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On generalizations of Gödel's Universe

by

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ABSTRACT. — The stationary and non-static generalizations of Gödel's universe filled with perfect and imperfect fluid distributions are considered. It is found that there exist stationary generalizations with perfect and imperfect fluid distributions. But the non-static generalization with perfect fluid is impossible in the scheme presented here, while that with imperfect fluid is permissible. One such model is investigated in detail.

I. — INTRODUCTION

The cosmological problem deals with the study of smoothed out metrical structure of our universe, the local details being neglected. The first systematic approach towards handling this problem was initiated by Einstein (1917), with the help of the General Theory of Relativity. Along the same line of thinking the standard relativistic non-static cosmological models of the universe were obtained under certain assumptions known as Weyl's postulate and the cosmological principle. But it is well-known that the growth of our knowledge in the field of astrophysics led to several difficulties regarding the age of the universe and the formation of galaxies. This fact gave rise to several attempts at modification of the above mentioned assumptions regarding the physical structure of the universe and one such attempt was to weaken Weyl's postulate by the introduction of shear and rotation.

Gödel [1] was the first to present a solution of Einstein's field equations exhibiting universal rotation. He obtained an interesting model of a homogeneous, anisotropic but stationary universe. His line-element can be written in the following form:

$$(1.1)\quad ds^2 = (dx^0)^2 + 2e^{x^1}(dx^0)(dx^2) + \frac{1}{2}e^{2x^1}(dx^2)^2 - a^2(dx^1)^2 - (dx^3)^2,$$

where a is a constant. It represents a rotating and shearless universe. Because of its stationary character it does not give rise to any red-shift and hence it is just a curiosity like the original Einstein's static model but it is of greatest interest for the interpretation of Einstein's theory of gravitation.

Heckmann and Schücking [2] extended these ideas to non-static models with a view of obtaining universes with non-singular origin, in particular, universes which oscillate between finite radii. The non-static model showing rotation and shear which they investigated has the following line-element:

$$(1.2)\quad \begin{aligned} ds^2 = (dx^0)^2 + 2e^{x^1}(dx^0)(dx^2) + \alpha C_{11}e^{2x^1}(dx^2)^2 \\ - C_{11}(dx^1)^2 - S^2(dx^3)^2 - 2C_{12}e^{x^1}(dx^1)(dx^2), \end{aligned}$$

where α is a constant and C_{11}, C_{12} and S^2 are the functions of time x^0. They studied this metric-form in connection with the incoherant matter but could not succeed in determining the behaviour of the unknown functions C_{11}, C_{12} and S^2 upto sufficient degree of requirement. In what follows we have attempted to study a line-element similar to the line-element (1.2) with a view to obtain possible generalizations of Gödel's universe. Our line-element is of the form

$$(1.3)\quad \begin{aligned} ds^2 = (dx^0)^2 + 2e^{x^1}(dx^0)(dx^2) + ae^{2x^1}(dx^2)^2 \\ + b(dx^1)^2 + c(dx^3)^2 + 2fe^{x^1}(dx^1)(dx^2) \end{aligned}$$

where a, b, c and f are the functions of time x^0.

II. — A STATIONARY SOLUTION

Consider the metric-form

$$(2.1)\quad \begin{aligned} ds^2 = (dx^0)^2 + 2e^{x^1}(dx^0)(dx^2) + ae^{2x^1}(dx^2)^2 \\ + b(dx^1)^2 - (dx^3)^2 + 2fe^{x^1}(dx^1)(dx^2), \end{aligned}$$

with a, b and f as unknown constants. For the space-time given by (2.1), the non-vanishing components of the Einstein tensor G_{ij} connected with the Ricci tensor of the space-time R_{ij} by the relation

$$G_{ij} = R_{ij} - \frac{1}{2} R g_{ij},$$

are given by

$$
\begin{aligned}
G_{00} &= \frac{(1-4a)}{4\alpha}, \\
G_{02} &= \frac{(1-4a)}{4\alpha} e^{x^1}, \\
G_{11} &= \frac{b}{4\alpha}, \\
G_{12} &= \frac{f}{4\alpha} e^{x^1}, \\
G_{22} &= -\frac{3a}{4\alpha} e^{2x^1}, \\
G_{33} &= \frac{(4a-3)}{4\alpha}
\end{aligned}
\tag{2.2}
$$

where $\alpha = b(a-1) - f^2$.

One can regard this metric to give the distribution of an imperfect fluid, the pressure in the x^3-direction being different from that in x^1 and x^2-directions. With the help of Lichnerowicz's [3] energy-momentum tensor for the anisotropic fluid distribution the required form of T_{ij} can be written as

$$
\begin{aligned}
T_{ij} &= (\rho + p)v_i v_j - p g_{ij} + (q - p)V_i V_j, \\
v_i v^i &= 1, \qquad V_i V^i = -1,
\end{aligned}
\tag{2.3}
$$

where q is the pressure in x^3-direction, p is the pressure in x^1 or x^2-directions and ρ is the density of matter.

From (2.2) and (2.3), with the help of the field equations of general relativity

$$G_{ij} = -8\pi T_{ij}, \tag{2.4}$$

we find that

$$
\begin{aligned}
8\pi\rho &= \frac{(4a-1)}{4\alpha}, \\
8\pi p &= \frac{1}{4\alpha}, \\
8\pi q &= \frac{(3-4a)}{4\alpha}, \\
v^i &= \delta^i_0, \\
V^i &= \delta^i_3.
\end{aligned}
\tag{2.5}
$$

The signature-requirement of the line-element (2.1) and the requirement that ρ is positive and p and q are non-negative lead to the following restrictions on various constants:

$$(2.6)\qquad \frac{1}{4} < a \leqslant \frac{3}{4}, \quad -\infty < b < 0, \quad 0 \leqslant f^2 < b(a-1).$$

If ω^i, θ and q_{ij} are respectively the angular velocity-vector, scalar of expansion and components of shear-tensor, we have

$$(2.7)\qquad \begin{aligned} \omega^i &= \frac{1}{2}(-g)^{-1/2}\epsilon^{ijkm}(v_{j;k} - v_{k;j})v_m, \\ \theta &= \frac{1}{3}v^i{}_{;i}, \\ q_{ij} &= \frac{1}{2}(v_{i;j} + v_{j;i}) - \frac{1}{3}(g_{ij} - v_i v_j)v^k{}_{;k}, \end{aligned}$$

where ϵ^{ijkm} is a totally skew symmetric tensor with $\epsilon^{0123} = 1$. In the above solution we find that

$$(2.8)\qquad \omega^i = \delta^i_3 \alpha^{-1/2}, \qquad \theta = 0, \qquad q_{ij} = 0.$$

From (2.6) and (2.8), it follows that the general solution (2.1) represents a family of universes which are homogeneous, anisotropic but stationary and filled with the fluid of constant density and pressure. These models are shearless and possess constant angular velocity.

If a lies in the range $\frac{1}{4} < a < \frac{1}{2}$ or $\frac{1}{2} < a \leqslant \frac{3}{4}$, this solution describes a family of Synge-type universes [4] filled with imperfect fluid. But when $a = \frac{1}{2}$, we find $p = q$ and it describes a family of Gödel-type universes filled with perfect fluid. In this case (2.1) can be transformed into the form

$$(2.9)\qquad \begin{aligned} ds^2 = A^2\Big[(dx^0)^2 &+ 2e^{x^1}(dx^0)(dx^2) + \frac{1}{2}e^{2x^1}(dx^2)^2 \\ &- (dx^1)^2 - (dx^3)^2 + 2\left(\frac{B}{1-B}\right)^{1/2}(dx^0)(dx^1)\Big], \end{aligned}$$

where A and B are constants such that $-\infty < A < +\infty$ and $0 \leqslant B < \frac{1}{2}$.

Every member of the family of universes given by (2.9) represents a rotating universe filled with perfect fluid, the pressure and density being given by

$$(2.10)\qquad 8\pi\rho = \frac{(1-B)}{(1-2B)} \cdot \frac{1}{2A^2} = 8\pi p.$$

It can be seen that Gödel's universe is a member of this family corresponding to $B = 0$ and is a member with least density [5].

III. — A NON-STATIC SOLUTION

Consider the metric form (1.3) with $f = 0$ i. e. the form

$$ds^2 = (dx^0)^2 + 2e^{x^1}(dx^0)(dx^2) + ae^{2x^1}(dx^2)^2 + b(dx^1)^2 + c(dx^3)^2, \tag{3.1}$$

here a, b and c are unknown functions of time x^0.

For the space-time given by (3.1), the non-vanishing components of the Einstein-tensor G_{ij} connected with the Ricci-tensor of the space-time R_{ij} by the relation

$$G_{ij} = R_{ij} - \frac{1}{2} R g_{ij}$$

are given by:

$$\begin{aligned}
G_{00} &= \frac{1}{2(a-1)}\left[\frac{\dot{b}^2}{2b^2} + \frac{\dot{c}^2}{2c^2} - \frac{\ddot{b}}{b} - \frac{\ddot{c}}{c} - \frac{(a-2)\dot{a}\dot{b}}{2b(a-1)} \right.\\
&\qquad\qquad \left. - \frac{(a-2)\dot{a}\dot{c}}{2c(a-1)} - \frac{a\dot{b}\dot{c}}{2bc} - \frac{(4a-1)}{2b}\right],\\
G_{10} &= \frac{1}{4(a-1)}\left[\frac{(2a-3)\dot{a}}{(a-1)} - \frac{(2a-1)\dot{b}}{b} + \frac{\dot{c}}{c}\right],\\
G_{20} &= \frac{a}{2(a-1)}\left[\frac{\dot{b}^2}{2b^2} + \frac{\dot{c}^2}{2c^2} - \frac{\ddot{b}}{b} - \frac{\ddot{c}}{c} + \frac{\dot{a}\dot{b}}{2ab(a-1)} \right.\\
&\qquad\qquad \left. + \frac{\dot{a}\dot{c}}{2ac(a-1)} - \frac{\dot{b}\dot{c}}{2bc} - \frac{(4a-1)}{2ab}\right] e^{x^1},\\
G_{11} &= \frac{ab}{2(a-1)}\left[\frac{\dot{a}^2}{2a(a-1)} + \frac{\dot{c}^2}{2c^2} - \frac{\ddot{a}}{a} - \frac{\ddot{c}}{c} - \frac{(a-2)\dot{a}\dot{c}}{2ac(a-1)} + \frac{1}{2ab}\right],\\
G_{21} &= \frac{a}{4(a-1)}\left[\frac{(a-2)\dot{a}}{a(a-1)} - \frac{\dot{b}}{b} + \frac{\dot{c}}{c}\right] e^{x^1}\\
G_{22} &= \frac{a^2}{2(a-1)}\left[\frac{\dot{b}^2}{2b^2} + \frac{\dot{c}^2}{2c^2} - \frac{\ddot{b}}{b} - \frac{\ddot{c}}{c} \right.\\
&\qquad\qquad \left. + \frac{\dot{a}\dot{b}}{2ab(a-1)} + \frac{\dot{a}\dot{c}}{2ac(a-1)} - \frac{\dot{b}\dot{c}}{2bc} - \frac{3}{2ab}\right] e^{2x^1},\\
G_{33} &= \frac{ac}{2(a-1)}\left[\frac{\dot{a}^2}{2a(a-1)} + \frac{\dot{b}^2}{2b^2} - \frac{\ddot{a}}{a} - \frac{\ddot{b}}{b} - \frac{(a-2)\dot{a}\dot{b}}{2ab(a-1)} - \frac{(4a-3)}{2ab}\right].
\end{aligned} \tag{3.2}$$

Here and in what follows an overhead dot denotes differentiation with regards to time x^0.

If we try to regard (3.2) as giving a perfect fluid distribution, we find that the function a has to satisfy two differential equations which are inconsistent.

However we can regard (3.2) as giving an imperfect fluid distribution, the pressure in x^3-direction being different from that in x^1 and x^2-directions. With the help of Lichnerowicz's energy-momentum tensor (2.3) for an anisotropic fluid distribution, it is easy to see that the field equations

$$G_{ij} = -8\pi T_{ij} \tag{3.3}$$

will be satisfied, if the unknown functions a, b and c are so chosen that they satisfy the relations

$$b = \alpha a, \quad c = \beta\left(\frac{a-1}{a}\right), \quad \frac{\dot{a}}{(a-1)} = \frac{A}{e^{1/a}}, \tag{3.4}$$

where α, β and A are constants of integration. The fact that V^i is a space-like vector and the signature-requirement of the line-element (3.1), respectively, require that

$$c < 0 \quad \text{and} \quad \alpha\beta < 0.$$

Then we shall find that

$$\begin{aligned}
&8\pi\rho = \frac{1}{4a}\left[(a+1)\frac{A^2}{e^{2/a}} + \frac{(4a-1)}{\alpha(a-1)}\right],\\
&8\pi p = -\frac{1}{4a}\left[(a+1)\frac{A^2}{e^{2/a}} - \frac{1}{\alpha(a-1)}\right],\\
&8\pi q = -\frac{1}{4a}\left[\frac{(3a^2+3a-4)}{a}\cdot\frac{A^2}{e^{2/a}} + \frac{(4a-3)}{\alpha(a-1)}\right],\\
&v^i = \delta^i_0,\\
&V^i = \delta^i_3\beta^{-1/2}a^{1/2}(1-a)^{-1/2},\\
&\omega^i = \delta^i_3(-\alpha\beta)^{-1/2}(a-1)^{-1},\\
&\theta = A/3e^{1/a},\\
&q_{11} = \frac{\alpha A(a-3)}{6e^{1/a}}, \quad q_{22} = \frac{A(a-1)}{6e^{1/a}}e^{2x^1}, \quad q_{33} = \frac{\beta A(a-1)(3-2a)}{6e^{1/a}a^2},
\end{aligned} \tag{3.5}$$

$q_{ij} = 0$ if $i \neq j$ and also if $i = j = 0$.

The surviving component of the angular velocity-vector ω^i implies that the matter rotates about x^3-axis.

Again we want that the pressures p and q be non-negative while the density ρ be positive. This fact firstly restricts a to the range $0 < a < 1$; so consequently we get

$$\beta > 0, \qquad \alpha < 0. \tag{3.6}$$

Next we observe that p, q and ρ are functions of a and $p = q = 0$ at $a = \xi$ where ξ is a real root of the cubic $x^3 - x^2 - 1 = 0$, its value lying between 0.754 and 0.755 while ρ vanishes at $a = \eta$ where η has a value between 0 and $\frac{1}{4}$. Thus in order that ρ be always positive and p and q non-negative, a must be restricted in the interval $\eta < a \leqslant \xi$. Further the constant $-\alpha A^2$ must be so chosen as to satisfy the following three inequalities:

$$\begin{array}{rll} 0 < -\alpha A^2 \leqslant \dfrac{e^{2/a}}{(1-a^2)} & \textit{if} & \eta < a \leqslant \dfrac{1}{4}, \\ 0 < -\alpha A^2 \leqslant \dfrac{e^{2/a}}{(1-a^2)} & \textit{if} & \dfrac{1}{4} \leqslant a \leqslant \dfrac{3}{4}, \\ \dfrac{a(4a-3)e^{2/a}}{(1-a)(4-3a-3a^2)} \leqslant -\alpha A^2 \leqslant \dfrac{e^{2/a}}{(1-a^2)} & \textit{if} & \dfrac{3}{4} \leqslant a < \xi. \end{array} \tag{3.7}$$

IV. — A PARTICULAR SOLUTION

As β is positive and not occurring in any g_{ij} except g_{33}, we can choose x^3 in such a way as to make $\beta = +1$. So the line-element (3.1) can be written in the form

$$ds^2 = (dx^0)^2 + 2e^{x^1}(dx^0)(dx^2) + ae^{2x^1}(dx^2)^2 + \alpha a(dx^1)^2 - \left(\frac{1-a}{a}\right)(dx^3)^2, \tag{4.1}$$

where

$$\dot{a}e^{1/a} = A(a-1). \tag{4.2}$$

We shall now take a particular case of the above non-static solution and work out the details. The line-element (4.1) has two disposable constants α and A. We choose these two constants by stipulating certain initial conditions.

Let us consider an expanding model. For such a model the scalar of expansion θ must be positive, that is A must be positive. Since $a < 1$, (4.2) shows that for an expanding model $\dot{a}$ must be negative and so a must decrease. Again equation (4.2) further shows that if $\dot{a}$ is once negative, it will always remain negative and vice versa. Therefore oscillating models are not included in the present scheme.

One can start with any value of a, say a_0, in the interval (η, ξ) and choose a negative initial value of $\dot{a}$. Then a will continually decrease from a_0. When a attains the value η, the model will cease to be physical. Further if we choose $a_0 = \frac{3}{4}$, the three inequalities (3.7) will, in this case, reduce to the following:

$$0 < -\alpha A^2 \leqslant \frac{16}{7} e^{8/3}. \tag{4.3}$$

We therefore choose $a_0 = \frac{3}{4}$ and stipulate that initially $p_0 = q_0$ [1]. We shall then find that the value of $-\alpha A^2$ is 31.43, correct upto two decimal places. This value of $-\alpha A^2$ satisfies inequality (4.3). As soon as $-\alpha A^2$ is determined one can find η, the lower limit of a from the expression for ρ in (3.5). It is found that η is 0.2477 approximately.

If the initial value of $\dot{a}$ is $-\dot{a}_0$, we further find from (4.2):

$$A \doteqdot (15.18)\dot{a}_0. \tag{4.4}$$

Hence the value of $-\alpha A^2$ implies that

$$-\alpha \doteqdot (0.1363)\dot{a}_0^{-2}. \tag{4.5}$$

Thus the line-element (4.1) is now completely determined in terms of initial values $\dot{a}_0$ and $a_0 = \frac{3}{4}$.

The model will cease to be physical when a reaches the value η at which ρ vanishes. The time-interval T for the function a to change from the initial value $\frac{3}{4}$ to the value η is given by

$$T = \frac{I}{A}, \tag{4.6}$$

where

$$I = \int_{\eta}^{\frac{3}{4}} \frac{e^{1/a} da}{(1-a)}. \tag{4.7}$$

Using the value 0.2477 for η, an approximate value of I is found to be 11.13. Consequently the time-interval T for the expansion of the model is approximately $(0.7331)\, \dot{a}_0^{-1}$. The model starts expanding at the time when $a = \frac{3}{4}$.

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[1] This stipulation is permissible only when $\frac{1}{2} < a_0 \leqslant \xi$.

The expansion can last upto the time before a equals η. At $a = \eta$, the density ρ vanishes and the model will loose physical significance.

As the function a tends towards η, the rate of expansion of the model gradually decreases and tends to the value

$$\frac{4}{3} e^{4/3} e^{-1/\eta} \dot{a}_0$$

which is, in this case, approximately equal to $(0.0978)\ \dot{a}_0$.

The magnitude of the surviving component of the angular velocity-vector is maximum when $a = \frac{3}{4}$ and decreases with a to a minimum value

$$\left| \left(\frac{22}{3}\right)^{1/2} \frac{\dot{a}_0}{(\eta - 1)} \right|,$$

that is approximately to the value $(3.6)\ \dot{a}_0$.

In case of the components of the shear-tensor, we observe that q_{11} is initially positive and decreases towards a positive value while q_{22} and q_{33} are initially negative and increase towards a negative value. This indicates that the shear of the model decreases along with the function a.

V. — CONCLUSION

We have given above a model of a rotating universe which is steadily expanding. Initially the three pressures are equal. If we assume an initial rate of expansion of the order of 10^{-29}, the model will have a life-time of the order of 10^{10} years, after which the density of the model will be negative and it looses physical significance. The model is filled with fluid whose pressure is non-isotropic. The geometry of the model is very much similar to that of Gödel's stationary model.

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