ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE

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Annales de la faculté des sciences de Toulouse 6e série, tome 9, nº 1 (2000), p. 113-131

<http://www.numdam.org/item?id=AFST_2000_6_9_1_113_0>

Annales de la Faculté des Sciences de Toulouse Vol. IX, n° 1, 2000
pp. 113–131

Weak Solutions to General Euler's Equations via Nonsmooth Critical Point Theory (*)

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RÉSUMÉ. — Au moyen de la théorie des points critiques non réguliers, nous démontrons l'existence de solutions faibles qui ne sont pas nulles pour une classe générale de problèmes elliptiques non linéaires.

ABSTRACT. — By means of nonsmooth critical point theory we prove existence of a nontrivial weak solution for a general class of nonlinear elliptic boundary value problems on a bounded domain Ω of $\mathbb{R}^n$.

1. Introduction

Since 1972, existence of weak solutions for the semilinear elliptic problem

$$\begin{cases} -\sum\limits_{i,j=1}^{n} D_j(a_{ij}(x)D_iu) = g(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

has been deeply investigated (see e.g. [1, 2, 3, 13, 21] and references therein) by means of classical critical point theory.

Later on, since 1994, also quasilinear elliptic problems of the type

$$\begin{cases} -\sum\limits_{i,j=1}^{n} D_j(a_{ij}(x,u)D_iu) + \frac{1}{2}\sum\limits_{i,j=1}^{n} D_s a_{ij}(x,u)D_iuD_ju = g(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

(*) Reçu le 30 novembre 1999, accepté le 21 mars 2000
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have been studied in [5, 9, 10, 12, 14, 22] via techniques of nonsmooth critical point theory. On the other hand, more recently, some results for the more general problem

$$\begin{cases} -\operatorname{div}\left(\nabla_\xi L(x,u,\nabla u)\right) + D_s L(x,u,\nabla u) = g(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

have been considered in [4] and [20].

The goal of this paper is to extend some of the results of [4, 20]. In order to solve (1.1), we shall look for critical points of functionals $f : W_0^{1,p}(\Omega) \to \mathbb{R}$ given by

$$f(u) = \int_\Omega L(x,u,\nabla u)\,dx - \int_\Omega G(x,u)\,dx. \tag{1.2}$$

In general, f is continuous but not even locally Lipschitzian unless L does not depend on u or L is subjected to some very restrictive growth conditions.

Then, we shall refer to the nonsmooth critical point theory developed in [15, 16, 17, 19] and in particular to [12] for what we shall recall in section 2.

We assume that $L : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is measurable in x for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$, of class C^1 in (s,ξ) for a.e. $x \in \Omega$, the function $L(x,s,\cdot)$ is strictly convex and $L(x,s,0) = 0$. Furthermore, we shall assume that:

(i) there exist $a \in L^1(\Omega)$ and $b_0, \nu > 0$ such that

$$\nu|\xi|^p \leqslant L(x,s,\xi) \leqslant a(x) + b_0|s|^p + b_0|\xi|^p, \tag{1.3}$$

for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$;

(ii) for each $\varepsilon > 0$ there exists $a_\varepsilon \in L^1(\Omega)$ such that

$$|D_s L(x,s,\xi)| \leqslant a_\varepsilon(x) + \varepsilon|s|^{p^*} + b_1|\xi|^p, \tag{1.4}$$

for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$, with $b_1 \in \mathbb{R}$ independent of ε.

Furthermore, there exists $a_1 \in L^{p'}(\Omega)$ such that

$$|\nabla_\xi L(x,s,\xi)| \leqslant a_1(x) + b_1|s|^{\frac{p^*}{p'}} + b_1|\xi|^{p-1}, \tag{1.5}$$

for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$;

(iii) there exists $R > 0$ such that

$$|s| \geqslant R \Longrightarrow D_s L(x,s,\xi)s \geqslant 0, \tag{1.6}$$

for a.e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$;

(*iv*) $G : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

$$G(x,s) \leqslant d(x)|s|^p + b|s|^{p^*} \tag{1.7}$$

$$\lim_{s \to 0} \frac{G(x,s)}{|s|^p} = 0 \tag{1.8}$$

for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$, where $d \in L^{\frac{n}{p}}(\Omega)$ and $b \in \mathbb{R}$. Moreover,

$$G(x,s) = \int_0^s g(x,\tau)\, d\tau$$

and there exist $c_1, c_2 > 0$ such that

$$|g(x,s)| \leqslant c_1 + c_2|s|^\sigma$$

for a.e. $x \in \Omega$ and each $s \in \mathbb{R}$, where $\sigma < p^* - 1$.

(*v*) there exist $q > p$ and $R' > 0$ such that for each $\varepsilon > 0$ there is $a_\varepsilon \in L^1(\Omega)$ with

$$|s| \geqslant R' \Longrightarrow 0 < qG(x,s) \leqslant g(x,s)s\,, \tag{1.9}$$

$$|s| \geqslant R' \Longrightarrow qL(x,s,\xi) - \nabla_\xi L(x,s,\xi) \cdot \xi - D_s L(x,s,\xi)s \geqslant \nu|\xi|^p - a_\varepsilon(x) - \varepsilon|s|^p \tag{1.10}$$

for a.e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$.

Under the previous assumptions, the following is our main result.

(1.11) THEOREM.— *The boundary value problem*

$$\begin{cases} -\operatorname{div}\left(\nabla_\xi L(x,u,\nabla u)\right) + D_s L(x,u,\nabla u) = g(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has at least one nontrivial weak solution $u \in W_0^{1,p}(\Omega)$.

This result is an extension of [4, Theorem 3.3], since instead of assuming that

$$\forall s \in \mathbb{R}: \ \ qL(x,s,\xi) - \nabla_\xi L(x,s,\xi) \cdot \xi - D_s L(x,s,\xi)s \geqslant \nu|\xi|^p\,,$$

for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^n$, we only request condition (1.10). In this way the proof of Lemma (3.15) becomes more difficult. The keypoint, to deal with the more general assumption, is constituted by Lemma (3.11).

Similarly, in [20, Theorem 1], a multiplicity result for (1.1) is proved, assuming that

$$\forall s \in \mathbb{R}: \ \ D_s L(x,s,\xi)s \geqslant 0\,,$$

$$\forall s \in \mathbb{R}: \ \ qL(x,s,\xi) - \nabla_\xi L(x,s,\xi) \cdot \xi - D_s L(x,s,\xi)s \geqslant \nu|\xi|^p\,,$$

for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^n$, which are both stronger than (1.6) and (1.10). In particular, the first inequality above and the more general condition (1.6) are involved in Theorem (3.4).

Finally, let us point out that the growth conditions (1.3) – (1.5) are a relaxation of those of [4, 20], where it is assumed that

$$\nu|\xi|^p \leqslant L(x,s,\xi) \leqslant \beta|\xi|^p, \qquad |D_s L(x,s,\xi)| \leqslant \gamma|\xi|^p, \tag{1.12}$$

$$|\nabla_\xi L(x,s,\xi)| \leqslant a_1(x) + b_1|s|^{p-1} + b_1|\xi|^{p-1}, \tag{1.13}$$

for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$.

2. Weak slope and weak solutions

In this section we want to recall the relationship between weak solutions to (1.1) and critical points of f in the sense of weak slope [12, Definition 1.1.1].

For the sake of generality, we shall here consider the more general setting of functionals on $W_0^{1,p}(\Omega,\mathbb{R}^N)$, with $N \geqslant 1$, subjected to growth conditions weaker than (1.3) – (1.5).

Let $a_0 \in L^1(\Omega)$, $b_0 \in \mathbb{R}$, $a_1 \in L^1_{loc}(\Omega)$ and $b_1 \in L^\infty_{loc}(\Omega)$ be such that

$$|L(x,s,\xi)| \leqslant a_0(x) + b_0|s|^{p^*} + b_0|\xi|^p, \tag{2.1}$$

$$|\nabla_s L(x,s,\xi)| \leqslant a_1(x) + b_1(x)|s|^{p^*} + b_1(x)|\xi|^p, \tag{2.2}$$

$$|\nabla_\xi L(x,s,\xi)| \leqslant a_1(x) + b_1(x)|s|^{p^*} + b_1(x)|\xi|^p. \tag{2.3}$$

for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R}^N \times \mathbb{R}^{nN}$.

Conditions (2.2) and (2.3) imply that for every $u \in W_0^{1,p}(\Omega,\mathbb{R}^N)$

$$\nabla_\xi L(x,u,\nabla u) \in L^1_{loc}(\Omega,\mathbb{R}^{nN}), \quad \nabla_s L(x,u,\nabla u) \in L^1_{loc}(\Omega,\mathbb{R}^N).$$

Therefore for each $u \in W_0^{1,p}(\Omega,\mathbb{R}^N)$ we have

$$-\text{div}\,(\nabla_\xi L(x,u,\nabla u)) + \nabla_s L(x,u,\nabla u) \in \mathcal{D}'(\Omega,\mathbb{R}^N).$$

(2.4) DEFINITION. — *We say that u is a* weak solution *to (1.1), if $u \in W_0^{1,p}(\Omega,\mathbb{R}^N)$ and*

$$-\text{div}\,(\nabla_\xi L(x,u,\nabla u)) + \nabla_s L(x,u,\nabla u) = g(x,u)$$

in $\mathcal{D}'(\Omega,\mathbb{R}^N)$.

Under more restrictive conditions, it would turn out f to be of class C^1 and

$$-\text{div}\ (\nabla_\xi L(x,u,\nabla u)) + \nabla_s L(x,u,\nabla u) \in W^{-1,p'}(\Omega,\mathbb{R}^N)\,,$$

for every $u \in W_0^{1,p}(\Omega,\mathbb{R}^N)$. In this regular setting, the classical $(PS)_c$ condition may be considered. On the other hand, in our nonsmooth context, it is convenient to introduce the following variant of the $(PS)_c$ condition :

(2.5) DEFINITION.— *Let $c \in \mathbb{R}$. A sequence $(u_h) \subseteq W_0^{1,p}(\Omega,\mathbb{R}^N)$ is said to be* a concrete Palais-Smale sequence at level c *($(CPS)_c$*–sequence, *in short) for f, if $f(u_h) \to c$,*

$$-\text{div}\ (\nabla_\xi L(x,u_h,\nabla u_h)) + \nabla_s L(x,u_h,\nabla u_h) \in W^{-1,p'}(\Omega,\mathbb{R}^N)\,,$$

eventually as $h \to +\infty$ and

$$-\text{div}\ (\nabla_\xi L(x,u_h,\nabla u_h)) + \nabla_s L(x,u_h,\nabla u_h) - g(x,u_h) \to 0\,,$$

strongly in $W^{-1,p'}(\Omega,\mathbb{R}^N)$.

We say that f satisfies the concrete Palais-Smale condition at level c *($(CPS)_c$ in short), if every $(CPS)_c$–sequence for f admits a strongly convergent subsequence.*

The next result connects the previous notions with abstract critical point theory.

(2.6) THEOREM.— *$f : W_0^{1,p}(\Omega,\mathbb{R}^N) \to \mathbb{R}$ is continuous and we have*

$$|df|\,(u) \geqslant$$

$$\geqslant \sup_{\substack{v\in C_c^\infty(\Omega,\mathbb{R}^N)\\ \|v\|_{1,p}\leqslant 1}} \left\{\int_\Omega [\nabla_\xi L(x,u,\nabla u)\cdot\nabla v + \nabla_s L(x,u,\nabla u)\cdot v - g(x,u)\cdot v]\,dx\right\}\,,$$

for every $u \in W_0^{1,p}(\Omega,\mathbb{R}^N)$. In particular, $|df|\,(u) < +\infty$ implies that

$$\|-\text{div}\ (\nabla_\xi L(x,u,\nabla u)) + \nabla_s L(x,u,\nabla u) - g(x,u)\|_{-1,p'} \leqslant |df|\,(u)\,.$$

Proof.— See, [12, Theorem 2.1.3] □

As a consequence, each critical point u of f is a weak solution to (1.1).

Let us finally recall from [12] a concrete versions of the Mountain Pass Theorem.

(2.7) THEOREM. — *Let (D, S) be a compact pair, $\psi : S \to W_0^{1,p}(\Omega, \mathbb{R}^N)$ a continuous map and let*

$$\Phi = \left\{ \varphi \in C(D, W_0^{1,p}(\Omega, \mathbb{R}^N)) : \varphi_{|S} = \psi \right\} .$$

Assume that there exists a closed subset A of $W_0^{1,p}(\Omega, \mathbb{R}^N)$ such that

$$\inf_A f \geqslant \max_{\psi(S)} f ,$$

$$A \cap \psi(S) = \emptyset \text{ and } A \cap \varphi(D) \neq \emptyset \text{ for all } \varphi \in \Phi.$$

If f satisfies the concrete Palais-Smale condition at level

$$c = \inf_{\varphi \in \Phi} \max_{\varphi(D)} f ,$$

then there exists a weak solution $u \in W_0^{1,p}(\Omega, \mathbb{R}^N)$ of (1.1) with $f(u) = c$.

3. The concrete Palais-Smale condition

We first recall a very useful consequence of Brezis-Browder's Theorem [8].

(3.1) PROPOSITION. — *Let $u, v \in W_0^{1,p}(\Omega)$, $\eta \in L^1(\Omega)$ and $w \in W^{-1,p'}(\Omega)$ with*

$$D_s L(x, u, \nabla u) v \geqslant \eta ,$$

and for all $\varphi \in C_c^\infty(\Omega)$

$$\langle w, \varphi \rangle = \int_\Omega \nabla_\xi L(x, u, \nabla u) \cdot \nabla \varphi \, dx + \int_\Omega D_s L(x, u, \nabla u) \varphi \, dx .$$

Then $D_s L(x, u, \nabla u) v \in L^1(\Omega)$ and

$$\langle w, v \rangle = \int_\Omega \nabla_\xi L(x, u, \nabla u) \cdot \nabla v \, dx + \int_\Omega D_s L(x, u, \nabla u) v \, dx .$$

Proof. — See e.g. [20, Proposition 1]. □

Let us point out that as a consequence of assumptions $(i) - (ii)$ and convexity of $L(x, s, \cdot)$, we can find $M > 0$ such that for each $\varepsilon > 0$ there is $a_\varepsilon \in L^1(\Omega)$ with

$$\nabla_\xi L(x, s, \xi) \cdot \xi \geqslant \nu |\xi|^p - a(x) - b_0 |s|^p, \tag{3.2}$$

$$|D_s L(x, s, \xi)| \leqslant M \nabla_\xi L(x, s, \xi) \cdot \xi + a_\varepsilon(x) + \varepsilon |s|^{p^*}, \tag{3.3}$$

for a.e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$.

We now come to a local compactness property, which is crucial for the $(CPS)_c$ condition to hold. This result improves [20, Lemma 2], since *(iii)* relaxes condition (8) in [20].

(3.4) THEOREM. — *Assume (i) – (iii), let (u_h) be a bounded sequence in $W_0^{1,p}(\Omega)$ and set*

$$\langle w_h, v\rangle = \int_\Omega \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla v \, dx + \int_\Omega D_s L(x, u_h, \nabla u_h) v \, dx\,, \quad (3.5)$$

for all $v \in C_c^\infty(\Omega)$. If (w_h) is strongly convergent to some w in $W^{-1,p'}(\Omega)$, then (u_h) admits a strongly convergent subsequence in $W_0^{1,p}(\Omega)$.

Proof. — Since (u_h) is bounded in $W_0^{1,p}(\Omega)$, we find a u in $W_0^{1,p}(\Omega)$ such that, up to a subsequence,

$$\nabla u_h \rightharpoonup \nabla u \text{ in } L^p(\Omega), \quad u_h \to u \text{ in } L^p(\Omega), \quad u_h(x) \to u(x) \text{ for a.e. } x \in \Omega\,.$$

By [7, Theorem 2.1], up to a subsequence, we have

$$\nabla u_h(x) \to \nabla u(x) \quad \text{for a.e. } x \in \Omega\,. \quad (3.6)$$

Therefore, by (1.5) we deduce that

$$\nabla_\xi L(x, u_h, \nabla u_h) \rightharpoonup \nabla_\xi L(x, u, \nabla u) \text{ in } L^{p'}(\Omega, \mathbb{R}^n)\,.$$

We now want to prove that u solves the equation

$$\forall v \in C_c^\infty(\Omega): \ \langle w, v\rangle = \int_\Omega \nabla_\xi L(x, u, \nabla u) \cdot \nabla v \, dx + \int_\Omega D_s L(x, u, \nabla u) v \, dx\,. \quad (3.7)$$

To this aim, let us test equation (3.5) with the functions

$$v_h = \varphi \exp\{-M(u_h + R)^+\}\,, \qquad \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)\,, \qquad \varphi \geqslant 0\,.$$

It results for each $h \in \mathbb{N}$

$$\int_\Omega \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla\varphi \exp\{-M(u_h + R)^+\} \, dx - \langle w_h, \varphi \exp\{-M(u_h + R)^+\}\rangle + \\ + \int_\Omega \left[D_s L(x, u_h, \nabla u_h) - M \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla(u_h + R)^+\right] \\ \varphi \exp\{-M(u_h + R)^+\} \, dx = 0\,.$$

Of course, for a.e. $x \in \Omega$, we obtain

$$\varphi \exp\{-M(u_h + R)^+\} \to \varphi \exp\{-M(u + R)^+\}\,.$$

Since by inequality (3.3) and (1.6) for each $\varepsilon > 0$ and $h \in \mathbb{N}$ we have

$$\left[D_s L(x, u_h, \nabla u_h) - M\nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla(u_h + R)^+\right] \varphi$$

$$\exp\{-M(u_h + R)^+\} - \varepsilon |u_h|^{p^*} \varphi \leqslant a_\varepsilon(x)\varphi,$$

Fatou's Lemma implies that for each $\varepsilon > 0$

$$\limsup_h \int_\Omega \left[D_s L(x, u_h, \nabla u_h) - M\nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla(u_h + R)^+\right] \varphi$$

$$\exp\{-M(u_h + R)^+\} - \varepsilon |u_h|^{p^*} \varphi \, dx \leqslant \int_\Omega \left[D_s L(x, u, \nabla u)\right.$$

$$\left. - M\nabla_\xi L(x, u, \nabla u) \cdot \nabla(u + R)^+\right] \varphi \exp\{-M(u + R)^+\} - \varepsilon |u|^{p^*} \varphi \, dx.$$

Since (u_h) is bounded in $L^{p^*}(\Omega)$, we find $c > 0$ such that for each $\varepsilon > 0$

$$\limsup_h \int_\Omega \left[D_s L(x, u_h, \nabla u_h) - M\nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla(u_h + R)^+\right] \varphi$$

$$\exp\{-M(u_h + R)^+\} \, dx \leqslant \int_\Omega \left[D_s L(x, u, \nabla u) - M\nabla_\xi L(x, u, \nabla u) \cdot \nabla(u + R)^+\right] \varphi$$

$$\exp\{-M(u + R)^+\} \, dx - c\varepsilon.$$

Letting $\varepsilon \to 0$, the previous inequality yields

$$\limsup_h \int_\Omega \left[D_s L(x, u_h, \nabla u_h) - M\nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla(u_h + R)^+\right] \varphi$$

$$\exp\{-M(u_h + R)^+\} \, dx \leqslant \int_\Omega \left[D_s L(x, u, \nabla u) - M\nabla_\xi L(x, u, \nabla u) \cdot \nabla(u + R)^+\right] \varphi$$

$$\exp\{-M(u + R)^+\} \, dx.$$

Note that we also have

$$\varphi \exp\{-M(u_h + R)^+\} \rightharpoonup \varphi \exp\{-M(u + R)^+\} \qquad \text{in } W_0^{1,p}(\Omega).$$

Moreover

$$\nabla\varphi \exp\{-M(u_h + R)^+\} \to \nabla\varphi \exp\{-M(u + R)^+\} \qquad \text{in } L^p(\Omega, \mathbb{R}^n),$$

so that

$$\int_\Omega \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla\varphi \exp\{-M(u_h + R)^+\} \, dx \to$$

$$\to \int_\Omega \nabla_\xi L(x, u, \nabla u) \cdot \nabla\varphi \exp\{-M(u + R)^+\} \, dx.$$

Therefore, we may conclude that

$$\int_\Omega \nabla_\xi L(x,u,\nabla u)\cdot\nabla\varphi\exp\{-M(u+R)^+\}\,dx - \langle w,\varphi\exp\{-M(u+R)^+\}\rangle +$$

$$+\int_\Omega \left[D_sL(x,u,\nabla u) - M\nabla_\xi L(x,u,\nabla u)\cdot\nabla(u+R)^+\right]\varphi\exp\{-M(u+R)^+\}\,dx \geqslant 0$$

Consider now the test functions

$$\varphi_k = \varphi H\left(\frac{u}{k}\right)\exp\{M(u+R)^+\}\,, \qquad \varphi\in C_c^\infty(\Omega)\,, \qquad \varphi\geqslant 0\,,$$

where $H\in C^1(\mathbb{R})$, $H=1$ in $\left[-\frac{1}{2},\frac{1}{2}\right]$ and $H=0$ in $]-\infty,-1]\cup[1,+\infty[$. It follows that

$$\int_\Omega \nabla_\xi L(x,u,\nabla u)\cdot\nabla\varphi_k\exp\{-M(u+R)^+\}\,dx - \left\langle w,\varphi H\left(\frac{u}{k}\right)\right\rangle +$$

$$+\int_\Omega \left[D_sL(x,u,\nabla u) - M\nabla_\xi L(x,u,\nabla u)\cdot\nabla(u+R)^+\right]\varphi H\left(\frac{u}{k}\right)\,dx \geqslant 0\,.$$

Furthermore, standard computations yield

$$\nabla\varphi_k = \exp\{M(u+R)^+\}\left[\nabla\varphi H\left(\frac{u}{k}\right) + H'\left(\frac{u}{k}\right)\frac{\varphi}{k}\nabla u + M\nabla(u+R)^+\varphi H\left(\frac{u}{k}\right)\right]\,.$$

Since $\varphi H\left(\frac{u}{k}\right)$ goes to φ in $W_0^{1,p}(\Omega)$, as $k\to+\infty$ we have

$$\left\langle w,\varphi H\left(\frac{u}{k}\right)\right\rangle \to \langle w,\varphi\rangle\,.$$

By the properties of H and the growth conditions on $\nabla_\xi L$, letting $k\to+\infty$ yields

$$\int_\Omega \nabla_\xi L(x,u,\nabla u)\cdot\nabla\varphi H\left(\frac{u}{k}\right)dx \to \int_\Omega \nabla_\xi L(x,u,\nabla u)\cdot\nabla\varphi\,dx\,,$$

$$\int_\Omega \nabla_\xi L(x,u,\nabla u)\cdot\nabla u\,H'\left(\frac{u}{k}\right)\frac{\varphi}{k}\,dx \to 0\,,$$

$$\int_\Omega M\nabla_\xi L(x,u,\nabla u)\cdot\nabla(u+R)^+\varphi H\left(\frac{u}{k}\right)dx \to \int_\Omega M\nabla_\xi L(x,u,\nabla u)\cdot\nabla(u+R)^+\varphi dx$$

Whence, we conclude that for all $\varphi\in C_c^\infty(\Omega)$

$$\varphi\geqslant 0 \Longrightarrow \langle w,\varphi\rangle \leqslant \int_\Omega \nabla_\xi L(x,u,\nabla u)\cdot\nabla\varphi\,dx + \int_\Omega D_sL(x,u,\nabla u)\varphi\,dx\,.$$

Choosing now as test functions

$$v_h = \varphi\exp\{-M(u_h-R)^-\}\,,$$

where as before $\varphi \geqslant 0$, we obtain the opposite inequality so that (3.7) is proven.

In particular, taking into account Proposition (3.1), we immediately obtain

$$\langle w, u\rangle = \int_\Omega \nabla_\xi L(x,u,\nabla u)\cdot \nabla u \, dx + \int_\Omega D_s L(x,u,\nabla u) u \, dx\,. \tag{3.8}$$

The final step is to show that (u_h) goes to u in $W_0^{1,p}(\Omega)$. Consider the function $\zeta : \mathbb{R} \to \mathbb{R}$ defined by

$$\zeta(s) = \begin{cases} Ms & \text{if } \; 0 < s < R \\ MR & \text{if } \; s \geqslant R \\ -Ms & \text{if } \; -R < s < 0 \\ MR & \text{if } \; s \leqslant -R, \end{cases} \tag{3.9}$$

and let us prove that

$$\begin{aligned} &\limsup_h \int_\Omega \nabla_\xi L(x,u_h,\nabla u_h)\cdot \nabla u_h \exp\{\zeta(u_h)\}\, dx \leqslant \\ &\leqslant \int_\Omega \nabla_\xi L(x,u,\nabla u)\cdot \nabla u \exp\{\zeta(u)\}\, dx\,. \end{aligned} \tag{3.10}$$

Since by Proposition (3.1), $u_h \exp\{\zeta(u_h)\}$ are admissible test functions for (3.5), we have

$$\int_\Omega \nabla_\xi L(x,u_h,\nabla u_h)\cdot \nabla u_h \exp\{\zeta(u_h)\}\, dx - \langle w_h, u_h \exp\{\zeta(u_h)\}\rangle +$$

$$+ \int_\Omega [D_s L(x,u_h,\nabla u_h) + \zeta'(u_h)\nabla_\xi L(x,u_h,\nabla u_h)\cdot \nabla u_h]\, u_h \exp\{\zeta(u_h)\}\, dx = 0\,.$$

Let us observe that (3.6) implies that

$$\nabla_\xi L(x,u_h,\nabla u_h)\cdot \nabla u_h \to \nabla_\xi L(x,u,\nabla u)\cdot \nabla u \qquad \text{for a.e. } x \in \Omega\,.$$

Since by inequality (3.3) for each $\varepsilon > 0$ and $h \in \mathbb{N}$ we have

$$[-D_s L(x,u_h,\nabla u_h) - \zeta'(u_h)\nabla_\xi L(x,u_h,\nabla u_h)\cdot \nabla u_h]\, u_h \exp\{\zeta(u_h)\} +$$

$$-R\exp\{MR\}\varepsilon |u_h|^{p^*} \leqslant R \exp\{MR\} a_\varepsilon(x)\,,$$

Fatou's Lemma yields

$$\limsup_h \int_\Omega [-D_s L(x,u_h,\nabla u_h) - \zeta'(u_h)\nabla_\xi L(x,u_h,\nabla u_h)\cdot \nabla u_h]\, u_h \exp\{\zeta(u_h)\}$$

$$-R\exp\{MR\}\varepsilon|u_h|^{p^*}\,dx \leqslant \int_\Omega [-D_sL(x,u,\nabla u) - \zeta'(u)\nabla_\xi L(x,u,\nabla u)\cdot\nabla u]\,u$$

$$\exp\{\zeta(u)\} - R\exp\{MR\}\varepsilon|u|^{p^*}\,dx\,.$$

Therefore, since (u_h) is bounded in $L^{p^*}(\Omega)$, we find $c > 0$ such that for all $\varepsilon > 0$

$$\limsup_h \int_\Omega [-D_sL(x,u_h,\nabla u_h) - \zeta'(u_h)\nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla u_h]\,u_h\exp\{\zeta(u_h)\}$$

$$\leqslant \int_\Omega [-D_sL(x,u,\nabla u) - \zeta'(u)\nabla_\xi L(x,u,\nabla u)\cdot\nabla u]\,u\exp\{\zeta(u)\} - c\varepsilon\,.$$

Taking into account that ε is arbitrary, we conclude that

$$\limsup_h \int_\Omega \nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla u_h\exp\{\zeta(u_h)\}\,dx =$$

$$= \limsup_h \left\{ \int_\Omega [-D_sL(x,u_h,\nabla u_h) - \zeta'(u_h)\nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla u_h]\,u_h \right.$$

$$\left. \exp\{\zeta(u_h)\}\,dx + \langle w_h, u_h\exp\{\zeta(u_h)\}\rangle \right\}$$

$$\leqslant \int_\Omega [-D_sL(x,u,\nabla u) - \zeta'(u)\nabla_\xi L(x,u,\nabla u)\cdot\nabla u]\,u\exp\{\zeta(u)\}\,dx$$

$$+ \langle w, u\exp\{\zeta(u)\}\rangle = \int_\Omega \nabla_\xi L(x,u,\nabla u)\cdot\nabla u\exp\{\zeta(u)\}\,dx\,.$$

In particular, we have

$$\int_\Omega \nabla_\xi L(x,u,\nabla u)\cdot\nabla u\exp\{\zeta(u)\}\,dx \leqslant \liminf_h \int_\Omega \nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla u_h$$

$$\exp\{\zeta(u_h)\}\,dx \leqslant \limsup_h \int_\Omega \nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla u_h\exp\{\zeta(u_h)\}\,dx \leqslant$$

$$\leqslant \int_\Omega \nabla_\xi L(x,u,\nabla u)\cdot\nabla u\exp\{\zeta(u)\}\,dx\,,$$

namely

$$\lim_h \int_\Omega \nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla u_h\exp\{\zeta(u_h)\}\,dx =$$

$$= \int_\Omega \nabla_\xi L(x,u,\nabla u)\cdot\nabla u\exp\{\zeta(u)\}\,dx\,.$$

Therefore, by (3.2), generalized Fatou's Lemma yields

$$\limsup_h \int_\Omega |\nabla u_h|^p \, dx \leqslant \int_\Omega |\nabla u|^p \, dx \,,$$

that implies the strong convergence of (u_h) to u in $W_0^{1,p}(\Omega)$. □

(3.11) LEMMA.— *Let $c \in \mathbb{R}$ and let (u_h) be a $(CPS)_c$-sequence in $W_0^{1,p}(\Omega)$. Then for each $\varepsilon > 0$ and $\varrho > 0$ there exists $K_{\varrho,\varepsilon} > 0$ such that for all $h \in \mathbb{N}$*

$$\int_{\{|u_h| \leqslant \varrho\}} \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx \leqslant$$

$$\leqslant \varepsilon \int_{\{\varrho < |u_h| < K_{\varrho,\varepsilon}\}} \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx + K_{\varrho,\varepsilon}$$

and

$$\int_{\{|u_h| \leqslant \varrho\}} |\nabla u_h|^p \, dx \leqslant \varepsilon \int_{\{\varrho < |u_h| < K_{\varrho,\varepsilon}\}} |\nabla u_h|^p \, dx + K_{\varrho,\varepsilon} \,.$$

Proof.— Let $\sigma, \varepsilon > 0$ and $\varrho > 0$. For all $v \in W_0^{1,p}(\Omega)$, we set

$$\langle w_h, v \rangle = \int_\Omega \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla v \, dx + $$
$$+ \int_\Omega D_s L(x, u_h, \nabla u_h) v \, dx - \int_\Omega g(x, u_h) v \, dx \,. \tag{3.12}$$

Let us now consider $\vartheta_1 : \mathbb{R} \to \mathbb{R}$ given by

$$\vartheta_1(s) = \begin{cases} s & \text{if } \ |s| < \sigma \\ -s + 2\sigma & \text{if } \ \sigma \leqslant s < 2\sigma \\ -s - 2\sigma & \text{if } \ -2\sigma < s \leqslant -\sigma \\ 0 & \text{if } \ |s| \geqslant 2\sigma \,. \end{cases} \tag{3.13}$$

Then, testing (3.12) with $\vartheta_1(u_h) \in L^\infty([-2\sigma, 2\sigma])$, we obtain

$$\int_\Omega \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla \vartheta_1(u_h) \, dx + \int_\Omega D_s L(x, u_h, \nabla u_h) \vartheta_1(u_h) \, dx \leqslant$$

$$\leqslant \int_\Omega g(x, u_h) \vartheta_1(u_h) \, dx + \|w_h\|_{-1,p'} \|\vartheta_1(u_h)\|_{1,p} \,.$$

Then, it follows that

$$\int_{\{|u_h| \leqslant \sigma\}} \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx - \int_{\{\sigma < |u_h| \leqslant 2\sigma\}} \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx +$$

$$+\int_{\{|u_h|\leqslant\sigma\}} D_sL(x,u_h,\nabla u_h)u_h\,dx+\int_{\{\sigma<|u_h|\leqslant 2\sigma\}} D_sL(x,u_h,\nabla u_h)\vartheta_1(u_h)\,dx \leqslant$$

$$\leqslant \int_\Omega \left(c_1+c_2|2\sigma|^{\frac{n(p-1)+p}{n-p}}\right)\sigma\,dx+\frac{4^{\frac{p'}{p}}}{p'p^{\frac{p'}{p}}\nu^{\frac{p'}{p}}}\|w_h\|^{p'}_{-1,p'}+\frac{\nu}{4}\|\vartheta_1(u_h)\|^p_{1,p}\,.$$

Let $K_0>0$ be such that $\|w_h\|_{-1,p'}\leqslant K_0$. Then, since by (3.2) we have

$$\nu\|\vartheta_1(u_h)\|^p_{1,p}\leqslant \int_{\{|u_h|\leqslant\sigma\}}\nu|\nabla u_h|^p\,dx+\int_{\{\sigma<|u_h|\leqslant 2\sigma\}}\nu|\nabla u_h|^p\,dx\leqslant$$

$$\leqslant \int_{\{|u_h|\leqslant\sigma\}}\nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx+\int_{\{\sigma<|u_h|\leqslant 2\sigma\}}\nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla u_h\,$$

$$+\int_{\{|u_h|\leqslant\sigma\}}a(x)\,dx+b_0\int_{\{|u_h|\leqslant\sigma\}}|u_h|^p\,dx+\int_{\{\sigma<|u_h|\leqslant 2\sigma\}}a(x)\,dx+$$

$$+b_0\int_{\{\sigma<|u_h|\leqslant 2\sigma\}}|u_h|^p\,dx\,,$$

taking into account (3.3), we get for a sufficiently small value of $\sigma>0$

$$\left(1-\sigma M-\frac{1}{4}\right)\int_{\{|u_h|\leqslant\sigma\}}\nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx\leqslant$$

$$\leqslant\left(1+\sigma M+\frac{1}{4}\right)\int_{\{\sigma<|u_h|\leqslant 2\sigma\}}\nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx+$$

$$+\int_\Omega\left(c_1+c_2|2\sigma|^{\frac{n(p-1)+p}{n-p}}\right)\sigma\,dx+\frac{4^{\frac{p'}{p}}}{p'p^{\frac{p'}{p}}\nu^{\frac{p'}{p}}}K_0^{p'}+$$

$$+\int_\Omega a_\varepsilon(x)\,dx+\left[b_0(2^p+1)\sigma^p+\varepsilon\sigma^{p^*+1}\right]\mathcal{L}^n(\Omega)\,.$$

Whence, we have shown an inequality of the type

$$\int_{\{|u_h|\leqslant\sigma\}}\nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx\leqslant$$

$$\leqslant K_1\int_{\{\sigma<|u_h|\leqslant 2\sigma\}}\nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx+K_2\,.$$

Let us now define for each $k\geqslant 1$ the functions $\vartheta_{2k}\,,\vartheta_{2k-1}:\mathbb{R}\to\mathbb{R}$ by setting

$$\vartheta_{2k}(s)=\begin{cases}0 & \text{if } |s|\leqslant k\sigma\\ s-k\sigma & \text{if } k\sigma<s<(k+1)\sigma\\ s+k\sigma & \text{if } -(k+1)\sigma<s<-k\sigma\\ -s+(k+2)\sigma & \text{if } (k+1)\sigma\leqslant s<(k+2)\sigma\\ -s-(k+2)\sigma & \text{if } -(k+2)\sigma<s\leqslant-(k+1)\sigma\\ 0 & \text{if } |s|\geqslant(k+1)\sigma\,,\end{cases}$$

and

$$\vartheta_{2k-1}(s) = \begin{cases} \frac{s}{k} & \text{if} \quad |s| \leqslant k\sigma \\ -s + (k+1)\sigma & \text{if} \quad k\sigma < s < (k+1)\sigma \\ s - (k+1)\sigma & \text{if} \quad (k+1)\sigma < s < (k+2)\sigma \\ -s - (k+1)\sigma & \text{if} \quad -(k+1)\sigma \leqslant s < -k\sigma \\ s + (k+1)\sigma & \text{if} \quad -(k+2)\sigma < s \leqslant -(k+1)\sigma \\ 0 & \text{if} \quad |s| \geqslant (k+1)\sigma \,. \end{cases}$$

Therefore, by iterating on k, we obtain the $k-$th inequality

$$\int_{\{|u_h| \leqslant k\sigma\}} \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx \leqslant \tag{3.14}$$

$$\leqslant K_1(k) \int_{\{k\sigma < |u_h| \leqslant (k+1)\sigma\}} \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx + K_2(k) \,.$$

Let now choose $k \geqslant 1$ such that $k\sigma \geqslant \varrho$ and $k\sigma \geqslant R$. Take $0 < \delta < 1$ and let $\vartheta_\delta : \mathbb{R} \to \mathbb{R}$ be the function defined by setting

$$\vartheta_\delta(s) = \begin{cases} 0 & \text{if} \quad |s| \leqslant k\sigma \\ s - k\sigma & \text{if} \quad k\sigma < s < (k+1)\sigma \\ s + k\sigma & \text{if} \quad -(k+1)\sigma < s < -k\sigma \\ -\delta s + \sigma + \delta(k+1)\sigma & \text{if} \quad (k+1)\sigma \leqslant s < (k+1)\sigma + \frac{\sigma}{\delta} \\ -\delta s - \sigma - \delta(k+1)\sigma & \text{if} \quad -(k+1)\sigma - \frac{\sigma}{\delta} < s \leqslant -(k+1)\sigma \\ 0 & \text{if} \quad |s| \geqslant (k+1)\sigma + \frac{\sigma}{\delta} \,. \end{cases}$$

As before, we get

$$\int_\Omega \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla \vartheta_\delta(u_h) \, dx + \int_\Omega D_s L(x, u_h, \nabla u_h) \vartheta_\delta(u_h) \, dx \leqslant$$

$$\leqslant \int_\Omega g(x, u_h) \vartheta_\delta(u_h) \, dx + \frac{1}{p' p^{\frac{p'}{p}} \delta^{\frac{p'}{p}}} \|w_h\|_{-1,p'}^{p'} + \delta \|\vartheta_\delta(u_h)\|_{1,p}^p \,.$$

Taking into account (1.6), by computations, we deduce that

$$\int_\Omega D_s L(x, u_h, \nabla u_h) \vartheta_\delta(u_h) \, dx \geqslant 0 \,.$$

Moreover we have as before

$$\|\vartheta_\delta(u_h)\|_{1,p}^p \leqslant \int_{\{k\sigma < |u_h| \leqslant (k+1)\sigma\}} |\nabla u_h|^p \, dx + \int_{\{|u_h| \geqslant (k+1)\sigma\}} |\nabla u_h|^p \, dx \leqslant$$

$$\leqslant \frac{1}{\nu} \int_{\{k\sigma < |u_h| \leqslant (k+1)\sigma\}} \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx +$$

$$
\begin{aligned}
&+\frac{1}{\nu}\int_{\{|u_h|\geqslant(k+1)\sigma\}}\nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx+\\
&+\frac{1}{\nu}\int_{\{k\sigma<|u_h|\leqslant(k+1)\sigma\}}a(x)\,dx+\frac{b_0}{\nu}\int_{\{k\sigma<|u_h|\leqslant(k+1)\sigma\}}|u_h|^p\,dx+\\
&+\frac{1}{\nu}\int_{\{|u_h|\geqslant(k+1)\sigma\}}a(x)\,dx+\frac{b_0}{\nu}\int_{\left\{(k+1)\sigma+\frac{\sigma}{\delta}\geqslant|u_h|\geqslant(k+1)\sigma\right\}}|u_h|^p\,dx\,,
\end{aligned}
$$

so that

$$
\begin{aligned}
&\left(1-\frac{\delta}{\nu}\right)\int_{\{k\sigma<|u_h|\leqslant(k+1)\sigma\}}\nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx\leqslant\\
&\leqslant\left(\delta+\frac{\delta}{\nu}\right)\int_{\{|u_h|>(k+1)\sigma\}}\nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx+\\
&+\int_\Omega\left(c_1+c_2\left|(k+1)\sigma+\frac{\sigma}{\delta}\right|^{\frac{n(p-1)+p}{n-p}}\right)\sigma\,dx+\frac{1}{p'p^{\frac{p'}{p}}\delta^{\frac{p'}{p}}}K_0^{p'}+\frac{2}{\nu}\int_\Omega a(x)\,dx+\\
&+\frac{b_0}{\nu}\left[(k+1)^p+\left((k+1)+\frac{1}{\delta}\right)^p\right]\sigma^p\mathcal{L}^n(\Omega)\,.
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
&\int_{\{k\sigma<|u_h|\leqslant(k+1)\sigma\}}\nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx\leqslant\\
&\leqslant\frac{\nu\delta+\delta}{\nu-\delta}\int_{\{|u_h|\geqslant(k+1)\sigma\}}\nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx+K_3(k,\delta)\,.
\end{aligned}
$$

Combining this inequality with (3.14) we conclude that

$$
\begin{aligned}
&\int_{\{|u_h|\leqslant\varrho\}}\nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx\leqslant\int_{\{|u_h|\leqslant k\sigma\}}\nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx\leqslant\\
&\leqslant K_1(k)\int_{\{k\sigma<|u_h|\leqslant(k+1)\sigma\}}\nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx+K_2(k)\leqslant\\
&\leqslant K_1(k)\frac{\nu\delta+\delta}{\nu-\delta}\int_{\{|u_h|>(k+1)\sigma\}}\nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx+K_1(k)K_3(k,\delta)+.\\
&\leqslant\varepsilon\int_{\{|u_h|>\varrho\}}\nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx+K_{\varrho,\varepsilon},
\end{aligned}
$$

where we have fixed $\delta>0$ in such a way that $K_1(k)\frac{\nu\delta+\delta}{\nu-\delta}\leqslant\varepsilon$. □

The next result is an extension of [20, Lemma 1], since (1.10) relaxes (9) of [20].

(3.15) LEMMA.— *Let $c \in \mathbb{R}$. Then each $(CPS)_c$−sequence for f is bounded in $W_0^{1,p}(\Omega)$.*

Proof.— First of all, we can find $a_0 \in L^1(\Omega)$ such that for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$

$$qG(x,s) \leqslant sg(x,s) + a_0(x)\,.$$

Now, let (u_h) be a $(CPS)_c$−sequence for f and let for all $v \in C_c^\infty(\Omega)$

$$\langle w, v\rangle = \int_\Omega \nabla_\xi L(x,u,\nabla u)\cdot\nabla v\,dx + \\ + \int_\Omega D_s L(x,u,\nabla u) v\,dx - \int_\Omega g(x,u_h) v\,dx\,. \tag{3.16}$$

According to Proposition (3.1) and Lemma (3.11), for each $\varepsilon > 0$ we have

$$-\|w_h\|_{-1,p'}\|u_h\|_{1,p} \leqslant \int_\Omega \nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx +$$

$$+\int_\Omega D_s L(x,u_h,\nabla u_h)u_h\,dx - \int_\Omega g(x,u_h)u_h\,dx \leqslant$$

$$\leqslant \int_\Omega \nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx + \int_\Omega D_s L(x,u_h,\nabla u_h)u_h\,dx +$$

$$-q\int_\Omega G(x,u_h)\,dx + \int_\Omega a_0\,dx \leqslant$$

$$\leqslant (1+\varepsilon)\int_{\{|u_h|>R'\}} \nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx + \int_\Omega D_s L(x,u_h,\nabla u_h)u_h\,dx +$$

$$-q\int_\Omega L(x,u_h,\nabla u_h)\,dx + qf(u_h) + \int_\Omega a_0\,dx + K_{R',\varepsilon}\,.$$

On the other hand, from Lemma (3.11) and (3.3), for each $\varepsilon > 0$ we obtain

$$\int_\Omega D_s L(x,u_h,\nabla u_h)u_h\,dx =$$

$$= \int_{\{|u_h|\leqslant R'\}} D_s L(x,u_h,\nabla u_h)u_h\,dx + \int_{\{|u_h|>R'\}} D_s L(x,u_h,\nabla u_h)u_h\,dx \leqslant$$

$$\leqslant \varepsilon M R' \int_{\{K_{R',\varepsilon}>|u_h|>R'\}} \nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx +$$

$$-\int_{\{|u_h|>R'\}} \nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx+q\int_\Omega L(x,u_h,\nabla u_h)\,dx+\int_\Omega a_\varepsilon(x)\,dx+$$

$$+\varepsilon\int_{\{|u_h|>R'\}}|u_h|^p\,dx-\nu\int_{\{|u_h|>R'\}}|\nabla u_h|^p\,dx+K_{R',\varepsilon}\,.$$

Taking into account Poincare and Young's inequalities, by (1.5) we find $c>0$ and $C_{R',\varepsilon}>0$ with

$$\int_\Omega D_sL(x,u_h,\nabla u_h)u_h\,dx\leqslant$$

$$\leqslant\varepsilon c\int_{\{|u_h|>R'\}}|\nabla u_h|^p\,dx-\int_{\{|u_h|>R'\}}\nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx+$$

$$+q\int_\Omega L(x,u_h,\nabla u_h)\,dx+\int_\Omega a_\varepsilon(x)\,dx-\nu\int_{\{|u_h|>R'\}}|\nabla u_h|^p\,dx+C_{R',\varepsilon}\,.$$

Therefore, for a sufficiently small $\varepsilon>0$, there exists $\vartheta_\varepsilon>0$ with

$$\vartheta_\varepsilon\int_{\{|u_h|>R'\}}|\nabla u_h|^p\,dx\leqslant\|w_h\|_{-1,p'}\|u_h\|_{1,p}+qf(u_h)+$$

$$+\int_\Omega a_0\,dx+\int_\Omega a_\varepsilon\,dx+K_{R',\varepsilon}+C_{R',\varepsilon}\,.$$

Moreover, it is

$$\int_\Omega|\nabla u_h|^p\,dx\leqslant(1+\varepsilon)\int_{\{|u_h|>R'\}}|\nabla u_h|^p\,dx+K_{R',\varepsilon}\,.$$

Since $w_h\to 0$ in $W^{-1,p'}(\Omega)$, the assertion follows. □

(3.17) LEMMA. — *Under assumptions (iv) we have*

$$\frac{\int_\Omega G(x,u_h)\,dx}{\|u_h\|_{1,p}^p}\to 0\quad\text{as } h\to+\infty\,,$$

for each (u_h) that goes to 0 in $W_0^{1,p}(\Omega)$.

Proof. — Let $(u_h)\subseteq W_0^{1,p}(\Omega)$ with $u_h\to 0$ in $W_0^{1,p}(\Omega)$. We can find $(\varrho_h)\subseteq\mathbb{R}$ and a sequence $(w_h)\subseteq W_0^{1,p}(\Omega)$ such that $u_h=\varrho_h w_h$, $\varrho_h\to 0$ and $\|w_h\|_{1,p}=1$. Taking into account (1.8), it follows

$$\lim_h\frac{G(x,u_h(x))}{\|u_h\|_{1,p}^p}=0\quad\text{for a.e. } x\in\Omega.$$

Moreover, for a.e. $x \in \Omega$ we have

$$\frac{G(x, u_h(x))}{\|u_h\|_{1,p}^p} \leqslant d|w_h|^p + b\varrho_h^{\frac{p^2}{n-p}} |w_h|^{p^*}.$$

If w is the weak limit of (w_h), since $d|w_h|^p \to d|w|^p$ in $L^1(\Omega)$ and $b\varrho_h^{\frac{p^2}{n-p}} |w_h|^{p^*} \to 0$ in $L^1(\Omega)$, (a variant of) Lebesgue's Theorem concludes the proof. □

We finally conclude with the proof of the main result of this paper.

Proof of Theorem (1.11). From Lemma (3.15) and Theorem (3.4) it follows that f satisfies the $(CPS)_c$ condition for each $c \in \mathbb{R}$. By (1.3) and (1.9) it easily follows that

$$\forall u \in W_0^{1,p}(\Omega)\backslash\{0\}: \lim_{t \to +\infty} f(tu) = -\infty.$$

Finally from Lemma (3.17) and (1.3) we deduce that 0 is a strict local minimum for f. From Theorem (2.7) the assertion follows.

(3.18) REMARK. — *In this paper we only consider existence of weak solutions to* (1.1). *However, as proved in [4, Lemma 1.4], each weak solution of* (1.1) *belongs to* $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ *provided that L and g satisfy suitable conditions. Then, some nice regularity results hold for various classes of integrands L. (see [18]).*

(3.19) ACKNOWLEDGMENTS. — The author wishes to thank Marco Degiovanni for providing helpful discussions.

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