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Commutants of Certain Operator Algebras on Fock Space (*)

by

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H. Ekstein recently proposed to replace the usual assumptions on invariance under a group of internal symmetries by a simple postulate on the commutant of the S matrix [*1*]. For this he needs the following theorem which we propose to prove in this note.

THEOREM. — Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ the tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$, and $\mathcal{H}^{\vee n}$ and $\mathcal{H}^{\wedge n}$ the respective Hilbert spaces of symmetric and antisymmetric tensors of order n over $\mathcal{H}$ (i. e., the symmetric and antisymmetric parts, respectively, of $\mathcal{H}^{\otimes n} = \mathcal{H} \otimes \mathcal{H} \otimes \ldots \otimes \mathcal{H}$, the tensor product in which $\mathcal{H}$ appears n times as a factor). Furthermore, let $\mathcal{H}^{\vee}$ and $\mathcal{H}^{\wedge}$, respectively, denote the symmetric and Grassmann algebras over $\mathcal{H}$ [*7*], i. e.,

$$\mathcal{H}^{\vee} = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\vee n}, \tag{1}$$

$$\mathcal{H}^{\wedge} = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\wedge n}, \tag{2}$$

where $\mathcal{H}^{\vee}$ and $\mathcal{H}^{\wedge}$ are, respectively, the Fock spaces of bosons and fermions with wave functions in $\mathcal{H}$. Also, let $B_1 = L_1 \otimes I_{\mathcal{H}_2}$ and $B_2 = I_{\mathcal{H}_1} \otimes L_2$ be the von Neumann algebras over $\mathcal{H}$, where L_1 and L_2 are arbitrary bounded operators on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively; and let $B_1^{\vee n}$, $B_1^{\wedge n}$ ($B_2^{\wedge n}$

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and $B_2^{\wedge n}$) denote the von Neumann algebras induced in $\mathcal{H}^{\vee n}$ and $\mathcal{H}^{\wedge n}$, respectively, by $B_1(B_2)$. Analogously, let $B_1^{\vee}$ and $B_1^{\wedge}$ ($B_2^{\vee}$ and $B_2^{\wedge}$) denote the von Neumann algebras induced in $\mathcal{H}^{\vee}$ and $\mathcal{H}^{\wedge}$, respectively, by $B_1(B_2)$. *Then* $B_1^{\vee n}$ *and* $B_2^{\vee n}$ (*and likewise* $B_1^{\vee}$ *and* $B_2^{\vee}$) *are the commutants of one another*; *andcorresponding statements hold for* $B_1^{\wedge n}$ *and* $B_2^{\wedge n}$ (*and for* $B_1^{\wedge}$ *and* $B_2^{\wedge}$). That is,

(3) $$(B_1^{\vee n})' = B_2^{\vee n} \quad \text{in } \mathcal{H}^{\vee n},$$

(4) $$(B_1^{\vee})' = B_2^{\vee} \quad \text{in } \mathcal{H}^{\vee},$$

(5) $$(B_1^{\wedge n})' = B_2^{\vee n} \quad \text{in } \mathcal{H}^{\wedge n},$$

(6) $$(B_1^{\wedge})' = B_2^{\wedge} \quad \text{in } \mathcal{H}^{\wedge}.$$

In fact, the result needed in [*1*] refers to the action on $\mathcal{H}_2$ of the group SU(p) (where p, supposed finite, is the dimension of $\mathcal{H}_2$) but the problem of reduction in irreducible tensors is known to be the same for SU(p) and the general linear group. If the group is SO(p) instead of SU(p), it is known that there is a further decomposition (as shown in [*4*] and in chap. 10 of [*5*]).

Proof of the Theorem. — Let $\mathcal{H}_1^{\otimes n}$ and $\mathcal{H}_2^{\otimes n}$ be the tensor nth powers of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then it is obvious that

(7) $$\mathcal{H}^{\otimes n} = \mathcal{H}_1^{\otimes n} \otimes \mathcal{H}_2^{\otimes n}.$$

On the other hand, if S and A denote the symmetrizing and antisymmetrizing operators, respectively (i. e., the Hermitean projectors whose action on $\mathcal{H}^{\otimes n}$ produces $\mathcal{H}^{\vee n}$ and $\mathcal{H}^{\wedge n}$, respectively), then

(8) $$\mathcal{H}^{\vee n} = S\mathcal{H}^{\otimes n},$$

(9) $$\mathcal{H}^{\wedge n} = A\mathcal{H}^{\otimes n}.$$

Now let $B_1^{\otimes n}$ and $B_2^{\otimes n}$, the tensor nth powers of B_1 and B_2, respectively, be von Neumann algebras on $\mathcal{H}^{\otimes n}$ (chap. I, § 2, section 4 of [*6*]). By definition, one has

(10) $$B_1^{\vee n} = (B_1^{\otimes n})_S, \qquad B_2^{\vee n} = (B_2^{\otimes n})_S,$$

(11) $$B_1^{\wedge n} = (B_1^{\otimes n})_A, \qquad B_2^{\wedge n} = (B_2^{\otimes n})_A.$$

Let $\mathfrak{A}$ be a von Neumann algebra on the Hilbert space H and let E be a Hermitean projector in H belonging either to $\mathfrak{A}$ or its commutant $\mathfrak{A}'$. Here Dixmier's symbol $\mathfrak{A}_E$ (defined in chap. I, § 2, section 1 of [*6*]) is used to denote that the set of operators E A E, in which A runs through $\mathfrak{A}$, is to be restricted to E H. In (10) and (11), it is clear that both A and S belong to the commutants $(B_1^{\otimes n})'$ and $(B_2^{\otimes n})'$ since $B_1^{\otimes n}$ and $B_2^{\otimes n}$ leave invariant the spaces $S\mathcal{H}^{\otimes n} = \mathcal{H}^{\vee n}$ and $A\mathcal{H}^{\otimes n} = \mathcal{H}^{\wedge n}$ of symmetric and antisymmetric tensors over $\mathcal{H}$.

We notice on the other hand that if $\mathcal{L}(H)$ denotes the set of all bounded operators on a Hilbert space H and C(H) denotes the multiples of the unit operator on H, then by Proposition 14 on p. 102 of [*6*] it follows that

$$B_1^{\otimes n} = \mathcal{L}(\mathcal{H}_1)^{\otimes n} \otimes C(\mathcal{H}_2^{\otimes n}), \tag{12}$$

$$B_2^{\otimes n} = C(\mathcal{H}_1^{\otimes n}) \otimes \mathcal{L}(\mathcal{H}_2)^{\otimes n}. \tag{13}$$

We can reduce the algebras $\mathcal{L}(\mathcal{H}_1)^{\otimes n}$ and $\mathcal{L}(\mathcal{H}_2)^{\otimes n}$ acting in the tensor spaces $\mathcal{H}_1^{\otimes n}$ and $\mathcal{H}_2^{\otimes n}$ by decomposing these spaces into spaces of tensors that are irreducible under permutations. Specifically, if $\mathcal{H}^n_{1_\chi}$, $(\mathcal{H}^n_{2_\chi})$ represents the subspace of $\mathcal{H}_1^{\otimes n}$, $(\mathcal{H}_2^{\otimes n})$ consisting of the tensors of symmetry character χ, where χ is any character of the symmetric group of *n* elements, then we have

$$\mathcal{H}_1^{\otimes n} = \bigoplus_{\text{all } \chi} \mathcal{H}^n_{1_\chi}, \tag{14}$$

$$\mathcal{H}_2^{\otimes n} = \bigoplus_{\text{all } \chi} \mathcal{H}^n_{2_\chi}. \tag{15}$$

Since the subspaces $\mathcal{H}^n_{1_\chi}$, $(\mathcal{H}^n_{2_\chi})$ are the irreducible stable subspaces for the algebras $\mathcal{L}(\mathcal{H}_1)^{\otimes n}$ and $\mathcal{L}(\mathcal{H}_2)^{\otimes n}$, we have

$$\mathcal{L}(\mathcal{H}_1)^{\otimes n} = \prod_{\text{all } \chi} \mathcal{L}(\mathcal{H}^n_{1_\chi}), \tag{16}$$

$$\mathcal{L}(\mathcal{H}_2)^{\otimes n} = \prod_{\text{all } \chi} \mathcal{L}(\mathcal{H}^n_{2_\chi}). \tag{17}$$

Here Π denotes a product of von Neumann algebras ([*6*], chap. I, § 2, section 2) (1).

According to (7) (14) and (15), we have

$$\mathcal{H}^{\otimes n} = \bigoplus_{\substack{\text{all pairs} \\ \chi_1 \chi_2}} \left(\mathcal{H}^n_{1_{\chi_1}} \otimes \mathcal{H}^n_{2_{\chi_2}}\right) \tag{18}$$

(1) It is *not* a tensor product, and is often called the direct sum of algebras.

and out of $\mathcal{H}^{\otimes n}$ we have to select the subspace $\mathcal{H}^{\vee n} = S\mathcal{H}^{\otimes n}$ ($\mathcal{H}^{\wedge n} = A\mathcal{H}^{\otimes n}$) of symmetric (antisymmetric) tensors. Since, according to (7), the representation of the symmetric group in $\mathcal{H}^{\otimes n}$ is evidently the tensor product of its representation in $\mathcal{H}^{\otimes n}$ and $\mathcal{H}^{\otimes n}$, we know (see Appendix) that the only terms in (18) that contain symmetric (antisymmetric) tensors are those for which $\chi_2 = \chi_1$ ($\chi_2 = \varepsilon\chi_1$, where ε is the alternating character). Let P and Q, respectively, denote the projections in $\mathcal{H}^{\otimes n}$ on these subspaces; i. e., let

$$\text{(19)} \qquad P\mathcal{H}^{\otimes n} = \bigoplus_{\text{all } \chi} \mathcal{H}^n_{1_\chi} \otimes \mathcal{H}^n_{2_\chi},$$

$$\text{(20)} \qquad Q\mathcal{H}^{\otimes n} = \bigoplus_{\text{all } \chi} \mathcal{H}^n_{1_\chi} \otimes \mathcal{H}^n_{2_{\varepsilon\chi}}.$$

One thus has

$$\text{(21)} \qquad P \geqq S, \qquad Q \geqq A$$

and therefore

$$\text{(22)} \qquad B_i^{\vee n} = (B_i^{\otimes n})_S = [(B_i^{\otimes n})_P]_S, \qquad i = 1, 2$$

$$\text{(23)} \qquad B_i^{\wedge n} = (B_i^{\otimes n})_A = [(B_i^{\otimes n})_Q]_A.$$

Obviously $B_i^{\otimes n}$ commutes with P and Q. To prove this formally, note that from (16) and (12) it follows that

$$\text{(24)} \qquad \begin{aligned} B_1^{\otimes n} &= \Big\{ \prod_{\chi_1} \mathcal{L}\big(\mathcal{H}^n_{1_{\chi_1}}\big) \Big\} \otimes C\big(\mathcal{H}_2^{\otimes n}\big), \\ &= \Big\{ \prod_{\chi_1, \chi_2} \mathcal{L}\big(\mathcal{H}^n_{1_{\chi_1}}\big) \Big\} \otimes C\big(\mathcal{H}^n_{2_{\chi_2}}\big), \end{aligned}$$

and analogous expressions follow for $B_2^{\otimes n}$. Now, we have

$$\text{(25)} \qquad (B_1^{\otimes n})_P = \prod_{\chi} \Big\{ \mathcal{L}\big(\mathcal{H}^n_{1_\chi}\big) \otimes C\big(\mathcal{H}^n_{2_\chi}\big) \Big\},$$

$$\text{(26)} \qquad (B_2^{\otimes n})_P = \prod_{\chi} \Big\{ C\big(\mathcal{H}^n_{1_\chi}\big) \otimes \mathcal{L}\big(\mathcal{H}^n_{2_\chi}\big) \Big\},$$

and hence

$$\text{(27)} \qquad [(B_1^{\otimes n})_P]' = (B_2^{\otimes n})_P.$$

Analogously,

$$(25') \qquad (B_1^{\otimes n})_Q = \prod_\chi \{ \mathcal{L}(\mathcal{H}_{1\chi}^n) \otimes C(\mathcal{H}_{2\varepsilon\chi}^n) \},$$

$$(26') \qquad (B_2^{\otimes n})_Q = \prod_\chi \{ C(\mathcal{H}_{1\chi}^n) \otimes \mathcal{L}(\mathcal{H}_{2\varepsilon\chi}^n) \},$$

and hence

$$(27') \qquad [(B_1^{\otimes n})_Q]' = (B_2^{\otimes n})_Q.$$

Now to (22) and (23) we apply the well known result $(\mathfrak{A}_E)' = (\mathfrak{A}')_E$ (Prop. 1(*i*) on p. 18 of [*6*]). Then

$$\begin{aligned}(B_1^{\vee n})' &= \{ [(B_1^{\otimes n})_P]_S \}' = \{ [(B_1^{\otimes n})_P]' \}_S \\ &= [(B_2^{\otimes n})_P]_S = B_2^{\vee n}\end{aligned}$$

and analogously

$$(B_1^{\wedge n})' = B_2^{\wedge n}.$$

Equations (4) and (6) now immediately result upon repeated application of Prop. 1(*i*) on p. 18 of [*6*] and making use of the remark that the projector on $\mathcal{H}^{\vee n}$ in $\mathcal{H}^{\vee}$ (on $\mathcal{H}^{\wedge n}$ in $\mathcal{H}^{\wedge}$) evidently belongs to the center of $B^{\vee}$ ($B^{\wedge}$).

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APPENDIX

In the course of the proof we have used the following well-known

LEMMA. — Let U_χ be the irreducible representation of the symmetric group of n objects corresponding to its character χ. The number of times the identical (or alternate) representation is contained in $U_\chi \otimes U_{\chi'}$, is equal to $\delta_{\chi,\chi'}$ (or $\delta_{\chi,\varepsilon\chi'}$), where

$$\delta_{\chi_1, \chi_2} = \begin{cases} 1 & \text{for} \quad \chi_1 = \chi_2 \\ 0 & \text{for} \quad \chi_1 \neq \chi_2 \end{cases}$$

and ε is the alternate character (equal to the parity of permutations).

Proof. — Let D_1 and D_2 be two representations of a compact group G of order h, and let χ_1 and χ_2 be the corresponding characters. The number of times n_1 that D_1 is contained in D_2 is given ([5], p. 105) by

$$n_1 = \frac{1}{h} \sum_{s \in G} \overline{\chi_2(s)}\chi_1(s).$$

In our particular case we have $h = n\,!$ and χ_2 is the character of $U_\chi \otimes U_{\chi'}$ and is equal to the product $\chi\chi'$. On the other hand, $\chi_1 = 1$ for the identity and $\chi_1 = \varepsilon$ for the alternate representation. The result then follows from the orthogonality relations of characters.

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