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On some classical space-time groups and their "SU6 generalizations"

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par

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ABSTRACT. — The Lorentz group, Poincaré group, de Sitter groups are defined as subgroups of the conformal group. The four-dimensional representation of the conformal group is investigated with the aid of the γ-Dirac algebra. This leads to a new interpretation of the γ_5 operator as the generator of the scale transformation of the Minkowski space. The usual currents $\overline{\psi}\gamma_\mu(1 + \gamma_5)\psi$ for massless particles are shown to belong to the adjoint representation of the Poincaré group (it is not the case for the $\overline{\psi}\gamma_\mu(1 - \gamma_5)\psi$ components). Natural « SU_6-generalizations » of all these groups are readily derived and some of their properties are discussed.

RÉSUMÉ. — Les groupes de Lorentz, Poincaré et de Sitter sont définis comme sous-groupes du groupe conforme. La représentation à 4 dimensions de ce groupe est étudiée en terme des matrices γ de Dirac. La matrice γ_5 est alors le générateur infinitésimal des dilatations dans l'espace de Minkowski. Les courants usuels $\overline{\psi}\,\gamma_\mu(1 + \gamma_5)\psi$ pour les particules de masse zéro appartiennent à la représentation adjointe du groupe de Poincaré (ce qui n'est pas le cas des composantes $\overline{\psi}\,\gamma_\mu(1 - \gamma_5)\psi$). Les généralisations « naturelles » de ces groupes pour l'invariance SU_6 sont établies et quelques propriétés sont discutées.

Finding SU_6 generalizations $\mathcal{T}_E$ of the Poincaré group $\mathcal{T}$ is not so easy, due to the fact that $\mathcal{T}$ is not a semi-simple group (i. e., it possesses an invariant Abelian subgroup, namely the translation group of space-time). In order

to avoid such a difficulty, it is interesting to connect the group $\mathcal{F}$ to a simple group G following a given rule :

(R) $\mathcal{F} \overset{(R)}{\to} G.$

Then, one enlarges the group G to a group G_E such that SU_6 is contained in G_E like the rotation group SU_2 is contained in G. One applies now the inverse rule to (R) in order to get a solution of our problem:

(R⁻¹) $G_E \overset{R^{-1}}{\to} \mathcal{F}_E$

The two following rules (R) can be used:

(R₁) $\mathcal{F}$ is the « limit » of G; the group G can be chosen as one of the two de Sitter groups $\mathcal{S}^+$ and $\mathcal{S}^-$, both built on Lie algebra C_2 (Cartan's notation) ;

(R₂) $\mathcal{F}$ is a subgroup of G; the natural choice for G is the conformal group, the Lie algebra of which is A_3 (Cartan's notation).

Such considerations led us to examine the connections between the conformal, de Sitter, Poincaré, Lorentz, and rotation groups.

The conformal group is usually defined as the largest group of space-time transformations which leaves invariant the Maxwell equations [1]. This group is generated by 15 operators, 11 of which correspond to linear transformations of the Minkowski space (namely, the six $M_{\mu\nu}$, the four P_μ and the scale transformation of space-time S) and the last four corresponding to the non-linear transformations (« accelerations »)

$$x^\mu \to x'^\mu = \frac{x^\mu + a^\mu(x^2)}{1 + 2a_\mu x^\mu + a^2 x^2} \tag{1}$$

where $x^2 = x_\mu x^\mu$.

This group is known to be simple; its Lie algebra is A_3 in Cartan's notation.

Starting from another point of view, we define in Section I the conformal group as a generalized Lorentz group and show how it contains as subgroups the Poincaré group (i. e., the inhomogeneous Lorentz group) and the two de Sitter groups (i. e., the groups of space-time transformations when the Minkowski space is replaced by a constant curvature space). A general framework for the investigation of the pseudo-orthogonal and pseudo-unitary groups is given.

In Section II, we examine the conformal group in a locally equivalent definition. A four-dimensional representation is given explicitly, using

the γ Dirac algebra. Some consequences concerning the Poincaré group are derived, namely:

i) the γ_5 operator can be identified with the operator S of scale transformations;

ii) given a Dirac spinor ψ the current $\bar{\psi}\,\gamma_\mu(1 + \gamma_5)\psi$ is shown to belong to the adjoint representation of the Poincaré group together with the $\bar{\psi}\sigma_{\mu\nu}\psi$ components.

In generalizing the conformal group in order to replace the rotation group SU_2 by the SU_6 group which has been proposed by several authors [2], natural generalizations of the de Sitter, Poincaré and Lorentz groups are derived. Some remarks concerning such groups are discussed. All questions connected with the SU_6 group are investigated in Section III.

I. Let us consider a real $(n + m)$ — dimensional vector space $M(n, m)$ where a pseudoeuclidean metric g is defined with the signature: n minus signs and m plus signs.

We define the two following groups:

— $L(n, m)$: the connected [3] group of $(n + m) \times (n + m)$ unimodular matrices which preserves the metric g, i. e., the quadratic form:

$$(x, x) = x^2 = - \sum_{i=1}^{n} x_i^2 + \sum_{i=n+1}^{n+m} x_i^2. \tag{2}$$

— $P(n, m)$: the corresponding inhomogeneous (connected) group, i. e., the group of transformations (Λ, a):

$$x \to x' = \Lambda x + a \tag{3}$$

where Λ is an element of $L(n, m)$ and a, x, x' vectors of $M(n, m)$.

To each pair of vectors a and b of $M(n, m)$ corresponds a dyadic linear operator written $a \otimes b$; such a dyad [4] acts on a given vector x as follows:

$$(a \otimes b)x = (b, x)a \tag{4}$$

where (b, x) denotes the scalar product of the two vectors b and x.

Conversely, it can be easily proved that every linear operator on M may be written in the form of a sum of such dyads. More precisely, given a basis e_α of the vector space M, every linear operator A may be put in the form

$$A = \sum_{\alpha,\beta} x^{\alpha\beta} e_\alpha \otimes e_\beta. \tag{5}$$

The following definitions will be useful:

a) given a non-singular linear operator A, there exists a linear associate operator $\tilde{A}$ such that

$$(\tilde{A}x, y) = (x, Ay) \tag{6}$$

for each pair of vectors x, y.
   Note that:

$$A = a \otimes b \ \Rightarrow \ \tilde{A} = b \otimes a.$$

b) A linear operator A will be called *pseudosymmetric* if:

$$\tilde{A} = A \tag{7}$$

and *pseudoantisymmetric* if:

$$\tilde{A} = - A. \tag{8}$$

Note that they are called, respectively, symmetric and antisymmetric if g is definite (n or m equals zero).   For instance, $a \otimes a$ is pseudosymmetric and

$$a \wedge b = a \otimes b - b \otimes a \tag{9}$$

is pseudoantisymmetric.

c) A non-singular linear operator Λ will be said pseudo-orthogonal if:

$$(\Lambda x, \Lambda y) = (x, y) \tag{10}$$

or, equivalently, if:

$$\tilde{\Lambda}\Lambda = 1 \quad \text{or} \quad \tilde{\Lambda} = \Lambda^{-1}. \tag{11}$$

The group of all these pseudo-orthogonal matrices is sometimes denoted by O(n, m).   It contains as a subgroup the unimodular pseudo-orthogonal group SO(n, m), the part which is connected to the unit transformation is a group denoted L(n, m) $\subseteq$ SO(n, m).
   Now, every operator Λ of L(n, m) may be put in the form

$$\Lambda = e^{A} \tag{12}$$

since L(n, m) is connected.   Moreover, according to Eq. (11), A has to be pseudoantisymmetric:

$$\left. \begin{array}{l} \Lambda^{-1} = (e^{A})^{-1} = e^{-A} \\ \tilde{\Lambda} = e^{\tilde{A}} \end{array} \right\} \Rightarrow \tilde{A} = - A. \tag{13}$$

Following Eq. (5) and (9), the generators of $L(n, m)$ can be written as

$$M_{\alpha\beta} = ie_\alpha \wedge e_\beta \qquad (14)$$

and from Eq. (4) and (9), we derive readily the well-known commutation rules:

$$[M_{\alpha\beta}, M_{\gamma\delta}] = i(g_{\beta\gamma}M_{\alpha\delta} - g_{\beta\delta}M_{\alpha\gamma} - g_{\alpha\gamma}M_{\beta\delta} + g_{\alpha\delta}M_{\beta\gamma}) \qquad (15)$$

where

$$g_{\alpha\beta} = (e_\alpha, e_\beta). \qquad (16)$$

It is perhaps important to note that the commutation rules do not depend on the kind of the basis: the e_α's are not necessarily orthogonal or normalized. The usual choice consists in putting:

$$g_{\alpha\beta} = \pm \delta_{\alpha\beta} \qquad (17)$$

such that basic vectors are either spacelike ($g_{\alpha\alpha} < 0$) or timelike ($g_{\alpha\alpha} > 0$). It will appear sometimes interesting to choose a basis with some isotropic vectors ($g_{\alpha\alpha} = 0$).

Our next task consists in proving the following theorem.

THEOREM. — *The subgroup of $L(n, m)$ which leaves invariant a given vector f is* [5]:

$$
\begin{array}{ll}
L(n-1, m) & \text{if } f \text{ is spacelike} \\
L(n, m-1) & \text{if } f \text{ is timelike} \qquad (18) \\
P(n-1, m-1) & \text{if } f \text{ is isotropic } (f^2 = 0).
\end{array}
$$

In view of physical applications, we will give the proof of the above theorem in the special case where $n = 4$ and $m = 2$ (There is no difficulty in giving a general proof for any values of n and m). The group $L(4,2)$ is the connected part of the conformal group. The three corresponding groups mentioned in (18) are, respectively, the two de Sitter groups $L(3,2)$ and $L(4, 1)$, and the Poincaré group $P(3,1)$. Note that the above theorem is well-known in the case where $L(n, m) = L(3,1)$ is the usual Lorentz group. The same theorem can be stated in another way: the little groups of $P(n, m)$ are $L(n-1, m)$, $L(n, m-1)$ and $P(n-1, m-1)$.

Let $e_{-1}, e_0, e_1, e_2, e_3, e_4$ be a « Lorentz » basis for the $M(4,2)$ space, i. e., a basis such that

$$g_{-1, -1} = g_{00} = 1 \qquad (19)$$

$$g_{11} = g_{22} = g_{33} = g_{44} = -1$$

and $g_{\alpha\beta} = 0$ for $\alpha \neq \beta$.

(Vectors e_0, e_1, e_2 and e_3 form the usual Lorentz basis for Minkowski space.)

Instead of a « Lorentz » basis, we choose the following one : e_0, e_1, e_2, e_3, l and l', where

$$l = e_4 - e_{-1} \tag{20}$$

$$l' = e_4 + e_{-1}. \tag{21}$$

These vectors satisfy the following relations:

$$l^2 = l'^2 = 0 \tag{22}$$

$$(l, l') = -2 \tag{23}$$

$$(e_\mu, l) = (e_\mu, l') = 0. \tag{24}$$

According to Eq. (14), the generators of L(4, 2) are

$$M_{\mu\nu} = ie_\mu \wedge e_\nu \tag{25}$$

$$P_\mu = il \wedge e_\mu \tag{26}$$

$$A_\mu = il' \wedge e_\mu \tag{27}$$

$$S = \frac{i}{2} l' \wedge l \tag{28}$$

where μ and ν run from 0 to 3.

Using the property:

$$[a \wedge b, c \wedge d]$$
$$= (b, c)a \wedge d - (b, d)a \wedge c - (a, c)b \wedge d + (a, d)b \wedge c \tag{29}$$

already used in order to get (15), we find the following commutation rules of the conformal group L(4,2):

$$[M_{\mu\nu}, M_{\rho\lambda}] = i(g_{\nu\rho}M_{\mu\lambda} - g_{\nu\lambda}M_{\mu\rho} - g_{\mu\rho}M_{\nu\lambda} + g_{\mu\lambda}M_{\nu\rho}) \tag{30.a}$$

$$[M_{\mu\nu}, P_\rho] = i(g_{\nu\rho}P_\mu - g_{\mu\rho}P_\nu) \tag{30.b}$$

$$[M_{\mu\nu}, A_\rho] = i(g_{\nu\rho}A_\mu - g_{\mu\rho}A_\nu) \tag{30.c}$$

$$[P_\mu, P_\nu] = [A_\mu, A_\nu] = 0 \tag{30.d}$$

$$[S, P_\mu] = iP_\mu \tag{30.e}$$

$$[S, A_\mu] = -iA_\mu \tag{30.f}$$

$$[A_\mu, P_\nu] = 2i(M_{\mu\nu} - g_{\mu\nu}S). \tag{30.g}$$

The $M_{\mu\nu}$'s and P_{μ}'s are the usual generators of the Poincaré group P(3,1), the A_{μ}'s are the generators of the transformations (1) and S is the scale transformation of the space-time.

We are interested in finding the subgroup of L(4,2) which leaves invariant a given vector f. Such a subgroup is generated by the operators $ia \wedge b$ which possess the property:

$$i(a \wedge b).f = 0. \tag{31}$$

Without loss of generality, we can choose as the vector f the following form

$$f = \frac{1}{4}(l + \beta l')$$

where β is some parameter. If

$$\beta > 0 \qquad f \text{ is spacelike}$$
$$\beta < 0 \qquad f \text{ is timelike}$$
$$\beta = 0 \qquad f \text{ is isotropic.}$$

Let us write how the generators act on f:

$$M_{\mu\nu}.f = i(e_{\mu} \wedge e_{\nu}).f = 0 \tag{32}$$

$$P_{\mu}.f = i(l \wedge e_{\mu}).f = \frac{i}{2}\beta e_{\mu} \tag{33}$$

$$A_{\mu}.f = i(l' \wedge e_{\mu}).f = \frac{i}{2}e_{\mu} \tag{34}$$

$$S.f = \frac{i}{2}(l' \wedge l).f = \frac{i}{4}(l - \beta l'). \tag{35}$$

According to (31), the subgroup of L(4,2) which leaves invariant the vector f is generated by the $M_{\mu\nu}$'s and the X^{η}'s defined as:

$$X_{\mu} = \frac{1}{2}(P_{\mu} - \beta A_{\mu}) \tag{36}$$

and the commutation rules of this subgroup are:

$$[M_{\mu\nu}, M_{\rho\lambda}] = i(g_{\nu\rho}M_{\mu\lambda} - g_{\nu\lambda}M_{\mu\rho} - g_{\mu\rho}M_{\nu\lambda} + g_{\mu\lambda}M_{\nu\rho}) \tag{37.a}$$

$$[M_{\mu\nu}, X_{\rho}] = i(g_{\nu\rho}X_{\mu} - g_{\mu\rho}X_{\nu}) \tag{37.b}$$

$$[X_{\mu}, X_{\nu}] = -i\beta M_{\mu\nu}. \tag{37.c}$$

One can easily verify that for $\beta = 1$, one gets L(3,2) as a subgroup and for $\beta = -1$, one gets L(4,1). For $\beta = 0$, Eq. (36) and (37.c) show that one gets the Poincaré group where the X_{μ}'s are the usual translation operators.

Eq. (37 *a*, *b*, *c*) are usually interpreted as follows: the X_μ's are the genera-
tors of space-time « translations » when the space-time is supposed curved
with a constant curvature $\sqrt{|\beta|}$. For a flat space-time $\beta \to 0$ and one
gets the Poincaré group as a limit.

One can readily generalize the above proof in order to state the above
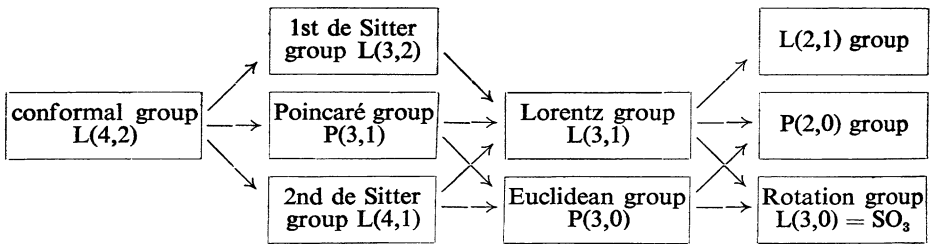theorem. The results can be summarized in the following chain:

Chain 1:
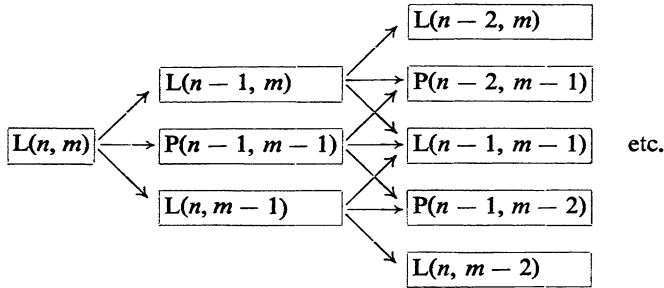
where an arrow means « contains as a subgroup ».

The « physical » part of this chain is:

Chain 2:

The last column of this chain contains the three little groups of the Poin-
caré group.

II. In view of a « SU_6-generalization » of the groups mentioned in the
chain 2, we have to build an analogous chain where the rotation group SO_3
is replaced by its covering group [6], namely the SU_2 group; in that case,
the Lorentz group is also replaced by its covering group, the group of uni-
modular complex 2×2 matrices SL(2,C). The generators of this group
are usually written in terms of the elements of the γ Dirac algebra: given
four matrices γ_μ obeying

$$[\gamma_\mu, \gamma_\nu]_+ = 2g_{\mu\nu} \tag{38}$$

the generators of SL(2,C) are the six matrices:

$$\sigma_{\mu\nu} = \frac{i}{2} [\gamma_\mu, \gamma_\nu]_-. \tag{39}$$

The $\sigma_{\mu\nu}$'s are 4×4 matrices; then we have a four-dimensional representation of the SL(2,C) group; this representation is not irreducible, in fact it is the direct sum of the two irreducible representations

$$D(\tfrac{1}{2}, 0) \oplus D(0, \tfrac{1}{2}).$$

Now, given a spinor ψ, the product $\overline{\psi}\psi$ is invariant under SL(2,C); this proves that SL(2,C) is a subgroup of the group which leaves invariant such a quadratic form:

$$\overline{\psi}\psi = |\psi_1|^2 + |\psi_2|^2 - |\psi_3|^2 - |\psi_4|^2. \tag{40}$$

This new group is very similar to U_4 (the group of 4×4 unitary matrices). Because of the signature of the « scalar product » (40), we shall call it the U(2,2) group. More generally, $U(n, m)$ will denote the group of all $(n + m) \times (n + m)$ matrices preserving the following scalar product [7]:

$$\overline{\psi}\psi = -\sum_{i=1}^{n} |\psi_i|^2 + \sum_{i=n+1}^{n+m} |\psi_i|^2. \tag{41}$$

The group SL(2,C) is then a subgroup of the group SU(2,2). This last group has fifteen parameters like the group SU_4 and like the conformal group. In fact, it is locally isomorphic [8] to the conformal group as can be seen from the following isomorphism:

$$M_{\mu\nu} \sim \frac{1}{2} \sigma_{\mu\nu}$$

$$P_\mu \sim \frac{1}{2} \gamma_\mu(1 + \gamma_5) \tag{42}$$

$$A_\mu \sim -\frac{1}{2} \gamma_\mu(1 - \gamma_5)$$

$$S \sim -\frac{i}{2} \gamma_5$$

where

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3. \tag{43}$$

Using the properties (38) and (39), we get

$$[\sigma_{\mu\nu}, \gamma_\rho]_- = i(g_{\nu\rho}\gamma_\mu - g_{\mu\rho}\gamma_\nu) \tag{44}$$

$$[\sigma_{\mu\nu}, \gamma_5]_- = 0 \tag{45}$$

$$[\gamma_\mu, \gamma_5]_+ = 0 \tag{46}$$

from which we verify that the matrices (42) satisfy the commutation rules of the conformal group (30).

The spinors ψ and $\bar\psi$ are not transformed in the same way under the SU(2,2) group. The two transformations correspond to two non-equivalent irreducible representations; we denote them, respectively, 4 and $\bar{4}$. The multiplication rule:

$$4 \times \bar{4} = 1 + 15 \tag{47}$$

is the same as in the SU_4 group (except, of course, the meaning of the bar).

The representation 1 in Eq. (47) contains the « scalar product » $\bar\psi\psi$; the fifteen components of the representation 15 are the :

$$\bar\psi\sigma_{\mu\nu}\psi, \quad \bar\psi\gamma_\mu\psi, \quad \bar\psi\gamma_\mu\gamma_5\psi, \quad \bar\psi\gamma_5\psi.$$

The adjoint representation 15 contains, of course, the adjoint representation 10 of the de Sitter groups, and the adjoint representation of the Poincaré group. For these, the rule is the following:

$$4 \times 4 = 1 + 5 + 10. \tag{48}$$

When β becomes zero, the irreductible representations 5 and 10 become reducible, *but not completely reducible*; consequently Eq. (48) can be considered as characterizing the three subgroups (de Sitter and Poincaré), but where 5 and 10 denote non completely reducible representations [9].

According to Eq. (42) the current components which belong to the adjoint representation of the Poincaré group are the $\bar\psi\gamma_\mu(1 + \gamma_5)\psi$ and the $\bar\psi\sigma_{\mu\nu}\psi$ alone. In résumé, we have:

TABLE I

U(2,2)	4	$\times \bar{4} =$	16			
SU(2,2)	4	$\times \bar{4} = 1$	15			
SL(2,C) inh	4	$\times \bar{4} = 1$	5		10	
SL(2,C)	$\begin{array}{c}4\\(D_{\frac{1}{2}0}+D_{0\frac{1}{2}})\otimes\\ \psi\end{array}$	$\begin{array}{c}\times 4 = 1\\(D_{\frac{1}{2}0}+D_{0\frac{1}{2}})=(D_{00})\\ \bar\psi \qquad =\bar\psi\psi\end{array}$	$\begin{array}{c}1\\(D_{00})\\ \bar\psi\gamma_5\psi\end{array}$	$\begin{array}{c}4'\\(D_{\frac{1}{2}\frac{1}{2}})\\ \bar\psi\gamma_\mu(1-\gamma_5)\psi\end{array}$	$\begin{array}{c}4'\\(D_{\frac{1}{2}\frac{1}{2}})\\ \bar\psi\gamma_\mu(1+\gamma_5)\psi\end{array}$	$\begin{array}{c}6\\(D_{10}+D_{01})\\ \bar\psi\sigma_{\mu\nu}\psi\end{array}$
SU₂	$(D_{\frac{1}{2}}+D_{\frac{1}{2}})\otimes(D_{\frac{1}{2}}+D_{\frac{1}{2}})=D_0$		D_0	D_0+D_1	D_0+D_1	D_1+D_1

It will be useful to give an explicit form of the representation 4 of the SU(2,2) group; such a representation can be obtained by choosing the following definition of the γ_μ's:

$$\gamma_\mu = \begin{pmatrix} 0 & \bar{\sigma}_\mu \\ \sigma_\mu & 0 \end{pmatrix} \tag{49}$$

where $\sigma_\mu = (1, -\vec{\sigma})$ and $\bar{\sigma}_\mu = (1, +\vec{\sigma})$. In that case, we get:

$$
\begin{aligned}
M_{ij} &= \frac{1}{2}\,\varepsilon_{ijk} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}
& M_{k0} &= \frac{1}{2} \begin{pmatrix} i\sigma_k & 0 \\ 0 & -i\sigma_k \end{pmatrix} \\[2ex]
S &= -\frac{i}{2}\,\gamma_5 = \frac{1}{2} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}
& & \\[2ex]
P_\mu &= \frac{\gamma_\mu(1+\gamma_5)}{2} = \begin{pmatrix} 0 & 0 \\ \sigma_\mu & 0 \end{pmatrix}
& A_\mu &= -\frac{\gamma_\mu(1-\gamma_5)}{2} = \begin{pmatrix} 0 & -\bar{\sigma}_\mu \\ 0 & 0 \end{pmatrix}
\end{aligned}
\tag{50}
$$

We are now able to give new definitions for the de Sitter and Poincaré groups: they are the subgroups of SU(2,2) which leave invariant the symplectic form:

$$h = \begin{pmatrix} \sigma_2 & \\ & \beta\sigma_2 \end{pmatrix} \tag{51}$$

where β has the same meaning as earlier.

If $\beta = 1$ (1st de Sitter group), the invariant form is $\bar{\psi}\sigma_2\psi$ or $\bar{\psi}\psi^c$;

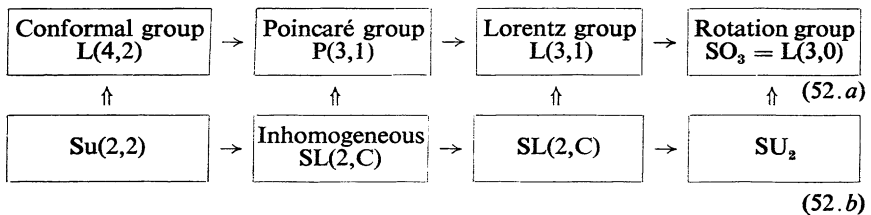If $\beta = -1$ (2nd de Sitter group), it is $\bar{\psi}\gamma_5\sigma_2\psi$ or $\bar{\psi}\gamma_5\psi^c$;

If $\beta = 0$ (Poincaré group), it is $\bar{\psi}(1 + \gamma_5)\sigma_2\psi$ or $\bar{\psi}(1 + \gamma_5)\psi^c$;

(ψ^c denotes the charge conjugate spinor).

The three quantities $\bar{\psi}\psi^c$, $\bar{\psi}\gamma_5\psi^c$ and $\bar{\psi}(1 + \gamma_5)\psi^c$ are invariant under the Lorentz group, since the Lorentz group is, in every case, a subgroup. Some supplementary details are given in Appendix B.

From the above discussion, we can consider the Poincaré group either as a « symplectic » group with the non-regular metric h [Eq. (51)] with $\beta = 0$, or as the inhomogeneous SL(2,C) group.

These two definitions lead to different SU_6 generalizations.

III. We are now able to try to find the SU_6 generalizations of the conformal, de Sitter and Poincaré groups. The following chain repeats some important results of Sections I and II:

Conformal group L(4,2)	→	Poincaré group P(3,1)	→	Lorentz group L(3,1)	→	Rotation group $SO_3 = L(3,0)$
⇑		⇑		⇑		⇑ (52.a)
Su(2,2)	→	Inhomogeneous SL(2,C)	→	SL(2,C)	→	SU_2

$$(52.b)$$

Here, a single arrow has the same meaning as above, namely A → B means « A contains B as a subgroup » and A ⇒ B means « A is a covering group of B ».

We are led, in a natural way, to the following generalization [10] of the chain (52.b):

$$SU(6,6) \rightarrow \boxed{\text{Inhomogeneous } SL(6,C)} \rightarrow SL(6,C) \rightarrow SU_6 \tag{53}$$

But, according to the remark at the end of Section II, it is also possible to consider the « symplectic » parts of the SU(6,6) group as the generalized de Sitter groups; then, taking the limit, we would be led to another generalized Poincaré group. As will be shown, such a generalization does not seem very attractive from a physical point of view.

In the two cases, we are led to an investigation of the SU(6,6) group. In order to get the generators of this group, we use the following property:

$$U(6,6) \rightarrow U(2,2) \otimes U_3. \tag{54}$$

U(6,6) possesses 144 generators as U(12,0). They are built on the same Lie algebra (A_{11} in the Cartan notation).

According to Eq. (42) and (54), each generator of the U(6,6) group can be put in the form of a direct product:

$$\Gamma \otimes \lambda$$

where Γ is one of the 16 matrices of the γ Dirac algebra, and λ one of the 9 generators of the group U_3 ($16 \times 9 = 144$).

The SU(6,6) group has 143 generators since we have to discard the generator $1 \otimes 1$. Therefore the generators of SU(6,6) can be written in the following 12-dimensional representation in analogy with (50):

35 generators of SU_6 $\rightarrow$

$$M_\alpha = \frac{1}{2} \left[\begin{array}{c|c} \Lambda_\alpha & 0 \\ \hline 0 & \Lambda_\alpha \end{array} \right] \tag{55.a}$$

35 generators of « pure Lorentz transformations »

$$N_\alpha = \frac{1}{2} \left[\begin{array}{c|c} i\Lambda_\alpha & 0 \\ \hline 0 & -i\Lambda_\alpha \end{array} \right] \tag{55.b}$$

36 « translation » operators $\tag{55 c}$

$$P_0 = \left[\begin{array}{c|c} 0 & 0 \\ \hline 1 & 0 \end{array} \right]$$

$$P_\alpha = \begin{array}{|c|c|} \hline 0 & 0 \\ \hline \Lambda_\alpha & 0 \\ \hline \end{array}$$

36 « inversion » operators   (55 d)

$$A_0 = \begin{array}{|c|c|} \hline 0 & -1 \\ \hline 0 & 0 \\ \hline \end{array}$$

$$A_\alpha = \begin{array}{|c|c|} \hline 0 & \Lambda_\alpha \\ \hline 0 & 0 \\ \hline \end{array}$$

1 scale transformation

$$S = \frac{1}{2} \begin{array}{|c|c|} \hline -i & \\ \hline & i \\ \hline \end{array}$$ (55.e)

where α runs from 1 to 35, and Λ_α are the traceless Hermitian 6×6 matrices generating the SU_6 group. According to these definitions, it is very easy to derive the commutation relations of the group. They are [11]:

$$[M_\alpha, M_\beta] = i f_{\alpha\beta\gamma} M_\gamma \tag{56.a}$$

$$[M_\alpha, N_\beta] = i f_{\alpha\beta\gamma} N_\gamma \tag{56.b}$$

$$[N_\alpha, N_\beta] = - i f_{\alpha\beta\gamma} M_\gamma \tag{56.c}$$

$$[M_\alpha, P_\beta] = i f_{\alpha\beta\gamma} P_\gamma \tag{56.d}$$

$$[M_\alpha, P_0] = 0 \tag{56.e}$$

$$[N_\alpha, P_\beta] = - i \delta_{\alpha\beta} P_0 - i d_{\alpha\beta\gamma} P_\gamma \tag{56.f}$$

$$[N_\alpha, P_0] = - i P_\alpha \tag{56.g}$$

$$[P_\alpha, P_\beta] = 0 \tag{56.h}$$

$$[P_\alpha, P_0] = 0 \tag{56.i}$$

$$[M_\alpha, A_\beta] = i f_{\alpha\beta\gamma} A_\gamma \tag{56.j}$$

$$[M_\alpha, A_0] = 0 \tag{56.k}$$

$$[N_\alpha, A_\beta] = - i \delta_{\alpha\beta} A_0 + i d_{\alpha\beta\gamma} A_\gamma \tag{56.l}$$

$$[N_\alpha, A_0] = 0 \tag{56.m}$$

$$[A_\alpha, \; A_\beta] = 0 \tag{56.n}$$

$$[A_\alpha, \; A_0] = 0 \tag{56.o}$$

$$[A_\alpha, \; P_\beta] = 2i(f_{\alpha\beta\gamma}M_\gamma - d_{\alpha\beta\gamma}N_\gamma) + 2iS \tag{56.p}$$

$$[S, \; M_\alpha] = 0 \tag{56.q}$$

$$[S, \; N_\alpha] = 0 \tag{56.r}$$

$$[S, \; P_\alpha] = iP_\alpha \tag{56.s}$$

$$[S, \; P_0] = iP_0 \tag{56.t}$$

$$[S, \; A_\alpha] = -iA_\alpha \tag{56.u}$$

$$[S, \; A_0] = -iA_0 \tag{56.v}$$

$$[A_\alpha, \; P_0] = -2iN_\alpha \tag{56.w}$$

$$[A_0, \; P_\alpha] = 2iN_\alpha \tag{56.x}$$

$$[A_0, \; P_0] = -2iS. \tag{56.y}$$

The $f_{\alpha\beta\gamma}$ and $d_{\alpha\beta\gamma}$ are defined from the Λ_α matrices as follows:

$$[\Lambda_\alpha, \; \Lambda_\beta]_- = 2if_{\alpha\beta\gamma}\Lambda_\gamma \tag{57}$$

$$[\Lambda_\alpha, \; \Lambda_\beta]_+ = 2\delta_{\alpha\beta} + 2d_{\alpha\beta\gamma}\Lambda_\gamma \tag{58}$$

as made by Gell-Mann for the SU_3 group [12]. The commutation rules (56) can be written in another form, using the indices of Lorentz and SU_3 groups (see Appendix C). They can also be generalized to the $SU(n, n)$ group [13].

The irreducible representations of $SL(6,C)$ are defined in a recent paper [14]; they are labelled by two numbers $D(n, m)$ where n and m denote dimensions [15] of two irreducible representations of SU_6. The 12-dimensional representation of Eqs. (55.a) and (55.b) is reducible, namely $D(6,1) \oplus D(1,6)$. The P_μ's and $P_{\mu(m)}$'s are components of an irreducible tensor $D(\overline{6},6)$; in fact, according to (55.c) they transform $D(6,1)$ into $D(1,6)$ following the rule:

$$D(\overline{6},6) \otimes D(6,1) = D(1,6) \oplus D(35,6) \tag{59}$$

On the contrary, the A_μ's and $A_{\mu(m)}$'s transform as the representation $D(6,\overline{6})$. The operator S is a scalar under $SL(6,C)$. The following Table is readily obtained [16]. In Table 2, λ_α denotes a generator of U_3.

Let us compare now Tables I et II.

i) Under the $SL(2,C)$ group, the representations 4 and $\overline{4}$ are equivalent; this is no longer the case in $SL(6,C)$ where $D(6,1) \oplus D(1,6)$ is not equivalent to $D(\overline{6},1) \oplus D(1,\overline{6})$.

ii) Under SL(2,C) the vector currents $\bar{\psi}\gamma_\mu(1+\gamma_5)\psi$ and $\bar{\psi}\gamma_\mu(1-\gamma_5)\psi$ belong to the same representation $D_{\frac{1}{2}\frac{1}{2}}$; under SL(6,C) they belong to two different irreducible representations, namely $D(\bar{6},6)$ and $D(6,\bar{6})$.

iii) Inhomogeneous SL(2,C) was shown to be a « symplectic » part of SU(2,2). On the contrary, inhomogeneous SL(6,C) is not a « symplectic » part of SU(6,6). In fact, for the symplectic groups [*17*] Sp_{12}, the adjoint representation is 78-dimensional (instead of 106). We will investigate such groups later.

TABLE II

U(6,6)	12 ×12̄=	144				
SU(6,6)	12 ×12̄=	1	143			
SL(6,C) inh	12 ×12̄=	1	37	106		
SL(6,C)	12 ×12̄=	1	1	36	36′	70
	$\begin{bmatrix} D(6,1) \\ +D(1,6) \end{bmatrix} \times \begin{bmatrix} D(\bar{6},1) \\ +D(1,\bar{6}) \end{bmatrix} =$	D(1,1)	D(1,1)	D(6,6̄)	D(6̄,6)	D(35,1)+D(1,35)
	ψ × $\bar{\psi}$	$\bar{\psi}\psi$	$\bar{\psi}\gamma_5\psi$	$\bar{\psi}\gamma_\mu(1-\gamma_5)\lambda_\alpha\psi$	$\bar{\psi}\gamma_\mu(1+\gamma_5)\lambda_\alpha\psi$	$\bar{\psi}\sigma_{\mu\nu}\lambda_\alpha\psi$
SU₆	$(6+6)\times(\bar{6}+\bar{6})$	1	1	35+1	35+1	35+35

The following question arises: is it possible to consider the « enlarged Poincaré group » inh SL(6,C) as a limit of an « enlarged de Sitter group » ρ Following considerations of Section I, we are led to see if the operators:

$$X_\alpha = \frac{1}{2}(P_\alpha - \beta A_\alpha) \qquad (60)$$

and

$$X_0 = \frac{1}{2}(P_0 - \beta A_0) \qquad (61)$$

can be adjoined to the SL(6,C) generators in order to get a Lie algebra. The answer is « no ». This can be easily seen in calculating the commutator $[N_\alpha, X_\beta]$. The coefficients *d* were equal to zero in SL(2,C) but it is no longer true in SL(6,C).

The reason why it is impossible to get a group in this way comes from

the fact that the P's and the A's do not belong to the same irreducible representation of SL(6,C) [see remark *ii*) above]. We can avoid this difficulty by « inhomogeneizing » SL(6,C) in another way, as it has already been suggested elsewhere [*14*]. Instead of 36 P's and 36 A's, wa can define 400 translation operators P and 400 operators A belonging both to the representation D(20,20) of SL(6,C). Such a choice leads to a 470 parameter group as an enlarged Poincaré group, but some difficulties concerning the parity operators can be avoided [*14*] [*16*].

In order to generalize the de Sitter groups, we can take the « symplectic part » of SU(6,6) as mentioned above. Unfortunately, such groups (as their « Poincaré » limit) do not seem interesting from the physical point of view, as we shall show.

According to results of Appendix B, sympletic parts Sp_{12} of SU(6, 6) contain SU_6 as a subgroup; the representation 12 of Sp_{12} reduces to $6 + \bar{6}$ with respect to SU_6. According to the Sp_{12} rule:

$$12 \times 12 = 1 + 65 + 78 \tag{62}$$

it can be proved that the representation 65 and the adjoint representation 78 reduce into:

$$65 = 35 + 15 + \overline{15} \tag{63}$$

$$78 = 35 + 21 + \overline{21} + 1 \tag{64}$$

with respect to SU_6. Then, we get the following Table:

TABLE III

U(6,6)	12	$\times \overline{12} =$		144	
SU(6,6)	12	$\times \overline{12} = 1$		143	
Sp_{12}	12	$\times 12 = 1$		65	78
SU_6	$(6 + \bar{6}) \times (6 + \bar{6}) = 1$			$35 + 15 + \overline{15}$	$35 + 21 + \overline{21} + 1$

By comparing Tables II and III, it appears that the SU_6 group contained in the symplectic part of SU(6,6) is not the same as that which is contained in SL(6,C). Moreover, Sp_{12} has 78 generators, 35 of which are those of SU_6:

the 43 other ones correspond to the decomposition $21 + \overline{21} + 1$. Among them we have to find the usual P_μ's; according to the $SU_3 \otimes SU_2$ decomposition:

$$21 = (\bar{3}, 1) + (6, 3)$$

it appears impossible to attribute to the four P_μ's the same behaviour with respect to the SU_3 group; for this reason we think that these generalized de Sitter groups cannot be seriously considered as candidates for physics [18]. The two groups which could be used in physics are the inhomogeneous SL(6,C) group and SU(6,6), already investigated in various ways [19]. The main advantage of SU(6, 6) is that it is a simple group.

ACKNOWLEDGEMENTS

Note added in proof : Some authors have already defined the lie algebra of SU(2, 2) in terms of γ Dirac algebra. See for instance H. A. KASTRUP, *Annalen der Phyisk*, 1962, **9**, 388.

APPENDIX A

LIE ALGEBRA OF THE PSEUDOUNITARY GROUPS

Let $M^*(n, m)$ be a complex vector space where a pseudohermitian metric g is defined:

$$(x, y) = - \sum_{i=1}^{n} x_i^* y_i + \sum_{i=n+1}^{n+m} x_i^* y_i. \tag{A.1}$$

To each vector x, the metric g associates an element $\bar{x}$ of the vector dual space, such that:

$$\bar{x}(y) = (x, y) \tag{A.2}$$

with the following properties:

$$\overline{\alpha x} = \alpha^* \bar{x} \tag{A.3}$$

$$(x, y) = (y, x)^* \tag{A.4}$$

where α is a complex number and the star denotes the complex conjugate.

It can be proved that every linear operator on M^* may be written as a sum of dyads $a\bar{b}$. Given a non-singular linear operator A acting on M^*, we define:

i) the adjoint operator A^+

$$(A^+ x, y) = (x, Ay). \tag{A.5}$$

The adjoint operator of $a\bar{b}$ is $b\bar{a}$

$$(a\bar{b})^+ = b\bar{a}. \tag{A.6}$$

[Note the property:

$$(\alpha A)^+ = \alpha^* A^+] \tag{A.7}$$

ii) a pseudohermitian operator A is an operator satisfying:

$$A^+ = A \tag{A.8}$$

$a\bar{a}$ is a pseudohermitian operator.

In the same manner, we define a pseudoantihermitian operator A as an operator satisfying:

$$A^+ = - A \tag{A.9}$$

$a\bar{b} - b\bar{a}$ is a pseudoantihermitian operator.

iii) A pseudounitary operator U is such that

$$(Ux, Uy) = (x, y) \tag{A.10}$$

or, equivalently,

$$U^+ = U^{-1}. \tag{A.11}$$

It can be easily proved (like in Section I for the pseudo-orthogonal groups) that a pseudounitary transformation U can be put in the form

$$U = e^{iA} \tag{A.12}$$

where A is a pseudohermitian operator. In fact, according to (A.7) and (A.11), we have:

$$U^+ = e^{-iA^+} = U^{-1} = e^{-iA}$$

$$A = A^+. \tag{A.13}$$

Given a basis e_i of the space $M^*(n, m)$, we can write the generators of the $U(n, m)$ group. They are

$$M_{ij} = i(e_i \bar{e}_j - e_j \bar{e}_i) = - M_{ji}$$
$$N_{ij} = \quad e_i \bar{e}_j + e_j \bar{e}_i = + N_{ji} \tag{A.14}$$

and the commutation rules can be readily obtained according to the rule:

$$(e_i \bar{e}_j)(e_k \bar{e}_l) = g_{jk} e_i \bar{e}_l \tag{A.15}$$

$$[M_{ij}, M_{kl}] = \quad i(g_{jk} M_{il} - g_{jl} M_{ik} - g_{ik} M_{jl} + g_{il} M_{jk})$$
$$[M_{ij}, N_{kl}] = \quad i(g_{jk} N_{il} + g_{jl} N_{ik} - g_{ik} N_{jl} - g_{il} N_{ik})$$
$$[N_{ij}, N_{kl}] = - i(g_{jk} M_{il} + g_{jl} M_{ik} + g_{ik} M_{jl} + g_{il} M_{jk}). \tag{A.16}$$

We readily see how $L(n, m)$ is a subgroup of $U(n, m)$ since the M_{ij}^*'s generate $L(n, m)$.

The group $SU(n, m)$ is a subgroup of $U(n, m)$ and is generated by the M_{ij}'s and the

$$N'_{ij} = e_i \bar{e}_j + e_j \bar{e}_i - \frac{2}{n} g_{ij} \sum_k e_k \bar{e}_k.$$

The group $SL(n + m, C)$ is generated by the M_{ij}'s, the N'_{ij}s' and the iM_{ij}'s and iN'_{ij}'s.

APPENDIX B

« DE SITTER » SUBGROUPS OF SU(n,n)

It has been shown in Section II that if we define SU(2,2) as the group of complex 4×4 matrices which leave invariant the pseudohermitian form $\bar{\psi}\psi = \bar{\psi}\gamma_0\psi$ with

$$\gamma_0 = \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & 0 \\ \hline \end{array} \qquad (B.1)$$

the two de Sitter subgroups S^β of SU(2,2) are those which leave invariant the symplectic metric:

$$h_\beta = \begin{array}{|c|c|} \hline \sigma_2 & \\ \hline & \beta\sigma_2 \\ \hline \end{array} \qquad (B.2)$$

where $\beta = \pm 1$. Moreover, for $\beta = 0$, one gets the Poincaré group.

The most natural generalization consists in defining the de Sitter subgroups $S^{\beta_1\beta_2\cdots\beta_n}$ of SU(n, n) by replacing γ_0 by:

$$\Gamma_0 = \begin{array}{|cccc|} \hline \gamma_0 & & & \\ & \gamma_0 & & \\ & & \ddots & \\ & & & \gamma_0 \\ \hline \end{array} \qquad (B.3)$$

and h_β by:

$$H_{\beta_1\beta_2\cdots\beta_n} = \begin{array}{|cccc|} \hline h_{\beta_1} & & & \\ & h_{\beta_2} & & \\ & & \ddots & \\ & & & h_{\beta_n} \\ \hline \end{array}$$

APPENDIX C

ANOTHER SET OF COMMUTATION RULES OF SU(6,6)

In order to normalize the 6×6 matrices generating SU_6, we choose the 35 following matrices:

$$1_3 \otimes \sigma^i, \qquad \sqrt{\frac{3}{2}} \lambda^m \otimes 1_2, \qquad \sqrt{\frac{3}{2}} \lambda^m \otimes \sigma^i. \qquad (C.1)$$

In these relations, 1_3 is the unit 3×3 matrix, 1_2 the unit 2×2 matrix, λ^m are the 3×3 matrices generating SU_3, σ^i the 2×2 matrices generating SU_2.

With such a choice, the generators of $SU(6,6)$ can be put in the form of the following 12×12 matrices.

a) *the 35 generators of* SU_6

$$M^{ij} = \frac{1}{2} \varepsilon^{ijk} \begin{pmatrix} \sigma^k \otimes 1_3 & 0 \\ 0 & \sigma^k \otimes 1_3 \end{pmatrix} \qquad (C.2)$$

$$M^{(m)} = \frac{1}{2} \sqrt{\frac{3}{2}} \varepsilon^{ijk} \begin{pmatrix} 1_2 \otimes \lambda^m & 0 \\ 0 & 1_2 \otimes \lambda^m \end{pmatrix} \qquad (C.3)$$

$$M^{ij(m)} = \frac{1}{2} \sqrt{\frac{3}{2}} \varepsilon^{ijk} \begin{pmatrix} \sigma^k \otimes \lambda^m & 0 \\ 0 & \sigma^k \otimes \lambda^m \end{pmatrix} \qquad (C.4)$$

b) *the 35 generators of « pure Lorentz » transformations*

$$M^{ok} = \frac{1}{2} \begin{pmatrix} i\sigma^k \otimes 1_3 & 0 \\ 0 & -i\sigma^k \otimes 1_3 \end{pmatrix} \qquad (C.5)$$

$$M^{ok(m)} = \frac{1}{2} \sqrt{\frac{3}{2}} \begin{pmatrix} i\sigma^k \otimes \lambda^m & 0 \\ 0 & -i\sigma^k \otimes \lambda^m \end{pmatrix} \qquad (C.6)$$

$$N^{(m)} = \frac{1}{2} \sqrt{\frac{3}{2}} \begin{pmatrix} i1_2 \otimes \lambda^m & 0 \\ 0 & -i1_2 \otimes \lambda^m \end{pmatrix} \qquad (C.7)$$

c) *the « translation » operators*

$$P^\mu = \begin{pmatrix} 0 & 0 \\ \sigma^\mu \otimes 1_3 & 0 \end{pmatrix} \qquad (C.8)$$

$$P^{\mu(m)} = \sqrt{\frac{3}{2}} \begin{pmatrix} 0 & 0 \\ \sigma^\mu \otimes \lambda^m & 0 \end{pmatrix} \qquad (C.9)$$

d) *the « acceleration » operators*

$$A^\mu = \begin{array}{|c:c|} \hline 0 & \overline{\sigma}^\mu \otimes 1_3 \\ \hdashline 0 & 0 \\ \hline \end{array} \qquad (\text{C}.10)$$

$${}^{(n)} = \sqrt{\frac{3}{2}} \begin{array}{|c:c|} \hline 0 & \overline{\sigma}^\mu \otimes \lambda^m \\ \hdashline 0 & 0 \\ \hline \end{array} \qquad (\text{C}.11)$$

e) *the scale transformation*

$$S = \frac{1}{2} \begin{array}{|c:c|} \hline -i & \\ \hdashline & i \\ \hline \end{array} \qquad (\text{C}.12)$$

In these formulae, $\sigma^\mu = (1, \vec{\sigma})$ and $\overline{\sigma}^\mu = (1, -\vec{\sigma})$. The following commutation rules can be derived:

$$[M^{\mu\nu}, \quad M^{\rho\lambda}] = i(g^{\nu\rho}M^{\mu\lambda} - g^{\nu\lambda}M^{\mu\rho} - g^{\mu\rho}M^{\nu\lambda} + g^{\mu\lambda}M^{\nu\rho})$$

$$[M^{\mu\nu}, \quad P^\rho] = i(g^{\nu\rho}P^\mu - g^{\mu\rho}P^\nu)$$

$$[P^\mu, \quad P^\nu] = 0$$

$$[P^\mu, \quad P^{\nu(m)}] = 0$$

$$[P^{\mu(m)}, \quad P^{\nu(n)}] = 0$$

$$[M^{\mu\nu}, \quad A^\rho] = i(g^{\nu\rho}A^\mu - g^{\mu\rho}A^\nu)$$

$$[A^\mu, \quad A^\nu] = 0$$

$$[A^\mu, \quad A^{\nu(m)}] = 0$$

$$[A^{\mu(m)}, \quad A^{\nu(n)}] = 0$$

$$[M^{(m)}, \quad M^{(n)}] = i\sqrt{\frac{3}{2}} f^{mnp} M_{(\,}$$

$$[M^{\mu\nu}, \quad M^{(m)}] = 0$$

$$[M^{\mu\nu}, \quad N^{(m)}] = 0$$

$$[P^\mu, \quad M^{(m)}] = 0$$

$$[A^\mu, \quad M^{(m)}] = 0$$

$$[N^{(m)}, \quad P^\rho] = iP^{\rho(m)}$$

$$[N^{(m)}, \quad A^\rho] = iA^{\rho(m)}$$

$$[M^{\mu\nu}, \quad M^{\rho\lambda(m)}] = i(g^{\nu\rho}M^{\mu\lambda(m)} - g^{\nu\lambda}M^{\mu\rho(m)} - g^{\mu\rho}M^{\nu\lambda(m)} + g^{\mu\lambda}M^{\nu\rho(m)})$$

$$[M^{\mu\nu(m)}, \quad M^{\rho\lambda(n)}] = i\delta^{mn}(g^{\nu\rho}M^{\mu\lambda} - g^{\nu\lambda}M^{\mu\rho} - g^{\mu\rho}M^{\nu\lambda} + g^{\mu\lambda}M^{\nu\rho})$$

$$+ i\sqrt{\frac{3}{2}}f^{mnp}(g^{\mu\rho}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\rho})M^{(p)}$$

$$+ i\sqrt{\frac{3}{2}}d^{mnp}\big(g^{\nu\rho}M^{\mu\lambda}{}_{(p)} - g^{\nu\lambda}M^{\mu\rho}{}_{(p)} - g^{\mu\rho}M^{\nu\lambda}{}_{(p)} + g^{\mu\lambda}M^{\nu\rho}{}_{(p)}\big)$$

$$[M^{\mu\nu(m)}, \quad M^{(n)}] = if^{mnp}M^{\mu\nu}{}_{(p)}$$

$$[M^{\mu\nu(m)}, \quad N^{(n)}] = \frac{i}{2}\sqrt{\frac{3}{2}}\,\varepsilon^{\mu\nu\rho\lambda}f^{mnp}M_{\rho\lambda(p)}$$

$$[M^{(m)}, \quad N^{(n)}] = i\sqrt{\frac{3}{2}}f^{mnp}N_{(p)} \qquad\qquad (C.13)$$

$$[N^{(m)}, \quad N^{(n)}] = -i\sqrt{\frac{3}{2}}f^{mnp}M_{(p)}$$

$$[M^{\mu\nu}, \quad P^{\rho(m)}] = i(g^{\nu\rho}P^{\mu(m)} - g^{\mu\rho}P^{\nu(m)})$$

$$[M^{\mu\nu}, \quad A^{\rho(m)}] = i(g^{\nu\rho}A^{\mu(m)} - g^{\mu\rho}A^{\nu(m)})$$

$$[M^{\mu\nu(m)}, \quad P^\rho] = i(g^{\nu\rho}P^{\mu(m)} - g^{\mu\rho}P^{\nu(m)})$$

$$[M^{\mu\nu(m)}, \quad A^\rho] = i(g^{\nu\rho}A^{\mu(m)} - g^{\mu\rho}A^{\nu(m)})$$

$$[M^{\mu\nu(m)}, \quad P^{\rho(n)}] = i\delta^{mn}(g^{\nu\rho}P^\mu - g^{\mu\rho}P^\nu) + i\sqrt{\frac{3}{2}}\,\varepsilon^{\mu\nu\rho\lambda}f^{mnp}P_{\lambda(p)}$$

$$+ i\sqrt{\frac{3}{2}}d^{mnp}\big(g^{\nu\rho}P^\mu{}_{(p)} - g^{\mu\rho}P^\nu{}_{(p)}\big)$$

$$[M^{\mu\nu(m)}, \quad A^{\rho(n)}] = i\delta^{mn}(g^{\nu\rho}A^\mu - g^{\mu\rho}A^\nu) - i\sqrt{\frac{3}{2}}\,\varepsilon^{\mu\nu\rho\lambda}f^{mnp}A_{\lambda(p)}$$

$$+ i\sqrt{\frac{3}{2}}d^{mnp}\big(g^{\nu\rho}A^\mu{}_{(p)} - g^{\mu\rho}A^\nu{}_{(p)}\big)$$

$$[M^{(m)}, \quad P^{\mu(n)}] = i\sqrt{\frac{3}{2}}f^{mnp}P^\mu{}_{(p)}$$

$$[M^{(m)}, \quad A^{\mu(n)}] = i\sqrt{\frac{3}{2}}f^{mnp}A^\mu{}_{(p)}$$

$$[N^{(m)}, \quad P^{\mu(n)}] = -i\delta^{mn}P^\mu - i\sqrt{\frac{3}{2}}d^{mnp}P^\mu{}_{(p)}$$

$$[N^{(m)}, \quad A^{\mu(m)}] = -i\delta^{mn}A^\mu - i\sqrt{\frac{3}{2}}d^{mnp}A^\mu{}_{(p)}$$

$$[S, \quad M^{\mu\nu}] = 0$$

$$[S, \quad M^{(m)}] = 0$$

$$[S, \quad N^{(m)}] = 0$$

$$[S, \quad M^{\mu\nu(m)}] = 0$$

$$[S, \quad P^{\mu}] = iP^{\mu}$$

$$[S, \quad A^{\mu}] = -iA^{\mu}$$

$$[S, \quad P^{\mu(m)}] = iP^{\mu(m)}$$

$$[S, \quad A^{\mu(m)}] = -iA^{\mu(m)}$$

$$[A^{\mu}, \quad P^{\nu}] = 2i(M^{\mu\nu} - g^{\mu\nu}S)$$

$$[A^{\mu}, \quad P^{\nu(m)}] = 2i(M^{\mu\nu(m)} + g^{\mu\nu}N^{(m)})$$

$$[A^{\mu(m)}, \quad P^{\nu}] = 2i(M^{\underline{\mu\nu(m)}} + g^{\mu\nu}N^{(m)})$$

$$[A^{\mu(m)}, \quad P^{\nu(n)}] = -2ig^{\mu\nu}\delta^{mn}S + 2i\sqrt{\frac{3}{2}}g^{\mu\nu}d^{mnp}N_{(p)} - 2i\sqrt{\frac{3}{2}}\bar{g}^{\mu\nu}f^{mnp}M_{(p)}$$

$$+ 2i\sqrt{\frac{3}{2}}d^{mnp}M^{\mu\nu}{}_{(p)} + 2i\delta^{mn}M^{\mu\nu} + i\sqrt{\frac{3}{2}}f^{mnp}\epsilon^{\mu\nu\rho\lambda}M_{\rho\lambda(p)}.$$

(C.13)

REFERENCES AND FOOTNOTES

[1] The conformal group was first introduced by H. BATEMAN, *Proc. Lond. Math. Soc.*, 8, 1910, 223 and E. CUNNINGHAM, *Proc. Lond. Math. Soc.*, 8, 1910, 77

Invariance of the Maxwell equations leads to the conformal group so far as space time is assumed to be flat.   For a more general case, see T. FULTON, F. ROHRLICH and L. WITTEN, *Rev. Mod. Phys.*, 34, 1962, 442.

See also J. WESS, *Nuovo Cimento*, 18, 1960, 1086.

[2] B. SAKITA, *Phys. Rev.*, 136, 1964, B1756.

F. GÜRSEY and L. A. RADICATI, *Phys. Rev. Letters*, 13, 1964, 173.

H. BACRY and J. NUYTS, *Phys. Letters*, 12, 156 and 13, 1964, 359.

[3] We discard transformations including « space » or « time » inversions.

[4] Dyads are very convenient in the investigation of the Lorentz group. See, for instance, H. BACRY, Thèse, *Annales de Physique*, 8, 1963, 197.

Note that from a rigorous point of view, a dyad is defined as follows.

To each vector x of the vector space $M(n, m)$ the metric g associates an element $\bar{x}$ of the dual space such that

$$\bar{x}(y) = (x, y).$$

A dyad is then defined as a product $x\bar{y}$ where $\bar{y}$ maps $M(n, m)$ on the field of real numbers R, and x maps R on $M(n, m)$.   With such notations the pseudoantisymmetric operator $a \wedge b$ has to be written

$$a\bar{b} - b\bar{a}.$$

See, for instance, J. M. SOURIAU, *Calcul linéaire* (Paris, Presses Universitaires, 1960).

[5] Obviously the case where $f = 0$ is not considered here.

[6] The covering group is locally isomorphic (see ref. [8]) but is simply connected.   Here, we have

$$SO_3 = SU_2/Z_2.$$

It can be easily proved that SO_3 is double connected.   See, for instance, D. SPEISER, Lectures given at the Istanbul Summer School (1962), to be published.

[7] In Appendix A we give a simple derivation of the Lie algebra of the U(n, m) groups.

[8] Two groups are locally isomorphic if they have the same Lie algebra. Examples: SO_3 and SU_2, L(3,1) and SL(2,C), L(4,2) and SU(2,2).

[9] The representation 5 of the Poincaré group is well known. Every transformation (Λ, a) can be put in the form of a 5×5 matrix

$$\begin{array}{c|c} \Lambda & a \\ \hline 0 & 1 \end{array}$$

[10] It is the natural generalization if we choose a « triplet model » of SU_3 like the quark model [M. GELL-MANN, *Phys. Letters*, 8, 1964, 214 and G. ZWEIG, Cern preprints TH. 401 and 412, 1964]. If we choose the eightfold way model based on SU_3/Z_3, we have to divide all groups of chain (53) by Z_3. In order to generalize the chain (52 a) we have to divide the groups of chain (53) by Z_6. See for such a discussion H. BACRY and J. NUYTS, ref. [2].

[11] The group generated by the operators $\Gamma \otimes \lambda$ of the current *densities* is then U(6,6) which is distinct from the group proposed and discussed by R. P. FEYNMAN, M. GELL-MANN and G. ZWEIG in *Phys. Rev. Letters* (to be published), « The group U(6) $\times$ U(6) generated by current components. »

[12] M. GELL-MANN, *Phys. Rev.*, 125, 1962, 1067.

[13] For the SU(n, n) group, we have to interpret the Λ_α matrices in (57) and (58) as the set of the ($n^2 - 1$) traceless Hermitian $n \times n$ matrices. The coefficient of $\delta_{\alpha\beta}$ in (58) is arbitrary and depends on the choice of the normalization of the Λ matrices. Only the factors of $\delta_{\alpha\beta}P_0$ in (56.f) of $\delta_{\alpha\beta}A_0$ in (56.l) and of S in (56.y) would be affected by this choice.

[14] H. BACRY and J. NUYTS, « Remarks on an enlarged Poincaré group: inhomogeneous SL(6,C) group » (to be published in *Nuovo Cimento*).

[15] Obviously, the dimension is not sufficient to characterize a representation of SU_6 but no confusion can be made in our paper.

[16] Another group of 142 parameters has been investigated by W. RÜHL (to be published), « Baryons and mesons in a theory which combines relativistic invariance with SU(6) symmetry ». The generators of this group are those of (56) except S and the commutation rules are those of (56) except that the A's commute with the P's. The operators A are denoted $\tilde{P}$ in Rühl's paper.

[17] We call symplectic groups all groups which are built on the Lie algebra C_l (in Cartan's notation). In the case which is considered in this paper $l = 6$.

[18] The same conclusion can of course be formulated for their « Poincaré limits ».

[19] See ref. [14] [16]. See also T. FULTON and J. WESS, *Phys. Letters*, 14, 1965, 57. R. DELBURGO, A. SALAM and J. STRATHDEE, « U(12) and broken SU(6) symmetry », « $\tilde{U}$(12) and the baryon form factor », preprints Trieste (January 1965). Note that the group investigated in Trieste is the group U(6,6). But it is not interpreted by these authors as a generalized conformal group.