0$ and $0 \leq \eta \leq 1$ so that:

(a) $\eta' \geq 0$, $\eta \circ f \not\equiv 0$;

(b) $\eta(t) = 0$ if $t \leq \beta + \varepsilon/3$;

(c) $\eta(t) = 1$ if $t \geq \beta + \varepsilon$.

Applying Stokes' theorem to (3.2) we get a contradiction.

Finally we will assume now that $x : M \to S^4$ is an isometric immersion with scalar curvature $\kappa \equiv 6$. Using Ricatti type equation coming from Codazzi equations it is possible to prove the impossibility of foliating (even locally) M^3 by totally geodesic 2-spheres. This reduces the case $\sigma_2 = K = 0$ to totally geodesic 3-spheres (see for example [BD]).

4. Proof of Theorem 1.4

Let $x : M \to S^4$ be an immersion into the standard 4-sphere S^4 satisfying the hypothesis of Theorem 1.4 and let $x^* : M \to S^4$ be its associated Gauss map. The principal curvatures of x^* are given by $k_i^* = 1/k_i$, $i = 1, 2, 3$, where k_1, k_2 and k_3 are the principal curvatures of M [L]. The scalar curvature and the Gauss–Kronecker curvature of x^* are constant and given by:

$$\kappa^* = 6 + \frac{2H}{K} \geq 0\,, \quad K^* = \frac{1}{K}\,. \tag{4.1}$$

It follows from Theorem 1.3 that the mean curvature $H^* = (\kappa - 6)/2K$ of x^* is a constant and M is isoparametric.

5. Examples and further comments

The results presented in this work in some sense characterize the isoparametric hypersurfaces of the 4-dimensional sphere S^4. From well known results we know that in S^4 there are only three families of isoparametric hypersurfaces. We will now describe explicitly those hypersurfaces.

Example 1 (Spheres). — Let $x : S^3(r) \to S^4$ be the isometric immersion given by $x(p) = (p, s)$, where $s^2 + r^2 = 1$. It is not difficult to see that $M = S^3(r)$ is all umbilic with principal curvatures $k_i = s/r$, $i = 1, 2, 3$. An elementary computation shows that the mean curvature (H), the scalar curvature (κ) and the Gauss–Kronecker curvature (K) satisfy the following relations:

$$H = 3K^{1/3}\,, \quad \kappa = 6(1 + K^{2/3}) = 6 + \frac{2H^2}{3}\,. \tag{5.1}$$

Example 2 (Clifford tori). — Let $x : S^2(r) \times S^1(s) \to S^4$ be the isometric immersion given by $X(p, q) = (p, q)$. It has principal curvatures given by $k_1 = -r/s$, $k_2 = k_3 = s/r$. The mean curvature (H), the scalar curvature κ and Gauss–Kronecker curvature K of the immersion x satisfy the equations:

$$H = -2K + \frac{1}{K}\,, \quad \kappa = 2(1 + K^2) = 3 + \frac{H^2 \pm H\sqrt{8 + H^2}}{4}\,. \tag{5.2}$$

Example 3 (Cartan's isoparametric family).— We must consider first the minimal immersion $\Psi : S^2(\sqrt{3}) \to S^4$ given by

$$\Psi(x,y,z) = \frac{1}{\sqrt{3}}\left(xy,\, xz,\, yz,\, \frac{1}{2}(x^2-y^2),\, \frac{1}{2\sqrt{3}}(x^2+y^2-2z^2)\right).$$

This defines an imbedding of the real projective plane into S^4. This real projective plane embedded in S^4 is called the Veronese surface. Let $N_1(\Sigma)$ denote the unit normal sphere bundle of the Veronese surface Σ. We can express $N_1(\Sigma)$ as

$$N_1(\Sigma) = \left\{(x,\nu) \in \Sigma \times S^4 \mid \nu \perp T_x\Sigma \text{ and } \nu \perp x\right\}.$$

At $(x,\nu) \in N_1(\Sigma)$ the principal curvatures of the Veronese surface are given by $k_1(x,\nu) = -k_2(x,\nu) = \sqrt{3}/3$. Therefore the Veronese surface is a minimal submanifold of S^4. We now take the *polar mapping* [L]:

$$\Phi : N_1(\Sigma) \longrightarrow S^4$$

given by $\Phi(x,\nu) = \nu$. The principal curvatures of the immersion Φ are $-\sqrt{3}$, 0, $\sqrt{3}$. Finally, we define the isoparametric family $\Phi_t : N_1(\Sigma) \to S^4$ by

$$\Phi_t(x,\nu) = \cos t\, \Phi(x,\nu) + \sin t\, x\,.$$

The principal curvatures of the immersion Φ_t are:

$$\frac{\sqrt{3}+\tan t}{1-\sqrt{3}\,\tan t},\quad \frac{\tan t-\sqrt{3}}{1+\sqrt{3}\,\tan t},\quad \tan t\,. \tag{5.3}$$

An easy computation shows that the mean curvature (H_t), the scalar curvature κ_t and Gauss–Kronecker curvature (K_t) fo the immersion Φ_t satisfy the following interesting relations:

$$\kappa_t \equiv 0\,, \quad H_t + 3K_t \equiv 0\,. \tag{5.4}$$

In [CDK], Chern–do Carmo–Kobayashi asked whether the value of the scalar curvature κ_M of a closed minimal hypersurface $M \subset S^{n+1}$ would determine the hypersurface up to a rigid motion of S^{n+1}. In their conjecture they assumed that κ_M was a constant function. They also asked if the values of the scalar curvature κ_M was a discrete set of real numbers. To solve the conjecture of Chern–do Carmo–Kobayashi for 3-dimensional hypersurfaces one just has to combine Theorem 1.1 together with the non-existence result in [C].

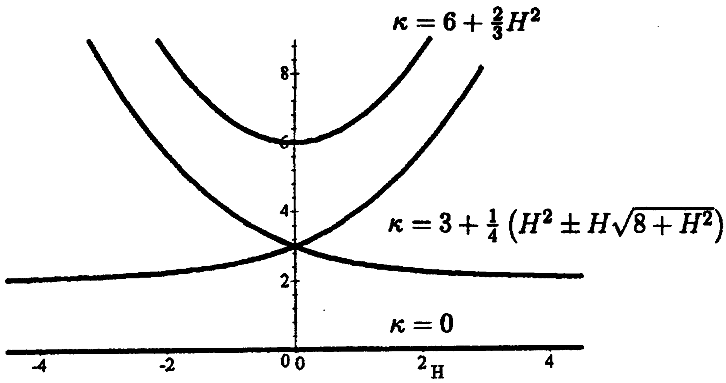


Fig. 1 (H, κ)

Remark 2.— Figure 1 shows the possible values for the mean curvature (H) and for the scalar curvature (κ) of a closed hypersurface $M \in \mathcal{F}_{1,2}$.

For fixed $n \geq 3$, we still denote by $\mathcal{F}_{r,s}$ the colection of all closed hypersurfaces $M \subset S^{n+1}$ having $\mathrm{d}H_r = \mathrm{d}H_s = 0$. The conjecture above is a particular case of the following more general question.

Question 1.— Determine $\mathcal{F}_{r,s}$ for all $r \neq s$.

In this note we considered only the case $n = 3$. It is interesting to note that, even in this particular case, Question 1 is not completely solved.

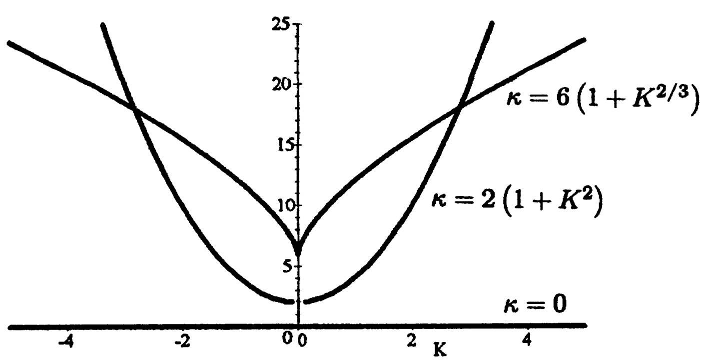


Fig. 2 (K, κ)

Remark 3.— Figure 2 shows the possible values for the Gauss–Kronecker curvature (K) and for the (unnormalized) scalar curvature (κ) of a closed hypersurface $M \in \mathcal{F}_{2,3}$ when $\kappa \geq 0$.

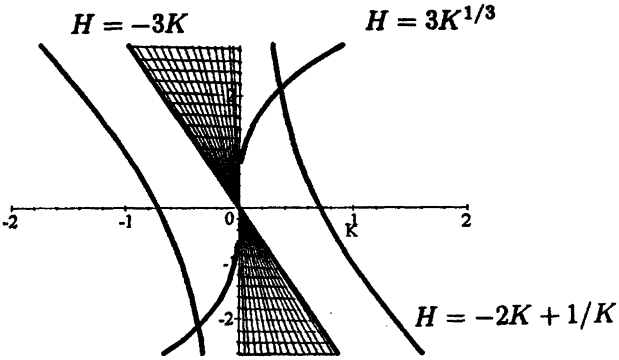


Fig. 3 (K, H)

Remark 4.— Figure 3 shows the possible values for the Gauss–Kronecker curvature (K) and for the mean curvature (H) of a closed hypersurface $M \in \mathcal{F}_{1,3}$. Only the shaded region was not considered in Theorem 1.4.

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