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Section A :
Physique théorique.

Group of invariance of a relativistic supermultiplet theory (*)

par

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Recently Sakita [*1*], Gürsey and Radicati [*2*] [*4*] and Pais [*3*] [*4*] have proposed a generalization of Wigner supermultiplet theory [*5*] for the nucleus to baryons and mesons [*6*]. This raises the question: what is a relativistic supermultiplet theory ? In this paper we shall consider only the problem of defining the invariance group G for such a theory [*7*].

We denote by P the connected Poincaré group. It is the semi-direct product $P = T \times L$ where T is the translation group and L is the homogeneous Lorentz group.

CONDITION 1. — The invariance group G of a relativistic theory contains P. We shall not discuss here the discrete invariance P, C, T, so we shall add.

CONDITION 2. — G is a connected topological group (with P as topological subgroup) [*8*].

Invariance under G is considered as the largest symmetry for strong coupling physics [*1*] [*2*] [*3*] [*4*]. The particles of a supermultiplet have

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the same mass and for a given energy momentum p, all possible states (spin, charges, etc.) of these particles form a finite dimensional Hilbert space which is the space of an irreducible unitary representation of a compact group S_p, the « little group » of p. From the classical Wigner analysis it is easy to translate this as conditions on G.

CONDITION 3. — The translation group T is invariant subgroup of G. The action of G on T (by its inner automorphisms) preserves the Minkowski metric and the little group (mathematicians say stabiliser or isotopy group) in G/T of a time-like translation $a \in T$ is a compact group S.

THEOREM. — If a group G satisfies condition 1, 2 and 3, then G/T is a direct product of $H \times L$.

Proof :

The condition 3 implies that for every $g \in G$, a ($\in T$) and its transformed $g(a) = gag^{-1}$ have same Minkowski length : $a.a = g(a).g(a)$. In the dual of T [i. e., the four dimensional vector space of energy momentum] the orbits of G are the connected sheets of mass hyperboloid. Denote f: $G \xrightarrow{f} \text{Aut}\, T$, the homomorphism of G which describes its action, by inner automorphisms, on its invariant subgroup T. It is easy to prove [*9*] that the connected group of continuous automorphisms of G which preserves the Minkowski metric is L. So the image of f is L: $\text{Im} f = L$. Since T is abelian $T \subset K = \text{Ker}\, f$, the kernel of f, and f is factorized into $g \circ p$, where p: $G \xrightarrow{p} G/T$ and g: $G/T \xrightarrow{g} L$. The restriction of g to the subgroup $L = P/T \subset G/T$, is an identity transformation. By definition of the semi-direct product, therefore, G/T is the semi-direct product $H \times L$ where $H = \text{Ker}\, g = K/T$. Furthermore, by definition of $H = \text{Ker}\, g$, H is an invariant subgroup of every stabilizer (little group) S for any a. For a time-like a, S_a is compact, this implies that its invariant subgroup H is compact, and from a theorem of Iwasawa [*10*] H compact and G/T connected implies that it is a central extension of kernel H. As we have seen, it is also the semi-direct product $H \times L$. Hence it is a direct product:

$$G/T = H \otimes L. \tag{1}$$

The proof of the theorem also gives conditions on the little group S, for a time-like translation. Indeed it must be isomorphic to the direct product $H \otimes R$ where R is the three dimensional rotation group. Of course, this excludes SU(6) or any simple Lie group for S.

A possible way to have a relativistic theory with supermultiplets of particles classified by irreducible unitary representation of SU(6), is to find a (connected Lie) group $\bar{G}$ with irreducible unitary representations characterized by $m > 0$ and those of SU(6), and such that $\bar{G} \supset \bar{P}$, the covering of the Poincaré group. We proceed now to build such a group $\bar{G}$.

Among all subgroups of the linear group with enumerable dimension let us look for the smallest group H such that:

$$\mathrm{SU}(6) \subset \mathrm{H}, \quad \mathrm{SL}(2, \mathrm{C}) \subset \mathrm{H}, \quad \mathrm{SU}(6) \cap \mathrm{SL}(2, \mathrm{C}) = \mathrm{SU}(2) \tag{2}$$

where SU(2) is the covering of R and SL(2, C) the covering of L. The smallest group H is the intersection of all groups which satisfy (2). SL(6, C) is one of them, so SU(6) $\subset$ H $\subset$ SL(6, C). But SU(6) is maximal subgroup of SL(6, C); this implies: H = SL(6, C). The elements $x \in \mathrm{H}$ are 6×6 matrices with determinant 1. They can be decomposed in a unique way into the product $x = hu$ where h is a 6×6 hermitian positive matrix of determinant 1 and u is a 6×6 unitary matrix with determinant 1. The matrices u generates SU(6) and the set $\{ h \}$ of matrices h is the homogenous space SL(6, C)/SU(6). The smallest Lie group generated by $\{ h \}$ is the additive group of the 6×6 hermitian matrices; it is the 36 real parameter simply connected abelian Lie group. We shall denote it T_{36}.

The group $\bar{G}$ is the semi-direct product of SL(6, C) by T_{36} with the action: $h \to xhx^*$ (indeed this contains the action of SL(2, C) on T_4, hence $\bar{G} \supset \bar{P}$). The orbit of $\bar{G}$ on T_{36} are characterized by det h and the sign of the eigenvalues. If $h > 0$, one can take as representative $h = m1$. Its little group is that of the matrices with determinant 1 such that $xhx^* = mxx^* = m1$; it is SU(6).

Hence the smallest connected Lie group which contains P and has unitary irreducible linear representations characterized by: $m > 0$ and the unitary representations of SU(6), is the group G we just defined. It is a 106 parameter Lie group [*11*]. As we shall explain elsewhere the use of such G as invariance group for a relativistic supermultiplet theory of elementary particle is possible, but we do not like it.

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[4] F. GÜRSEY, A. PAIS and L. RADICATI, *Phys. Rev. Letters*, t. 13, 1964, p. 299.
[5] E. P. WIGNER, *Phys. Rev.*, t. 51, 1937, p. 106.
[6] More than twenty papers on this subject have been published or mimeographed.
[7] For physicists who may find our simple rigorous proof too abstract we are writing a more detailed paper on the subject in terms of Lie algebra. We will also show in this paper that the invariance properties of such a theory cannot be reduced to the study of a group.
[8] This is a purely technical condition; with the work of E. C. ZEEMAN, *J. Math. Phys.*, t. 5, 1964, p. 491, we can obtain that G/T is the semi-direct product H $\times$ L without condition 2.
[9] Indeed ZEEMAN [8], has proven it without the assumptions of continuity and automorphisms.
[10] K. IWASAWA, *Ann. Math.*, t. 50, 1949, p. 507.
[11] This group has been mentioned at the end of reference [1].

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