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Traveling front solutions for a diffusive epidemic model with external sources (*)

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RÉSUMÉ. — Dans ce travail, on s'intéresse à l'étude d'un système de deux équations différentielles non autonomes du second ordre à valeurs propres et dépendant de paramètres. Ce système concerne les solutions de type ondes progressives pour un système de réaction-diffusion. Ce dernier décrit la propagation d'une maladie infectieuse au sein d'une population confinée dans une région donnée; le modèle étudié tient en compte la présence de sources extérieures possibles. Selon l'importance relative de celles-ci, nous posons la question de l'existence d'ondes progressives au système de réaction-diffusion correspondant. Nous donnons ensuite une revue de toutes les situations rencontrées pour lesquelles nous montrons comment les difficultées mathématiques peuvent être surmontées. Utilisant des méthodes topologiques de type Leray-Schauder, nous prouvons l'existence de solutions fronts d'ondes pour toute vitesse de l'onde de propagation.

ABSTRACT. — This work is devoted to the study of parameters dependent system of two second order non-autonomous ODE's with eigenvalues. The system is concerned with the traveling wave solutions to a reaction diffusion system. The latter describes the propagation of an infectious disease within a population confined in a given region; the model to be dealt with takes into account the presence of possible external sources. According to the relative importance of these ones, we address the question of existence of traveling waves to the corresponding reaction-diffusion system. Then, we give a survey of all encountered situations for which we show how the mathematical difficulties may be overcome. Using some topological methods of Leray and Schauder degree type, we prove existence of traveling front solutions for any wave speed.

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1. Introduction

In this work, we aim to discuss the existence of traveling wave solutions to a reaction-diffusion system; the latter arises as a model from epidemiology the kinetic of which is given by the so-called Kermack-McKendrick model [9]. We consider a population living in a region $\Omega \subset \mathbb{R}^n$ and consisting of two components, one of infectives and the other of susceptibles who are capable of becoming infected. The spread of an infectious disease through this population is governed by the following reaction-diffusion equations

$$\begin{cases} u_t - \nabla.\big(a(u)\nabla u\big) = -\lambda u + vg(u) + q_1(x,t) \\ v_t - \nabla.\big(b(v)\nabla v\big) = -\mu v - vg(u) + q_2(x,t), \end{cases} \tag{1.1}$$

with the initial conditions

$$u(x,0) = u_0(x); \quad v(x,0) = v_0(x), \tag{1.2}$$

and the Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \qquad \text{on } \partial\Omega, \tag{1.3}$$

where $x \in \Omega$, $t > 0$. The unknowns $u(x,t)$ and $v(x,t)$ denote the spatial densities of infectives and susceptibles respectively; the positive functions a and b refer to the diffusion coefficients while λ and μ are the removal rate constants; q_1 and q_2 stand for possible external sources. λ, μ, q_1 and q_2 are assumed non-negative. The nonlinearity g is supposed to be a regular positive function; it represents the interaction term or the force of infection. The special case $g(s) = s$, $q_1 = q_2 \equiv 0$ corresponds to the Kermack-McKendrick model [9]; more details on the biological background can be found in [2], [4] and [12]. We refer the reader to [5], [13] and [14] for the mathematical investigation of problem (1.1)-(1.3).

For the one-dimensional case, we remind that traveling epidemic waves of constant shape are particular solutions of (1.1) having the form

$$u(x,t) = \hat{u}(x+ct) \text{ and } v(x,t) = \hat{v}(x+ct); \tag{1.4}$$

here, we have assumed a wave to propagate from right to left (see [1], [7] for more generalities on traveling waves to reaction-diffusion systems). Then, a pulse of infectives $\hat{u}$ moves into a population of susceptibles $\hat{v}$ with speed $c > 0$. If we substitute (1.4) into (1.1), we get a system of differential equations with respect to the stretched variable $\xi = x + ct$; q_1 and q_2 being replaced by $\hat{q}_1(x+ct) = q_1(x,t)$ and $\hat{q}_2(x+ct) = q_2(x,t)$ respectively (external sources moving with speed c). For constant diffusion coefficients, let us write down, when $a = b = 1$, the derived equations, in which we have dropped the hats while $'$ refers to the derivative $\frac{d}{d\xi}$,

$$\begin{cases} -u'' + cu' + \lambda u = vg(u) + q_1 & (1.5a) \\ -v'' + cv' + \mu v = -vg(u) + q_2. & (1.5b) \end{cases} \tag{1.5}$$

The restrictions $a = b = 1$ are only imposed for simplicity; the results obtained in this paper are easily extended to the case $a(s) \equiv a > 0$, $b(s) \equiv b > 0$. Mathematically, by traveling wave solutions to system (1.1), we understand nontrivial solutions to system (1.5) lying in the positive cone

$$\mathcal{C}^+ = \{(u, v) \in [BC^1]^2;\ u, v \geqslant 0 \text{ on } \mathbb{R}\},$$

that is classical solutions biologically meaningful. Recall that $BC^k = BC^k(\mathbb{R})$ denotes, as usual, the space of all functions $u(.)$ with derivatives $u^{(j)}(.)$ bounded and continuous on $\mathbb{R}$ for $j = 0, 1, ..., k$.

In the sequel, we are interested in the question of existence of such solutions even when one of the parameters λ, μ or the functions q_1, q_2 is identically zero. In each case, suitable boundary conditions will be associated with system (1.5); in fact, they may often be derived from it.

The organization of the paper is as follows. In Section 2, we first make some assumptions on the functions g and q_1, q_2; next, we classify four basic types of systems related to system (1.5) and then infer the boundary conditions to be associated with. Section 3 is concerned with the discussion of those distinct types. Existence results as well as methods of solvability are presented. Sections 4-6 are devoted to the proofs of the existence theorems.

The following preliminary lemma, the proof of which is rather classical, will be useful in the sequel.

LEMMA 1.1.— (a) *Let $f\colon \mathbb{R}^+ \to \mathbb{R}$ be a uniformly continuous function such that $\int^\infty f(x)\,dx$ converges. Then* $\lim_{x\to+\infty} f(x) = 0$.

(b) *Let $f\colon \mathbb{R}^+ \to \mathbb{R}$ be a C^2 class function such that $f'' \in L^\infty(\mathbb{R}^+)$ and $\lim_{x\to+\infty} f(x)$ exists. Then* $\lim_{x\to+\infty} f'(x) = 0$.

2. Assumptions and settings of the problems

Given $c > 0$, let us assume λ, $\mu \geqslant 0$; they may be viewed as eigenvalues of system (1.5). The nonlinear term $g\colon \mathbb{R} \to \mathbb{R}$ as well as the functions q_1, q_2: $\mathbb{R} \to \mathbb{R}^+$ are continuous and satisfy global Lipschitz condition. The relevant assumptions read

$$q_i \in L^1(\mathbb{R}), \quad i = 1, 2. \tag{2.1}$$

$$\begin{cases} g(\mathbb{R}_*^+) \subset \mathbb{R}_*^+; \\ g(s) = -g(-s), \ \forall\, s \leqslant 0. \end{cases} \tag{2.2}$$

Owing to (2.1) and Lemma 1.1, the functions q_i $(i = 1, 2)$ necessarily satisfy $q_i \in L^\infty(\mathbb{R})$ and $\lim_{|x|\to+\infty} q_i(x) = 0$.

The notations used herein are mostly standard. The norm of the Lebesgue space $L^p(\mathbb{R})$ is defined in the usual way and is denoted by $|.|_p$ $(1 \leqslant p \leqslant \infty)$ while the norm in $W^{r,\infty}(\mathbb{R})$ is denoted by $\|.\|_{r,\infty}$. We will shorten these spaces to L^p and $W^{r,\infty}$ respectively.

In regard to system (1.5), we deal basically with four types of distinct systems which we classify as follows:

Type I:	$q_1 = q_2 \equiv 0$.
Type II:	q_1 and $q_2 \not\equiv 0$.
Type III:	$q_1 \not\equiv 0$ and $q_2 \equiv 0$.
Type IV:	$q_1 \equiv 0$ and $q_2 \not\equiv 0$.

When λ or/and μ equals to zero, corresponding sub-cases will be also taken into consideration. Even if, in the biological context, λ and μ are known to be positive, the cases $\lambda = 0$ or $\mu = 0$ appear to be of substantial mathematical interest. Furthermore, we will show that each type gives rise to some specific difficulties. In order to set suitable boundary conditions, let us primarily mention and prove the following

LEMMA 2.1.— *Let $(u, v) \in \mathcal{C}^+$ be a traveling wave solution to system (1.1). Then*

(a) λ *(resp.* μ*)* $> 0 \Rightarrow \lim_{|x|\to+\infty} u(x) = 0$ *(resp.* $\lim_{|x|\to+\infty} v(x) = 0$*).*

(b) $\lambda = 0$ *(resp.* $q_2 \equiv 0$*)* $\Rightarrow u$ *(resp.* v*) increasing (resp. decreasing).*

(c) $\lim_{|x|\to+\infty} u^{(k)}(x) = \lim_{|x|\to+\infty} v^{(k)}(x) = 0$, *for* $k = 1, 2$.

Proof.— (a) Set $y := u + v$ and $q := q_1 + q_2$; then, adding (1.5a) to (1.5b), and integrating the resulting equation from 0 to x, we get:

$$\lambda \int_0^x u(t)dt + \mu \int_0^x v(t)dt = \int_0^x q(t)dt + \theta(x), \tag{2.3}$$

where $\theta(x) := (y'-cy)(x)-(y'-cy)(0)$. Since $\theta \in W^{1,\infty}$ and u, v are positive, both integrals in the left-hand side of (2.3) are bounded. Moreover, they are increasing for $x > 0$ (resp. decreasing for $x < 0$), and hence convergent. Then, Lemma 1.1, (a) completes the first claim of the lemma.

(b) Assume $\lambda = 0$; Equation (1.5a) may be written as $-\big(u'(x)e^{-cx}\big)' = \big(vg(u) + q_1\big)(x)e^{-cx} > 0$ over $\mathbb{R}$. Integrating backwards from $+\infty$ and noting that $u' \in L^\infty$, we get $u' > 0$ over $\mathbb{R}$; the case $q_2 \equiv 0$ is treated in a similar manner.

(c) • First, assume λ and μ positive. As u'', $v'' \in L^\infty$, part (a) both with Lemma 1.1, (b) imply $\lim\limits_{|x|\to+\infty} u'(x) = \lim\limits_{|x|\to+\infty} v'(x) = 0$.

• When $\lambda = 0$, u is bounded and monotone increasing by part (b); then $\lim\limits_{|x|\to+\infty} u(x)$ exist. Now, either $\mu > 0$ and so $\lim\limits_{|x|\to+\infty} v(x) = \lim\limits_{|x|\to+\infty} v'(x) = 0$ or $\mu = 0$ and then adding the two equations in (1.5) yields $-y''+cy' = q \geqslant 0$. Thanks to Maximum Principal, y is increasing, hence admits some limits at infinity; thus, the existence of $\lim\limits_{|x|\to+\infty} v(x)$ follows immediately and, in turn, implies $\lim\limits_{|x|\to+\infty} v'(x) = 0$.

• It remains to consider the case in which $\lambda > 0$ and $\mu = 0$; here $\lim\limits_{|x|\to+\infty} u(x) = \lim\limits_{|x|\to+\infty} u'(x) = 0$. Integrating between 0 and x the equation $-y'' + cy' + \lambda u = q$, we find that $(-y' + cy)$ converges, as $x \to +\infty$, and so does $(-v' + cv)$. Now multiply Equation (1.5b) by v and integrate again from 0 to $x > 0$; we get the following identity:

$$\int_0^x |v'(t)|^2 dt - v'(x)v(x) + \frac{c}{2}|v(x)|^2 = \frac{c}{2}|v(0)|^2 - v'(0)v(0) - \int_0^x v^2(t)g(u(t))dt + \int_0^x v(t)q_2(t)\, dt.$$

Whence,

$$0 < \int_0^x |v'(t)|^2 dt < -v(0)v'(0) + \frac{c}{2}|v(0)|^2 + |v'|_\infty |v|_\infty + |v|_\infty \int_0^{+\infty} q_2(t)dt;$$

therefore the integral $\int_0^x |v'(t)|^2 dt$ converges. Then, Lemma 1.1, part (a) still applies and shows that $\lim\limits_{x\to+\infty} |v'(x)|^2 = 0$; since $(-v' + cv)$ converges, $\lim\limits_{x\to+\infty} v(x)$ exist; the same holds at $-\infty$.

At last, the vanishing values of the second derivatives follow from system (1.5) itself. The proof of Lemma 2.1 is thereby completed.

Remark 2.2. — As shown in Lemma 2.1, the behaviour of the solutions at infinity is rather related to the positions of the parameters λ and μ with respect to 0. Particularly, when $\lambda = \mu = 0$ the following conservation relation, in which $q := q_1 + q_2$, holds true:

$$(u+v)(+\infty) = (u+v)(-\infty) + \frac{1}{c}|q|_1. \tag{2.4}$$

Now, u is monotone increasing, then $v(+\infty) = 0$ follows as a consequence of $vg(u)(+\infty) = 0$ both with (2.2); in addition, if $v(-\infty) = v_-$ for some non-negative constant v_-, then (2.4) yields $u(+\infty) = u(-\infty) + v_- + \frac{1}{c}|q|_1$. In addition, when v_- is positive (this holds in particular if $q_2 \equiv 0$), then $u(-\infty) = 0$ so that $u(+\infty) = v_- + \frac{1}{c}|q|_1$.

However, in general, we will only set $0 \leqslant v_+ = v(+\infty) < v(-\infty) = v_-$ whenever $q_2 \equiv 0$ and $0 \leqslant u_- = u(-\infty) < u(+\infty) = u_+$ in case $\lambda = 0$.

3. Presentation of the results and description of the proofs

3.1. Type I Problems ($q_1 = q_2 \equiv 0$)

The study of the autonomous system

$$\begin{cases} -u'' + cu' + \lambda u = vg(u) \\ -v'' + cv' + \mu v = -vg(u). \end{cases} \tag{3.1}$$

is available in the literature. So, we briefly review the main known results.

3.1.1. *Sub-case $\mu > 0$, $\lambda \geqslant 0$*

From Lemma 2.1, $v(\pm\infty) = 0$ while v is decreasing which is absurd; this case is thus of no use whatever.

3.1.2. *Sub-case $\lambda = \mu = 0$*

System (3.1) supplemented by the following boundary conditions

$$\begin{cases} u(-\infty) = v(+\infty) = 0 \\ v(-\infty) = u(+\infty) = \beta, \end{cases} \tag{3.2}$$

with some $\beta > 0$, reduces to the well known Fisher problem. Concerning this case, it holds

THEOREM 3.1. ([11], [15]). — *Assume (2.2) and g is increasing; then, there exits $c_0 > 0$ such that problem (3.1)-(3.2) has a solution $(u, v) \in C^+$*

if and only if $c \geqslant c_0$. *Moreover,* u *(resp.* $v = \beta - u$*) is increasing (resp. decreasing).*

3.1.3. *Sub-case* $\lambda > 0$, $\mu = 0$

In [6], the author studied a similar system with a slightly different reaction term having the form $uvh(v)$ instead of $vg(u)$. Using a topological method, an existence result has been proved assuming $\lim_{s \to 0} sh(s) = 0$, the function $s \hookrightarrow sh(s)$ is non decreasing and that it holds:

$$\exists\, k_0 > 0, \quad \frac{1}{x^3} \int_0^x s(x-s)h(x-s)ds \geqslant k_0, \quad \forall\, x > 0.$$

Here, the reaction term $vg(u)$ is linear in terms of v; the first two hypotheses are then obvious; however, following [6], we may require, instead of the third assumption, the following one

$$\exists\, k_0 > 0, \quad \frac{1}{x^3} \int_0^x (x-s)g(s)ds \geqslant k_0, \quad \forall\, x > 0. \tag{3.3}$$

Then, we can prove again the following existence result

THEOREM 3.2. — *For any* $\beta > 0$, *there exists* $0 < \underline{c} < \overline{c}$ *such that, for any* $c \in]\underline{c}, \overline{c}]$, *system (3.1) has a solution* $(u, v, \lambda) \in \mathcal{C}^+ \times]0, +\infty[$ *satisfying* $u(\pm\infty) = 0$ *and* $0 \leqslant v(+\infty) < v(-\infty) \leqslant \beta$.

We omit here the details of the proof. Note that in the model case $g(s) \equiv s$, equality holds in (3.3) with $k_0 = \frac{1}{6}$. For this particular case, Hosono & Ilyas proved, by a shooting type argument, the following

THEOREM 3.3. ([8]). — *For any* $\beta > 0$, *there exists* $\overline{\lambda} = \overline{\lambda}(\beta) > 0$ *and* $\alpha \in [0, \beta[$ *such that* $\forall \lambda \in]0, \overline{\lambda}[$, $\exists\, \underline{c} = \underline{c}(\lambda) > 0$ *such that system (3.1) has a solution* $(u, v) \in [C^1(\mathbb{R})]^2$ *satisfying* $u(\pm\infty) = 0$, $v(-\infty) = \beta$ *and* $v(+\infty) = \alpha$ *for any* $c \geqslant \underline{c}$.

3.2. Type II Problems (q_1, $q_2 \not\equiv 0$)

We will prove

THEOREM 3.4. — *Assume (2.1)-(2.2) hold true. Then, for any* $c > 0$ *and* λ, $\mu \geqslant 0$, *system (1.5) admits at least one solution* $(u, v) \in \mathcal{C}^+$ *subject to one of the following boundary conditions:*

- *Case* λ, $\mu > 0$: $u(\pm\infty) = v(\pm\infty) = 0$.

• *Case* $\lambda > 0$, $\mu = 0$: $u(\pm\infty) = 0$; $v(-\infty) = v_-$, $v(+\infty) = v_+$ *for some* $0 \leqslant v_-, v_+$.

• *Case* $\lambda = 0$, $\mu > 0$: $u(-\infty) = u_-$, $u(+\infty) = u_+$; $v(\pm\infty) = 0$ *for some* $0 \leqslant u_- < u_+$.

• *Case* $\lambda = \mu = 0$: $u(-\infty) = u_-$, $u(+\infty) = u_+$; $v(-\infty) = u_+ - u_- - (1/c)|q|_1$, $v(+\infty) = 0$ *for some* $0 \leqslant u_- < u_+$.

It is to point out that Type II problems are easier to solve than Type III or IV ones; the reason is that trivial solutions are ruled out by the presence of q_1 and q_2. Here is a sketch of the proof. Considering the problem in a bounded domain of the real line, the problem is then reduced to a fixed point formulation. The use of classical properties of Leray-Schauder degree [10] leads to an existence result. Subsequently, a priori estimates allow us to pass to the limit on the size of the domain and get a solution defined on the whole real line; to this end, we only need to invoke the compactness of the embedding $H^2_{loc}(\mathbb{R}) \hookrightarrow C^1_{loc}(\mathbb{R})$.

3.3. Type III and IV Problems

The method described above to prove Theorem 3.4 will be used here again. However, when $q_1 \equiv 0$ (resp. $q_2 \equiv 0$), the function $u \equiv 0$ (resp. $v \equiv 0$) is a trivial solution. So, one may expect this trivial situation to arise when passing to the limit from a bounded domain to the full real line; to avoid such a situation, we may resort to an additional prescribed condition, say $u(0) = \gamma$ or $v(0) = \eta$ with some γ, $\eta > 0$, at the cost of introducing some additional fictitious unknown, say ν or of considering another unknown among the parameters c, λ or μ. Thus, the latter may be regarded as an eigenvalue of the problem and therefore must be estimated; such estimates are not always easy to be performed; in the simple case $q_1 = q_2 = \lambda = \mu = 0$, estimates of c has been undertaken successfully in [3], [11]. As for the case $\mu = q_1 \equiv q_2 \equiv 0$, estimating the eigenvalue $\lambda > 0$ is not much more difficult [6]. Anyway, we will, even so, prove the following results, under the sole assumptions (2.1)-(2.2):

THEOREM 3.5.— *($q_2 \equiv 0$; $q_1 \not\equiv 0$, $\lambda \geqslant 0$, $\mu = 0$). For any $c > 0$, there exists a solution $(u, v) \in \mathcal{C}^+$ to system (1.5) satisfying one of the boundary conditions:*

• *Case* $\lambda = 0$: $u(-\infty) = v(+\infty) = 0$; $v(-\infty) = v_-$, $u(+\infty) = u_+$ *for some* $0 < u_+, v_-$.

• *Case* $\lambda > 0$: $u(\pm\infty) = 0$; $v(-\infty) = v_-$, $v(+\infty) = v_+$ *for some* $0 \leqslant v_+ < v_-$.

THEOREM 3.6.— *($q_1 \equiv 0$; $q_2 \not\equiv 0$, $\lambda \geqslant 0$, $\mu \geqslant 0$). For any $c > 0$, there exists a solution $(u, v) \in \mathcal{C}^+$ to system (1.5) satisfying one of the boundary conditions:*

- *Case* $\lambda,\ \mu > 0 : u(\pm\infty) = v(\pm\infty) = 0$.

- *Case* $\lambda = 0,\ \mu > 0 : u(-\infty) = u_-,\ u(+\infty) = u_+;\ v(\pm\infty) = 0$ *for some* $0 \leqslant u_- < u_+$.

- *Case* $\lambda > 0,\ \mu = 0 : u(\pm\infty) = 0;\ v(-\infty) = v_-,\ v(+\infty) = v_+$ *for some* $0 < v_-, v_+$.

- *Case* $\lambda = \mu = 0 : u(-\infty) = u_-,\ u(+\infty) = u_+; v(-\infty) = v_-,\ v(+\infty) = 0$ *for some* $0 < u_-, v_-, v_+$ *with* $u_- v_- = 0$.

Remark 3.7.— (a) The techniques used to prove Theorems 3.5 and 3.6 are slightly different.

(b) As indicated in Section 3.1.1, the conditions $q_2 \equiv 0$ and $\mu > 0$ are incompatible.

(c) Contrary to Theorems 3.1-3.3, we can observe that the existence of traveling waves is obtained with no condition prescribed upon the wave speed; this is due to the presence of external sources in system 1.1.

4. Proof of theorem 3.4 ($q_1, q_2 \not\equiv 0, \lambda, \mu \geqslant 0$)

4.1. The general framework

Given some positive real parameter $a > 0$, we first intend to solve system (1.5) in the open interval $I_a :=]-a, +a[$. The setting is as used in [3], [6] and [11]. Let $C^k(\overline{I}_a)$ be the space of functions whose k^{th} derivatives are continuous on $[-a, +a]$ and consider the Banach space $X = [C^1(\overline{I}_a)]^2$, both endowed with the supremum norm. Let λ, μ be non-negative constants. For any $t \in [0, 1]$, define the mapping K_t: $X \longrightarrow X$ which sends each element $(u, v) \in X$ onto the element (U, V) where (U, V) is the unique solution to the following linear boundary value problem

$$\begin{cases} -U'' + cU' + \lambda U = tv|g(u)| + q_1 \\ -V'' + cV' + \mu V = -tv|g(u)| + q_2. \end{cases} \tag{4.1}$$

$$\begin{cases} U'(-a) = r_2 U(-a), \quad V'(-a) = s_2 V(-a); \\ U(a) = V(a) = 0. \end{cases} \tag{4.2}$$

Here $r_1 \leqslant 0 < r_2$ (resp. $s_1 \leqslant 0 < s_2$) are the roots of the characteristic equation $r^2 - cr - \lambda = 0$ (resp. $s^2 - cs - \mu = 0$). Searching $(u, v) \in X$ solution to system (1.5) is then equivalent to finding a fixed point to the mapping K_1. For this purpose, we begin by defining a Leray-Schauder topological degree with respect to an open set Ω, viz. $\deg(I - K_t, \Omega, 0)$; when the latter is nonzero for some t, it will be so, by Invariance Property by Homotopy of the degree, for any $t \in [0,1]$; this is what we intend to do when $t = 0$. On behalf of Non-vanishing Property of the latter [10], we infer the existence of a sought solution.

Remark 4.1. — The boundary conditions (4.2) are purely technical. When $\lambda = 0$, one may expect an increasing solution u on the whole real line; so, the homogeneous Dirichlet condition $u(a) = 0$ seems to be unrealistic; in fact this condition helps to estimate the sought solutions without altering the sign of u' on $\mathbb{R}$ for the convergence, as $a \to +\infty$, is only of local type. Likewise, mixed conditions at $-a$ are not related to those expected to be obtained at $-\infty$.

4.2. Definition of a topological degree

First, let us start with

LEMMA 4.2. — *Let $(u, v) \in X$ be a fixed point of the mapping K_t and set $q := q_1 + q_2$, $\kappa := |q|_\infty + \frac{|\lambda-\mu|}{c}|q_2|_1$. Then, for any $x \in I_a$, the following pointwise estimates hold true:*

(a) $0 < u(x), v(x)$.

(b) $v(x) < \frac{|q_2|_1}{c}$; $-|q_2|_1 < v'(x) < 2|q_2|_1$.

(c) *Case* $\lambda > 0:$ $u(x) < \frac{\kappa}{\lambda} + \frac{|r_2 - s_2||q_2|_1}{c(r_2 - r_1)}$;

$$|u'(x)| < 2\frac{\kappa}{-r_1} + 2|q_2|_1 + 2|r_2 - s_2|\frac{|q_2|_1}{c}.$$

(d) *Case* $\lambda = 0:$ $u(x) < \frac{|q|_1}{c}$; $-2|q_2|_1 - |q|_1 < u'(x) < |q|_1$.

Proof. — (a) The positiveness of v is an easy consequence of Strong Maximum Principle for

$$\begin{cases} -v'' + cv' + \big(\mu + t|g(u)|\big)v \geqslant 0; \\ v(-a)v'(-a) > 0, \quad v(a) = 0. \end{cases}$$

Once v is positive, so is u for the same reason. One can note the introduction of the absolute value which will disappear as a result of the positiveness of u both with assumption (2.2).

(b) Starting from the inequality

$$-v'' + cv' \leqslant q_2, \tag{4.3}$$

we make an integration from $-a$ to x after observing that $(-v' + cv)(-a) = (c - s_2)v(-a) < 0$; we get

$$-v'(x) + cv(x) < \int_{-a}^{x} q_2(t)\, dt; \tag{4.4}$$

hence $v'(x) > -\int_{-a}^{x} q_2(t)\, dt > -|q_2|_1$. On the other hand, $v'(a) < 0$ since $v(a) = 0$ and $v > 0$ on I_a. Then, integrating (4.3) from x to a yields the following estimate

$$v'(x) - cv(x) < \int_{x}^{a} q_2(t)\, dt < |q_2|_1. \tag{4.5}$$

Now, (4.4) may be written as

$$-\left(v(x)e^{-cx}\right)' < e^{-cx} \int_{-a}^{x} q_2(t)\, dt,$$

which we integrate again from x to a to get

$$0 < v(x) < e^{cx} \int_{x}^{a} e^{-ct}\, dt \int_{-a}^{t} q_2(s)\, ds \leqslant \frac{|q_2|_1}{c}. \tag{4.6}$$

Inserting this in (4.5) proves part (b) of the lemma.

(c) Set $\varphi(s) := q(s) + (\lambda - \mu)v(s)$; then the function $y := u + v$ satisfies the equation $-y'' + cy' + \lambda y = \varphi$ which is explicitly solvable:

$$y(x) = e^{r_2 x} \int_{x}^{a} e^{(r_1 - r_2)t} dt \int_{-a}^{t} \varphi(s) e^{-r_1 s}\, ds + \theta(x), \tag{4.7}$$

$$y'(x) = r_2 y(x) - e^{r_1 x} \int_{-a}^{x} \varphi(s) e^{-r_1 s}\, ds + \theta'(x); \tag{4.8}$$

with

$$\theta(x) := \frac{(r_2 - s_2)v(-a)e^{r_1 a}}{r_2 - r_1} e^{r_2 x}\left[e^{(r_1 - r_2)x} - e^{(r_1 - r_2)a}\right].$$

Assume $\lambda > 0$; using (4.6), (4.7), it is easy to check that, for any $x \in I_a$, $0 < y(x) < \frac{|\varphi|_\infty}{\lambda} + \frac{|r_2 - s_2|}{r_2 - r_1}\frac{|q_2|_1}{c}$. We infer

$$0 < u(x) < \frac{|q|_\infty}{\lambda} + \frac{|\lambda - \mu|}{\lambda c}|q_2|_1 + \frac{|r_2 - s_2||q_2|_1}{c(r_2 - r_1)}.$$

Turning back to (4.8) and noting that $0 < e^{r_1 x} \int_{-a}^{x} |\varphi|(s)e^{-r_1 s}\, ds < \frac{|\varphi|_\infty}{-r_1}$, we deduce

$$-\frac{|r_2 - s_2||q_2|_1}{c} + \frac{|\varphi|_\infty}{r_1} < y'(x) < \frac{2|\varphi|_\infty}{-r_1} + \frac{2|r_2 - s_2||q_2|_1}{c}.$$

Estimating u' then follows from part (b); the case $\lambda = 0$ is treated as in (b) since $-y'' + cy' \leqslant q$. The proof of the lemma is now complete.

COROLLARY 4.3.— *Let I be the identity operator on X. Then, there exists an open set $\Omega \subset X$ such that, for any $t \in [0,1]$, the Leray-Schauder topological degree* $\deg(I - K_t, \Omega, 0)$ *is well defined.*

Proof.— From Lemma 4.2, we have got some positive constant M not depending on t and a such that $\|u\|_{2,\infty}, \|v\|_{2,\infty} \leqslant M$ whenever $(u, v) \in X$ is a fixed point of the mapping K_t. Then, take Ω the open ball of radius $R = M + 1$ in X. It is not difficult to show that K_t is uniformly continuous with respect to t; in addition, it is compact by the compactness of the embedding $W^{2,\infty}(I_a) \hookrightarrow C^1(\overline{I}_a)$; therefore, our claim follows.

4.3. Computation of the degree

We have

PROPOSITION 4.4.— $\forall t \in [0,1]$, $\deg(I - K_t, \Omega, 0) = +1$.

Proof.— As mentioned in Sub-section 4.1, it is enough to prove the formula for $t = 0$. In this case, an explicit computation yields

$I - K_0\colon X \to X\colon (u, v) \mapsto (u - U_0, v - V_0)$ with

$$U_0(x) = e^{r_2 x} \int_x^a e^{(r_1 - r_2)t} dt \int_{-a}^{t} q_1(s) e^{-r_1 s}\, ds,$$
$$V_0(x) = e^{s_2 x} \int_x^a e^{(s_1 - s_2)t} dt \int_{-a}^{t} q_2(\xi) e^{-s_1 \xi}\, d\xi.$$

The claim of the proposition is then a consequence of Multiplicative Property of the degree since $\deg(I, \Omega, 0) = +1$.

4.4. Passage to the limit $a = +\infty$

Thanks to Proposition 4.4, K_1 admits at least one fixed point $(u_a, v_a) \in X$, which we extend to the whole real line by setting $\tilde{u}_a(x) = u_a(-a)$, $\tilde{v}_a(x) = v_a(-a)$, $\forall x \leqslant -a$ and $\tilde{u}_a(x) = \tilde{v}_a(x) = 0$, $\forall x \geqslant a$. System (4.1) both with Corollary 4.3 show that $\tilde{u}_a$, $\tilde{v}_a$ are bounded independently of a in $W^{2,\infty}_{loc}(\mathbb{R})$;

the latter is compactly embedded in $C^1_{loc}(\mathbb{R})$. As a consequence, there is a sequence $(a_n) \nearrow^{\infty}_{n\infty}$ such that, as $n \to +\infty$, $(\tilde{u}_{a_n}, \tilde{v}_{a_n})$ converges, in the topology of $\left[C^1_{loc}(\mathbb{R})\right]^2$ to (u, v) solution to system (3.1), ending the proof of Theorem 3.4.

5. Proof of theorem 3.5 ($q_2 \equiv 0$; $q_1 \not\equiv 0$, $\lambda \geqslant 0$, $\mu = 0$)

5.1. Preliminaries

The proof runs parallel to the one of Theorem 3.4; nevertheless, we will introduce a new unknown and get the degree computed differently. The function q_1 as well as the parameters $\lambda \geqslant 0$ and $c > 0$ being given, consider two real constants $0 < \eta < \beta$ and define the function $f_1 : \mathbb{R} \to \mathbb{R}$ by

$$f_1(s) = \begin{cases} 0, & s \leqslant \frac{\eta+\beta}{2} \\ 2s - (\eta + \beta), & s \geqslant \frac{\eta+\beta}{2}. \end{cases} \tag{5.1}$$

On a bounded open interval $I_a =]-a, +a[\ (a > 0)$, we define, as usual, a family of linear mappings K_t, $(t \in [0,1])$, on the Banach space $X := [C^1(\overline{I}_a)]^2 \times \mathbb{R}$ such that $K_t(u, v, \nu) = (U, V, v(0) + \nu - \eta - f_1(\nu))$ and (U, V) is the solution to the system

$$\begin{cases} -U'' + cU' + \lambda U = tv|g(u)| + q_1 + (1-t)\chi_1 \mathrm{sgn}(v) \\ -V'' + cV' = -tv|g(u)| - (1-t)\chi_1\ \mathrm{sgn}(v). \end{cases} \tag{5.2}$$

where $\chi_1 \in L^1(\mathbb{R})$ is a positive, Lipschitz continuous auxiliary function to be chosen later on. With (5.2), appropriate boundary conditions are

$$\begin{cases} U'(-a) = r_2 U(-a), \quad V'(-a) = c[V(-a) - \nu]; \\ \qquad U(a) = V(a) = 0. \end{cases} \tag{5.3}$$

Our purpose is to find a fixed point (u, v, ν) for the mapping K_1. As undertaken in Section 4, we define and compute a topological degree of Leray and Schauder type. Since, as in Corollary 4.3, K_t is compact and uniformly continuous with respect to the parameter t, it remains to find an open set containing all desirable solutions and such that exactly one solution exists for $t = 0$.

Remark 5.1. — The parameterization used in (5.2)-(5.3) is not standard; that used in Section 4 will be ineffective in the sequel. Here, one can note the introduction of the fictitious unknown ν which need to be bounded. The definition (5.1) will play a key role in its estimate. Moreover, its presence justifies the condition $v(0) = \eta + f_1(\nu)$ for any fixed point which allows us to avoid the trivial solution $v \equiv 0$ when passing to the limit as $a \to +\infty$.

5.2. A priori bounds

First, begin with

LEMMA 5.2.— *Let $(u, v, \nu) \in X$ $(\nu \geqslant 0)$ be a fixed point of the mapping K_t. Then, for any $x \in I_a$, it holds*

(a) $\eta < \nu < \beta$.

(b) $u(x)$, $v(x) > 0$.

(c) $v'(x) < 0$, $v(x) < \beta$, $v'(x) > -c\beta$.

(d) $u(x) < \beta + \frac{|q_1|_1}{c}$; $|u'(x)| < c\beta + |q_1|_1$.

(e) $0 < u(x) + v(x) < \beta + \frac{|q_1|_1}{c}$.

Proof.— It mimics that of Lemma 4.2. Assuming $\nu \geqslant 0$, we can easily check that $v \geqslant 0$; hence, from Maximum Principle both with left boundary condition in (5.3), we infer $\nu > 0$; then we just observe that $v(a) = 0$ and $v > 0$ on I_a imply $v'(a) < 0$ so that an integration over (x, a) of the inequality $(v'(x)e^{-cx})' \geqslant 0$ yields $v'(x) < 0$, $\forall x \in I_a$; in particular $v'(-a) < 0$. Then $v(0) < v(-a) < \nu$ follows, that is $f_1(\nu) < \nu - \eta$. Definition (5.1) yields $\eta < \nu < \beta$ (Fig. 1) so that $0 < v(x) < v(-a) < \beta$ for any $x \in I_a$. In addition, the function $v' - cv$ is increasing; then $v'(x) - cv(x) > -c\nu > -c\beta$ so that $v'(x) > -c\beta$ and $v(x) < \nu < \beta$; whence part (c). Let $y := u + v$; an upper bound for u' is achieved by noticing that $-y'' + cy'(x) \leqslant q_1(x)$; then, using e^{-cx} as an integrand factor and integrating from x to a yields $y'(x) < |q_1|_1$ $\forall x \in I_a$; the same inequality integrated over $-a$ to x implies $-y' + cy < |q_1|_1 + c\beta$, and so $y'(x) > -|q_1|_1 - c\beta$, for any $x \in I_a$; on the other hand, multiplying by e^{-cx} and integrating again from x to a yields part (e) of the lemma; note that all bounds are independent of the parameters $t \in [0, 1]$ and $a > 0$.

In an analogous manner to the one used to prove Corollary 4.3, we deduce some positive constants k_1, k_2 which enable us to define an open subset of X, say

$$\Omega := \{(u, v, \nu) \in X;\ 0 < u, v < k_1;\ 0 < |u'|, |v'| < k_2;\ \eta < \nu < \beta\}$$

such that $\deg(I - K_t, \Omega, 0)$ is well defined for any $t \in [0, 1]$; now, we have to compute it in the case $t = 0$.

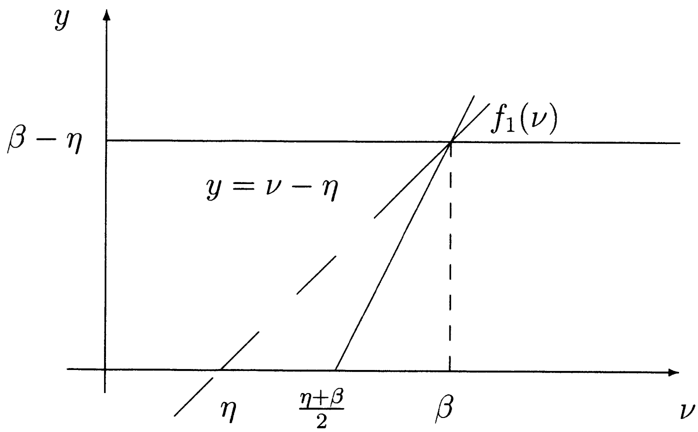

Figure 1. – Graph of the function f_1

5.3. Computation of the degree

Let us set

$$\mathcal{J}_1 := \frac{1}{c}\int_{-\infty}^{+\infty} \chi_1(s)\, ds, \quad \mathcal{J}_2 := \int_0^{+\infty} e^{-ct}\, dt \int_{-\infty}^{t} \chi_1(s)\, ds. \tag{5.4}$$

For $a > 0$, we will use the notations

$$\mathcal{J}_{1,a} := \frac{1}{c}\int_{-a}^{+a} \chi_1(s)\, ds, \quad \mathcal{J}_{2,a} := \int_0^{+a} e^{-ct}\, dt \int_{-a}^{t} \chi_1(s)\, ds. \tag{5.5}$$

and assume that

$$\exists\, a_0 > 0, \quad \mathcal{J}_2 < \mathcal{J}_{1,a_0} - \eta \tag{5.6}$$

$$\mathcal{J}_2 < \beta - \mathcal{J}_1. \tag{5.7}$$

Remark 5.3. — As an example, hypotheses (5.6), (5.7) are fulfilled by means of the function

$$\chi_1(s) = \begin{cases} e^{\delta s}, & s \leqslant 0 \\ e^{-\delta s}, & s \geqslant 0 \end{cases}$$

for some $0 < \delta_1 < \delta < \delta_2$ with

$$\delta_1 = \frac{4 - \beta c^2 + \sqrt{(\beta c^2 - 4)^2 + 12\beta c^2}}{2\beta c} \quad \text{and} \quad \delta_2 = \frac{-\eta c + \sqrt{\eta^2 c^2 + 4\eta}}{2\eta};$$

since $\lim_{\eta \to 0_+} \delta_2(\eta) = +\infty$, $0 < \delta_1 < \delta_2$ hold true for sufficiently small η. In fact, (5.6) $\Leftrightarrow \frac{1}{\delta(c+\delta)} > \eta$ for a large enough, while (5.7) $\Leftrightarrow \frac{3c+4\delta}{c\delta(c+\delta)} < \beta$. At present, we are ready to prove

PROPOSITION 5.4.— *There exists* $\overline{a} > 0$ *such that for any* $t \in [0,1]$, *and any* $a \geqslant \overline{a}$, $\deg(I - K_t, \Omega, 0) \neq 0$.

Proof.— By virtue of Invariance Property by Homotopy of the degree together with the classical properties of Schauder Index, it is enough to prove that the mapping K_0 has exactly one fixed point (u_0, v_0, ν_0) in Ω [10]; however, let us notice that such a fixed point is obtained equivalently as the unique positive solution to the linear boundary value problem

$$\begin{cases} -u'' + cu' + \lambda u = \chi_1 + q_1 \\ -v'' + cv' = -\chi_1. \end{cases} \tag{5.8}$$

$$\begin{cases} u'(-a) = r_2 u(-a), \quad v'(-a) = c[v(-a) - \nu]; \\ u(a) = v(a) = 0. \end{cases} \tag{5.9}$$

$$v(0) = \eta + f_1(\nu). \tag{5.10}$$

r_1, r_2 being as defined in Section 4.1, a straightforward computation yields

$$u(x) = e^{r_2 x} \int_x^a e^{(r_1 - r_2)t} dt \int_{-a}^t (\chi_1 + q_1)(s) e^{-r_1 s} ds;$$

$$v(x) = \nu\big(1 - e^{c(x-a)}\big) - e^{cx} \int_x^a e^{-ct} dt \int_{-a}^t \chi_1(s) ds.$$

In addition, let us remark that

$$v \geqslant 0 \iff v'(a) < 0 \iff \nu > \mathcal{J}_{1,a}, \tag{5.11}$$

while the condition (5.10) reads

$$f_1(\nu) = \nu(1 - e^{-ac}) - \eta - \mathcal{J}_{2,a}. \tag{5.12}$$

So, it remains to prove that there exists uniquely one solution $\nu \in]\eta, \beta[$ which fulfills both (5.11) and (5.12). Adding (5.6) and (5.7) and noting that $0 < \mathcal{J}_{1,a_0} < \mathcal{J}_1$, we infer $\mathcal{J}_2 < \frac{\beta - \eta}{2}$ and so

$$\eta < \frac{\eta + \mathcal{J}_{2,a}}{1 - e^{-ac}} < \frac{\eta + \mathcal{J}_2}{1 - e^{-ac}} < \frac{\eta + \beta}{2}, \tag{5.13}$$

whenever $a \geqslant a_1 := -\frac{1}{c} \ln \left(1 - \frac{\eta + \mathcal{J}_2}{(\eta + \beta)/2}\right)$.

Now, consider the line (Δ) the equation of which in terms of ν reads $y = \nu(1-e^{-ac})-\eta-\mathcal{J}_{2,a}$; by virtue of (5.13), it intersects the ν-axis through the point $\frac{\eta+\mathcal{J}_{2,a}}{1-e^{-ac}} \in]\eta, \frac{\eta+\beta}{2}[$ with a slope less than unity. By definition (5.1), (Δ) must also intersect the graph of the function f_1 in two points $\eta < \nu_{1,a} < \frac{\eta+\beta}{2} < \nu_{2,a} < \beta$ (Fig. 2) solutions of the algebraic equation (5.12); more explicitly,

$$\nu_{1,a} := \frac{\eta + \mathcal{J}_{2,a}}{1 - e^{-ac}} \quad \text{and} \quad \nu_{2,a} := \frac{\beta - \mathcal{J}_{2,a}}{1 + e^{-ac}}.$$

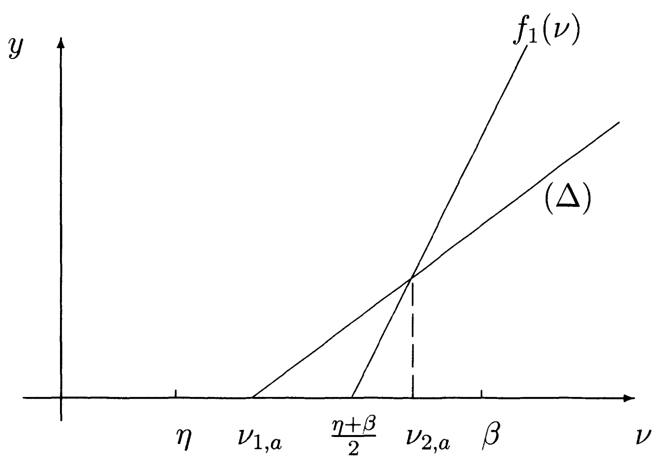

Figure 2. – Position of f_1 with respect to (Δ)

Let $a_2 := -\frac{1}{c} \ln\left(\frac{\beta-\mathcal{J}_1-\mathcal{J}_2}{\mathcal{J}_1}\right)$; then

$$\begin{aligned} a \geqslant a_2 &\Rightarrow (1 + e^{-ac})\mathcal{J}_1 < \beta - \mathcal{J}_2 \\ &\Rightarrow (1 + e^{-ac})\mathcal{J}_{1,a} < \beta - \mathcal{J}_{2,a} \\ &\Rightarrow \nu_{2,a} > \mathcal{J}_{1,a}. \end{aligned}$$

Moreover, if $a_3 := -\frac{1}{c} \ln\left(1 - \frac{\eta+\mathcal{J}_2}{\mathcal{J}_{1,a_0}}\right)$, then

$$\begin{aligned} a \geqslant \max(a_0, a_3) &\Rightarrow \frac{\eta + \mathcal{J}_{2,a}}{1 - e^{-ac}} < \frac{\eta + \mathcal{J}_2}{1 - e^{-ac}} < \mathcal{J}_{1,a_0} < \mathcal{J}_{1,a} \\ &\Rightarrow \nu_{1,a} < \mathcal{J}_{1,a}. \end{aligned}$$

Finally, assume $a \geqslant \bar{a} := \max(a_0, a_1, a_2, a_3)$; we conclude that only $\nu_{2,a}$ satisfies (5.11) ensuring positiveness of v; it then corresponds to a unique solution (u_a, v_a, ν_a) lying in the open set Ω, ending the proof of the proposition. As in Sub-section 4.4, we pass to the limit, when $a \to +\infty$, and achieve the proof of Theorem 3.5. Note that the restriction $v(0) \geqslant \eta$ does not allow the trivial solution $v \equiv 0$ to arise.

6. Proof of theorem 3.6 ($q_1 \equiv 0$; $q_2 \not\equiv 0$, $\lambda \geqslant 0$, $\mu \geqslant 0$)

We aim to show again existence of solution (u, v) for any wave speed $c > 0$; to this end, we introduce, as in Section 5, a new unknown $\nu > 0$ while λ and μ are considered fixed. Then, we look for fixed points (u, v, ν) for the mapping K_t defined by $K_t(u, v, \nu) = (U, V, u(0)+\nu-\gamma-f_2(\nu)$ where

$$\begin{cases} -U'' + cU' + \lambda U = tv|g(u)| + (1-t)\chi_2 \operatorname{sgn}(v) \\ -V'' + cV' + \mu V = -tv|g(u)| + q_2 - (1-t)\chi_2 \operatorname{sgn}(v). \end{cases} \tag{6.1}$$

and

$$\begin{cases} U'(-a) = r_2 U(-a),\ U(a) = 0 \\ V'(-a) = s_2[V(-a) - \nu],\ V(a) = 0. \end{cases} \tag{6.2}$$

where χ_2 is a smooth positive function to be selected conveniently.

Are given two constants γ and β required to satisfy some conditions which we present at the end of the proof; in particular, we assume

$$\frac{1}{c}|q_2|_1 \leqslant \gamma < \beta. \tag{6.3}$$

As for the real function f_2, it is defined by

$$f_2(s) = \begin{cases} 0, & s \leqslant \gamma \\ m(s-\gamma), & s \geqslant \gamma; \end{cases} \tag{6.4}$$

with slope $m := \frac{1}{\beta-\gamma}\left(\frac{s_2}{c}\beta - \gamma + \frac{|q_2|}{c}\right)$ (note that, with (6.3), $s_2 > c \Rightarrow m > 0$).

6.1. A priori bounds

As in Lemma 5.2, we can show the following estimates the proofs of which are omitted.

LEMMA 6.1.— *Let $(u, v, \nu) \in X$ ($\nu \geqslant 0$) be a fixed point of the mapping K_t. Then, for any $x \in I_a$, it holds*

(a) $\frac{c}{s_2}(\gamma - \frac{|q_2|_1}{c}) < \nu < \beta$.

(b) $0 < u(x), v(x) < \frac{s_2}{c}\beta + \frac{|q_2|_1}{c}$.

(c) $-s_2\beta - |q_2|_1 < v'(x) < |q_2|_1$.

(d) $|u'(x)| < s_2\beta + 2|q_2|_1$.

6.2. Computation of the degree

Our purpose is to prove

PROPOSITION 6.2. — *There exists positive constants $0 < \gamma < \beta$ and a positive function χ_2 such that the system*

$$\begin{cases} -u'' + cu' + \lambda u = \chi_2 \\ -v'' + cv' + \mu v = -\chi_2 + q_2, \end{cases} \tag{6.5}$$

subject to (6.2), admits exactly one solution (u, v, ν) satisfying $u(0) = \gamma + f_2(\nu)$ as well as the statements of Lemma 6.1, provided a is large enough.

Proof. — Setting $h := q_2 - \chi_2$ and solving (6.2), (6.5), we get

$$u(x) = e^{r_2 x} \int_x^a e^{(r_1 - r_2)t} dt \int_{-a}^t \chi_2(s) e^{-r_1 s} ds;$$

$$v(x) = e^{s_2 x} \int_x^a e^{(s_1 - s_2)t} dt \int_{-a}^t h(\xi) e^{-s_1 \xi} d\xi + \frac{s_2 \nu e^{s_1 a} e^{s_2 x}}{s_1 - s_2} [e^{a(s_1 - s_2)} - e^{x(s_1 - s_2)}].$$

Moreover $v \geqslant 0 \iff v'(a) < 0 \iff s_2 \nu e^{s_1 a} + \int_{-a}^a h(t) e^{-s_1 t} dt > 0$; this holds particularly true if the unknown ν is positive and

$$\int_{-a}^a h(t) e^{-s_1 t} dt > 0. \tag{6.6}$$

However, the condition $u(0) = \gamma + f_2(\nu)$ reads

$$f_2(\nu) = \int_0^a e^{(r_1 - r_2)t} dt \int_{-a}^t \chi_2(s) e^{-r_1 s} ds - \gamma.$$

Owing to the definition (6.4), this equation admits uniquely one solution $\nu \in]\gamma, \beta[$ if and only if (Fig. 3)

$$\gamma < \int_0^a e^{(r_1 - r_2)t} dt \int_{-a}^t \chi_2(s) e^{-r_1 s} ds < \frac{s_2}{c} \beta + \frac{|q_2|_1}{c}. \tag{6.7}$$

Now, (6.3), (6.6) and (6.7) are simultaneously fulfilled whenever the following sufficient conditions hold true (note that $s_2 > c$)

$$\frac{1}{c} |q_2|_1 \leqslant \gamma, \tag{6.8}$$

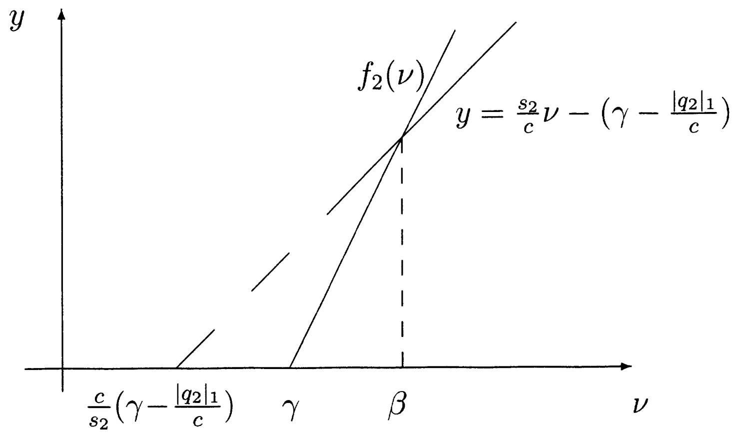

Figure 3. – Graph of the function of f_2

$$\int_{-\infty}^{+\infty} h(t)e^{-s_1 t}dt > 0, \tag{6.9}$$

$$\gamma < \int_0^{+\infty} e^{(r_1-r_2)t}dt \int_{-\infty}^{t} \chi_2(s)e^{-r_1 s}ds < \beta, \tag{6.10}$$

and the parameter a is large enough. Furthermore, sufficient conditions for (6.10) read

$$\gamma < \frac{1}{r_2 - r_1}\int_{-\infty}^{0} \chi_2(t)e^{-r_1 t}dt < \frac{1}{r_2 - r_1}\int_{-\infty}^{+\infty} \chi_2(t)e^{-r_1 t}dt < \beta. \tag{6.11}$$

But (6.8) and (6.11) lead to the following necessary condition

$$\frac{1}{c}\int_{-\infty}^{+\infty} q_2(s)ds < \frac{1}{r_2 - r_1}\int_{-\infty}^{0} \chi_2(t)e^{-r_1 t}dt. \tag{6.12}$$

In order to show how the conditions (6.8), (6.9) and (6.11), (6.12) may be accomplished, let us suppose, for instance, the continuous function h to be defined in the following way:

$$h(s) = h_{k,\delta}(s) = \begin{cases} -e^s, & s \leqslant -1; \\ p_k(s), & -1 \leqslant s \leqslant 0; \\ e^{-\delta s}, & 0 \leqslant s; \end{cases} \tag{6.13}$$

in which $p_k(s) = (k - 1 - e^{-1})s^2 + ks + 1$ with $k \in \mathbb{R}$ and $\delta > 0$ to be selected suitably further on. Inequality (6.12) is then equivalent to

$$\int_0^{+\infty} q_2(s)ds + \int_{-\infty}^0 q_2(s)\Big(1 - \frac{ce^{-r_1 s}}{r_2 - r_1}\Big)ds < -\frac{c}{r_2 - r_1}\int_{-\infty}^0 h(t)e^{-r_1 t}dt. \tag{6.14}$$

Thanks to (6.13), the integral in the right-hand side of (6.14) equals $C + k\int_{-1}^0 (t^2+t)e^{-r_1 t}dt$, where C is some constant not depending on k, δ; now, since $\int_{-1}^0 (t^2+t)e^{-r_1 t}dt < 0$, (6.14) is satisfied for k large enough.

At last, let us split up the integral in (6.9)

$$\int_{-\infty}^{+\infty} h(t)e^{-s_1 t}dt = \int_{-\infty}^0 h(t)e^{-s_1 t}dt + \int_0^{+\infty} h(t)e^{-s_1 t}dt. \tag{6.15}$$

In order to check (6.9), let us distinguish between two cases:

(i) $\mu > 0$: From definition (6.13), the first integral in (6.15) is bounded while the second one diverges to $+\infty$ whenever $0 < \delta < -s_1 < 0$; (6.9) then follows.

(ii) $\mu = 0$: Here $s_1 = 0$ and $\int_{-\infty}^{+\infty} h(t)dt = \frac{1}{\delta} - \frac{k}{6} - \frac{4e^{-1}-2}{3} > 0$ if and only if

$$\frac{1}{\delta} > \frac{k}{6} + \frac{4e^{-1}-2}{3}. \tag{6.16}$$

k being taken so large that (6.14) is satisfied, we may choose $\delta > 0$ small enough to guarantee (6.16).

Therefore, in both cases, (6.9) and (6.12) are fulfilled by selecting k large and afterwards $\delta > 0$ small enough; a function $\chi_2 = q_2 - h$ is thus chosen. At last, we pick out two positive constants γ, β satisfying

$$\frac{1}{c}\int_{-\infty}^{+\infty} q_2(s)ds < \gamma < \frac{1}{r_2 - r_1}\int_{-\infty}^0 \chi_2(s)e^{-r_1 s}ds;\ \beta > \frac{1}{c}\int_{-\infty}^0 \chi_2(s)ds$$

to obtain (6.8), (6.11) and then the desirable conditions for sufficiently large $a > 0$, ending the proof of Lemma 6.2.

As previously undertaken in Sections 4-5, we then easily conclude, from Lemma 6.1 and Proposition 6.2, the proof of Theorem 3.6.

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Bibliographie

[1] ARONSON (D.G.) & WEINBERGER (H.F.). — *Multidimensional nonlinear diffusion arising in population genetics,* Adv. Math. (1978), pp. 33-76.

[2] BAILEY (N.T.). — *The Mathematical Theory of Infectious Diseases,* Griffin, 1975.

[3] BERESTYCKI (H.), NICOLAENKO (B.) & SCHEURER (B.). — *Traveling wave solutions to combustion models and their singular limits,* SIAM J. Math. Anal., Vol. 16, No 6 (1985), pp. 1207-1242.

[4] BRITTON (N.F.). — *Reaction-Diffusion Equations and their Applications to Biology,* Academic Press, 1986.

[5] MOTTONI (P.) de, ORLANDI (E.) & TESEI (A.). — *Asymptotic behavior for a system describing epidemics with migration and spatial spread of infection,* Nonlinear Analysis. Theory, Methods and Applications. Vol. 3 (1979), pp. 663-675.

[6] DJEBALI (S.). — *Traveling wave solutions to a reaction-diffusion system arising in epidemiology,* Nonlinear Analysis: Real World Applications, Vol. 2, No 4 (2001), pp. 417-442.

[7] FIFE (P.C.). — *Mathematical Aspects of Reacting and Diffusing Systems,* in Lecture Notes in Biomathematics, No 28, Springer Verlag, 1979.

[8] HOSONO (Y.) & ILYAS (B.). — *Traveling waves for a simple diffusive epidemic model,* Math. Models and Meth. in Applied Sciences, Vol. 5, No 7 (1995), pp. 935-966.

[9] KERMACK (W.O.) & MCKENDRICK (A.G.). — *Contributions to the mathematical theory of epidemics,* Pro. Roy. Soc., A115 (1927), pp. 700-721.

[10] LLOYD (N.G.). — *Degree Theory,* Cambridge University Press, 1978.

[11] MARION (M.). — *Qualitative properties of a nonlinear system for laminar flames without ignition temperature,* Nonlinear Analysis. Theory, Methods and Applications. Vol. 9, No 11 (1985), pp.1269-1292.

[12] MURRAY (J.D.). — *Mathematical Biology,* Springer Verlag, 1989.

[13] PAO (C.V.). — *On nonlinear reaction-diffusion systems,* Journal of Mathematical Analysis and Applications, Vol. 87 (1982), pp. 165-198.

[14] WALTMAN (P.). — *Deterministic Threshold Models in the Theory of Epidemics,* Lecture Notes in Biomathematics, Vol. 1, Springer-Verlag, Berlin, New York, 1974.

[15] UCHIYAMA (K.). — *The behaviour of some nonlinear diffusion equations for large time,* J. Kyoto Univ., Vol. 18, (1978) pp. 453-508.