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On the theory of unitary representations of the $SL(2, \mathbb{C})$ group

by

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ABSTRACT. — The canonical basis of the unitary representations of the group $SL(2, \mathbb{C})$ are constructed in the explicit form on the base of homogeneous functions. The matrix elements for finite transformations are found. The maximal degenerate principal series of $SL(n, \mathbb{C})$ is also considered.

§ 1. INTRODUCTION

In a series of papers (Barut, Budini and Fronsdal [1], Dothan, Gell-Mann and Ne'eman [2], Fronsdal [3], Delbourgo, Salam and Strathdee [4], Ruhl [5], Michel [6], Todorov [7] and Nguyen van Hieu [8]) the possibility of using the unitary representations of non-compact groups to classify the elementary particles was discussed. It was shown that in the symmetry theory with the group G

$$G = \mathfrak{P} \boxtimes S, \quad S \supset SL(2, \mathbb{C})$$

which is the semi-direct product of the Poincaré group \mathfrak{P} and some internal symmetry non-compact group S containing some $SL(2, \mathbb{C})$ subgroup there exists no contradiction with the unitary condition for S -matrix [8], and in this theory we can introduce the field operators in such a manner that the free field operators obey the normal commutation of anticommutation relations (with the normal connection between spins and statistics).

Before to study the experimental consequences of this new symmetry theory we must solve some mathematical problems:

1. To study the irreducible unitary representations of the internal symmetry non-compact group S and the splitting of these representations into direct sum of the irreducible finite-dimensional representations of the maximal compact subgroup of S .
2. To calculate for the unitary representations the matrix elements of the finite transformations of the group S which correspond to the pure Lorentz transformations.

These problems were considered in all above mentioned papers, some partial results were obtained there, but none of them was solved finally.

In the present work we study the irreducible unitary representations of the $SL(2, C)$ group and calculate the matrix elements of the finite transformations for these representations. The method developed here can be also generalized to study the group $SL(n, C)$ and $SU(p, q)$. We note that the theory of unitary representations of these groups was developed in the work of Gelfand and Neimark [9]. The Gelfand-Neimark theory is a rigorous one from the mathematical point of view. However, the method used by Gelfand and Neimark is not convenient for the physical applications. Here we apply an other method based on using the homogeneous functions to realize the irreducible unitary representations of non-compact groups. This method is very convenient for the applications to physics. The possibility of using the homogeneous functions to study the representations of non-compact groups was discussed in references [10] [3] [5] [11]. Within this method we can obtain the Gelfand-Neimark results in a very simple manner.

§ 2. UNITARY REPRESENTATIONS OF THE $SL(2, C)$ group

$SL(2, C)$ is the group of all 2×2 complex matrices with determinant equal to 1. We now realize the representations of this group in the Hilbert space of the functions $f(z_1, z_2)$ depending on two complex variables z_1 and z_2 . For every matrix $g \in SL(2, C)$ we define a corresponding operator T_g in the given Hilbert space of functions $f(z_1, z_2)$:

$$g \rightarrow T_g \tag{2.1}$$

$$T_g f(z_1, z_2) = f(z'_1, z'_2); \quad z'_a = z_b g_{ba} \mid a, b = 1, 2 \mid$$

It is not difficult to prove that

$$T_{g_1} T_{g_2} = T_{g_1 g_2}$$

Therefore the corresponding $g \rightarrow T_g$ is a representation of the group $SL(2, C)$. We define now the scalar product in the Hilbert space of functions $f(z_1, z_2)$:

$$\langle f_1, f_2 \rangle = \left(\frac{i}{2}\right)^2 \int f_1(z_1, z_2) \overline{f_2(z_1, z_2)} \cdot dz_1 d\bar{z}_1 \cdot dz_2 d\bar{z}_2 \quad (2.2)$$

In this case our Hilbert space consists of all square integrable functions of two complex variables. It is not difficult to prove that with this scalar product (2.2) the operators T_g are unitary:

$$\langle T_g f_1, T_g f_2 \rangle = \langle f_1, f_2 \rangle$$

Thus we obtain an unitary representation of the $SL(2, C)$ group, which is not yet an irreducible one, however. In order to get the irreducible representations we use the homogeneous functions. A function $f(z_1, z_2)$ is called a homogeneous function of degree (λ_1, λ_2) where λ_1 and λ_2 are complex numbers, if for any complex number $\sigma \neq 0$ we have

$$f(\sigma z_1, \sigma z_2) = \sigma^{\lambda_1} \bar{\sigma}^{\lambda_2} f(z_1, z_2) \quad (2.3)$$

This definition makes sense only if the difference $\lambda_1 - \lambda_2$ is an integer number. From this definition and from (2.1) it follows that if $f(z_1, z_2)$ is a homogeneous function of degree (λ_1, λ_2) then $T_g f(z_1, z_2)$ is also a homogeneous function with the same degree. Thus the Hilbert space D_λ of homogeneous functions of some degree $\lambda = (\lambda_1, \lambda_2)$ realizes a representation of the group $SL(2, C)$.

For the homogeneous functions we cannot use the scalar product defined as in (2.2). Indeed, each function $f(z_1, z_2)$ from D_λ is determined uniquely by a corresponding function of one variable $f(z) \equiv f(z, 1)$. Since

$$f(z_1, z_2) = z_2^{\lambda_1} \bar{z}_2^{\lambda_2} f\left(\frac{z_1}{z_2}, 1\right) = z_2^{\lambda_1} \bar{z}_2^{\lambda_2} f\left(\frac{z_1}{z_2}\right) \quad (2.4)$$

and the integral in the right-hand side of (2.2) can be written in the form of the product of two independent integrals

$$\begin{aligned} & \int f_1(z_1, z_2) \overline{f_2(z_1, z_2)} \cdot dz_1 d\bar{z}_1 \cdot dz_2 d\bar{z}_2 \\ &= \int |z_2^{\lambda_1} \bar{z}_2^{\lambda_2}|^2 f_1\left(\frac{z_1}{z_2}\right) \overline{f_2\left(\frac{z_1}{z_2}\right)} \cdot |z_2|^2 \cdot d\left(\frac{z_1}{z_2}\right) \cdot d\left(\frac{z_1}{z_2}\right) \cdot dz_2 d\bar{z}_2 \\ &= \int f_1(z) \overline{f_2(z)} dz d\bar{z} \cdot \int |z_2^{\lambda_1 + 1} \bar{z}_2^{\lambda_2}|^2 dz_2 d\bar{z}_2, \end{aligned}$$

It is not difficult to show that the second integral tends to infinity.

Thus, in the Hilbert space D_λ we must define the scalar product in another manner. Since the homogeneous functions $f(z_1, z_2)$ effectively depend only on one of variables then to define the scalar product we must use the complex path integral instead of the complex surface integral. Namely, we define the scalar product in the following manner:

$$\langle f_1, f_2 \rangle = \frac{i}{2} \int f_1(z_1, z_2) \overline{f_2(z_1, z_2)} d\omega_z, \quad (2.5)$$

where $d\omega_z$ is some measure invariant under the transformations

$$z_a \rightarrow z'_a = z_b g_{ba}, \quad \det g = 1$$

It is easy to see that such a measure can be of the form

$$d\omega_z = (\overline{z_2} d\overline{z_1} - \overline{z_1} d\overline{z_2})(z_2 dz_1 - z_1 dz_2) \quad (2.6)$$

As in the case of (2.2) from the invariance of the measure it follows that all the operators T_g are unitary with respect to the scalar product (2.5), and the representation of the $SL(2, C)$ group in the Hilbert space D_λ is an unitary one.

The norm of an element $f(z_1, z_2) \in D_\lambda$ is determined according to (2.5):

$$\|f\|^2 \equiv \langle f | f \rangle = \frac{i}{2} \int |f(z_1, z_2)|^2 d\omega_z \quad (2.7)$$

Putting in the integral in the right-hand side of (2.7) $z_a = \sigma z'_a$ and using (2.3) we get

$$\|f\|^2 = |\sigma^{\lambda_1 + \overline{\lambda_1} + 2}|^2 \|f\|^2 \quad (2.8)$$

for any complex $\sigma \neq 0$. This relation shows that λ_1 and λ_2 must satisfy the equation

$$\lambda_1 + \overline{\lambda_1} + 2 = 0 \quad (2.9)$$

whose solution is

$$\lambda_1 = \nu_0 + \frac{i\rho}{2} - 1$$

$$\lambda_2 = -\nu_0 + \frac{i\rho}{2} - 1,$$

where ν_0 and ρ are real numbers. Since $\lambda_1 - \lambda_2 = 2\nu_0$ must be an integer number, then ν_0 is an integer or half-integer number. Thus we have obtained the unitary representations of the $SL(2, C)$ group in the Hilbert space D_λ

of homogeneous functions of degrees $\lambda = \left(\nu_0 + \frac{i\rho}{2} - 1, -\nu_0 + \frac{i\rho}{2} - 1 \right)$ where ν_0 is any integer or half-integer number, ρ is any real number. These representations will be denoted by $\mathfrak{C}_{\nu_0\rho}$. They are irreducible [10] and form the so called principal series. Together with this principal series there exists also the supplementary one [9] [10] [12] which can be considered in a similar manner. However, we do not study this series here.

Finally we note that from (2.6) and (2.9) it follows that for the representations of the principal series the scalar product defined by (2.5) is identical to the scalar product introduced by Gelfand and Neimark

$$\langle f_1, f_2 \rangle = \frac{i}{2} \int f_1(z) \overline{f_2(z)} dz d\bar{z} \quad (2.10)$$

§ 3. EQUIVALENT REPRESENTATIONS. SPLITTING OF THE UNITARY REPRESENTATIONS OF $SL(2, C)$ GROUP INTO DIRECT SUMS OF THE REPRESENTATIONS OF $SU(2)$ SUBGROUP

Let $\mathfrak{C}_{\nu_0\rho}$ (with operators T_g) and $\mathfrak{C}_{\nu'_0\rho'}$ (with operators T'_g) be two irreducible unitary representations of the group $SL(2, C)$ which are realized in the Hilbert spaces of homogeneous functions

$$D_{\lambda = \left(\nu_0 + \frac{i\rho}{2} - 1, -\nu_0 + \frac{i\rho}{2} - 1 \right)} \text{ and } D_{\lambda' = \left(\nu'_0 + \frac{i\rho'}{2} - 1, -\nu'_0 + \frac{i\rho'}{2} - 1 \right)}$$

respectively. Now we find the conditions for the equivalence of these two representations. The representations $\mathfrak{C}_{\nu_0\rho}$ and $\mathfrak{C}_{\nu'_0\rho'}$ are called equivalent if there exists such an operator A which realizes an one-to-one mapping D_λ onto $D_{\lambda'}$ that

$$T'_g A = A T_g \quad (3.1)$$

for any $g \in SL(2, C)$. From this definition we see immediately that if $\lambda = \lambda'$ (i. e. $\nu_0 = \nu'_0$, $\rho = \rho'$) then $\mathfrak{C}_{\nu_0\rho}$ and $\mathfrak{C}_{\nu'_0\rho'}$ are equivalent. This case is a trivial one and it is not of any interest, because here D_λ and $D_{\lambda'}$ coincide, $A = 1$.

Let

$$f(\xi_1, \xi_2) \in D_\lambda, \quad f'(\eta_1, \eta_2) \in D_{\lambda'}.$$

We represent the operator A in the form of an integral transformation with some kernel K :

$$f'(\eta_1, \eta_2) = Af(\xi_1, \xi_2) \equiv \frac{i}{2} \int K(\eta_1, \eta_2; \xi_1, \xi_2) f(\xi_1, \xi_2) d\omega_\xi \quad (3.2)$$

Then the condition of equivalence (3.1) can be rewritten explicitly in the form

$$T'_g f'(\eta_1, \eta_2) = \frac{i}{2} \int K(\eta_1, \eta_2; \xi_1, \xi_2) T_g f(\xi_1, \xi_2) d\omega_\xi \quad (3.3)$$

Using Eq. (2.1) we can rewrite last equation in the form:

$$f'(\eta'_1, \eta'_2) = \frac{i}{2} \int K(\eta_1, \eta_2; \xi_1, \xi_2) f(\xi'_1, \xi'_2) d\omega_{\xi'} \quad (3.4)$$

where $\eta'_a = \eta_b g_{ba}$, $\xi'_a = \xi_b g_{ba}$. Comparing Eqs (3.2) and (3.4) we get:

$$K(\eta'_1, \eta'_2; \xi'_1, \xi'_2) = K(\eta_1, \eta_2; \xi_1, \xi_2).$$

Thus, the kernel K must be an invariant function of ξ_a and η_a . As it was well known in the theory of spinor representations of the group $SL(2, C)$ from the variables ξ_a and η_a we can form the following invariant

$$\xi_1 \eta_2 - \xi_2 \eta_1 = \text{inv.}$$

The kernel K must be a function of this invariant combination $(\xi_1 \eta_2 - \xi_2 \eta_1)$. It must be also a homogeneous function since $f'(\eta_1, \eta_2)$ is a homogeneous one. Let K be a homogeneous function on two variables (ξ_1, ξ_2) of degree (μ_1, μ_2) . Then putting $\xi_a = \sigma \xi'_a$ into (3.2) we get:

$$\begin{aligned} & \int K(\eta_1, \eta_2; \xi_1, \xi_2) f(\xi_1, \xi_2) d\omega_\xi \\ &= (\sigma)^{\mu_1 + \lambda_1 + 2} (\bar{\sigma})^{\mu_2 + \lambda_2 + 2} \int K(\eta_1, \eta_2; \xi'_1, \xi'_2) f(\xi'_1, \xi'_2) d\omega_{\xi'} \end{aligned}$$

for any complex $\sigma \neq 0$. Therefore we must have

$$\begin{aligned} \mu_1 &= -\lambda_1 - 2 = -\nu_0 - \frac{i\rho}{2} - 1 \\ \mu_2 &= -\lambda_2 - 2 = +\nu_0 - \frac{i\rho}{2} - 1 \end{aligned} \quad (3.5)$$

Thus, $K(\eta_1, \eta_2; \xi_1, \xi_2)$ must be an homogeneous function of (ξ_1, ξ_2) of degree $\left(-\nu_0 - \frac{i\rho}{2} - 1, \nu_0 - \frac{i\rho}{2} - 1\right)$. On the other hand the kernel $K(\eta_1, \eta_2; \xi_1, \xi_2)$ is a function of the combination $(\xi_1 \eta_2 - \xi_2 \eta_1)$ and therefore

it is also a homogeneous function of (η_1, η_2) of the same degree $\left(-\nu_0 - \frac{i\rho}{2} - 1, \nu_0 - \frac{i\rho}{2} - 1\right)$. This means that if $\nu'_0 = -\nu_0$, $\rho' = -\rho$ then the representations $\mathfrak{C}_{\nu_0\rho}$ and $\mathfrak{C}_{\nu'_0\rho'}$ are equivalent.

Consider now the splitting of irreducible unitary representations of the group $SL(2, C)$ into direct sums of the irreducible representations of the maximal compact subgroup $SU(2)$. In the theory of spinor representations of the group $SL(2, C)$ we know that if z_a is transformed as a spinor φ_a then \bar{z}_a is transformed as a spinor $\dot{\varphi}^a$ and the sum $z_a\bar{z}_a$ is transformed as the sum $\varphi_a\dot{\varphi}^a$ i. e. is invariant of the $SU(2)$ subgroup. In the following for the convenience we denote \bar{z}_a by \bar{z}^a or \bar{z}^a .

For the clarity we illustrate now our method on some simple examples. The general case will be considered in the following section. Consider firstly the representation $\mathfrak{C}_{0\rho}$. This representation is realized in the Hilbert space $D\left(\frac{i\rho}{2}-1, \frac{i\rho}{2}-1\right)$ of homogeneous functions $f(z_1, z_2)$ of degree $\left(\frac{i\rho}{2}-1, \frac{i\rho}{2}-1\right)$. One of these functions is

$$f_{00}(z_1, z_2) \sim (z_1\bar{z}^1 + z_2\bar{z}^2)^{\frac{i\rho}{2}-1} \tag{3.6}$$

As it was noted, this function is invariant under the $SU(2)$ subgroup and therefore characterizes the spin zero state. In order to get the functions corresponding to the states with non-zero spin we must construct them in such a manner that they contain some factors z_a and \bar{z}^b without summation. It is not difficult to see that for the spin 1 states we have the following functions:

$$f_{11}(z_1, z_2) \sim (z_1\bar{z}^1 + z_2\bar{z}^2)^{\frac{i\rho}{2}-2} z_1\bar{z}^2 \quad \text{for } j = 1, m = 1 \tag{3.9}$$

$$f_{10}(z_1, z_2) \sim (z_1\bar{z}^1 + z_2\bar{z}^2)^{\frac{i\rho}{2}-2} (z_2\bar{z}^2 - z_1\bar{z}^1) \quad \text{for } j = 1, m = 0 \tag{3.8}$$

$$f_{1,-1}(z_1, z_2) \sim (z_1\bar{z}^1 + z_2\bar{z}^2)^{\frac{i\rho}{2}-2} z_2\bar{z}^1 \quad \text{for } j = 1, m = -1 \tag{3.9}$$

(j and m denote the spin and its projection on the $z -$ axis).

Consider now the representations $\mathfrak{C}_{\nu_0\rho}(\nu_0 \neq 0)$. Since the representations $\mathfrak{C}_{\nu_0\rho}$ and $\mathfrak{C}_{-\nu_0, -\rho}$ are equivalent then we can assume that $\nu_0 > 0$. We choose the basis elements of the representations $\mathfrak{C}_{\nu_0\rho}$ in the form of the products of the quantity $(z_1\bar{z}^1 + z_2\bar{z}^2)^{\frac{i\rho}{2}-n}$ ($n \geq \nu_0 + 1$) and some factors z_a

and \bar{z}^b without summation. The products with the minimal number of free factors z_a and \bar{z}^b are of the form

$$f(z_1, z_2) \sim (z_1 \bar{z}^1 + z_2 \bar{z}^2)^{\frac{i\rho}{2} - 1 - \nu_0} z_{a_1} z_{a_2} \dots z_{a_{2\nu_0}} \quad (3.10)$$

These functions describe the states with spin $j_0 = \nu_0$. The other functions correspond to the states with spins $j = \nu_0 + 1, \nu_0 + 2, \dots$. Thus, the representation $\mathfrak{C}_{\nu_0\rho}$ splits into the direct sum of irreducible finite-dimensional representations of the SU(2) subgroup. Each of them is contained in given representation $\mathfrak{C}_{\nu_0\rho}$ once and describes state with definite spin $j = j_0 + n, n = 0, 1, 2, \dots$

§ 4. MATRIX ELEMENTS OF FINITE TRANSFORMATIONS

As it was noted in the introduction, in studying the structure of the vertex parts and the scattering amplitudes we must use the matrix elements of the finite transformations of the group S and in particular of the group SL(2, C). Note that this problem was first considered in the paper by Dolgihinov and Topyghin [13] for the case with $\nu_0 = 0$. These authors choose the analytic continuations of the 4-dimensional spherical functions as the basis functions (see [13] [14]). Our method is based on the results obtained in § 2.

The matrix element $D_{jm; j'm'}^{\nu_0\rho}(g)$ corresponding to the representation $g \rightarrow U_g$ is defined in the following manner:

$$U_g | \nu_0\rho; jm \rangle = \sum_{j'm'} D_{jm; j'm'}^{\nu_0\rho}(g) | \nu_0\rho; j'm' \rangle, \quad (4.1)$$

where $| \nu_0\rho; jm \rangle$ is the canonical basis of representation $\mathfrak{C}_{\nu_0\rho}$; j and m are the spin and its projection on the z -axis. Generalizing the obtained results (see (3.6)-(3.10)) we determine firstly the canonical basis in the space of homogeneous functions $D_{\lambda = (\nu_0 + \frac{i\rho}{2} - 1, -\nu_0 + \frac{i\rho}{2} - 1)}$ in the following manner:

$$| \nu_0\rho; jj \rangle \rightarrow f_{jj}^{\nu_0\rho}(z_1, z_2) = c_{jj} (z_1 \bar{z}^1 + z_2 \bar{z}^2)^{\frac{i\rho}{2} - 1 - j} (z_1)^{j + \nu_0} (\bar{z}^2)^{j - \nu_0} \quad (4.2)$$

where c_{jj} are the normalization constants. Using (2.10) we get:

$$c_{jj} = \left\{ \frac{i}{2} \int | f_{jj}^{\nu_0\rho}(z) |^2 dz d\bar{z} \right\}^{-\frac{1}{2}} = \frac{1}{\sqrt{\pi}} \left\{ \frac{(2j+1)!}{(j+\nu_0)! (j-\nu_0)!} \right\}^{1/2} \quad (4.3)$$

From $| \nu_0 \rho; jj \rangle$ we can find $| \nu_0 \rho; jm \rangle$:

$$| \nu_0 \rho; jm \rangle \rightarrow f_{jm}^{\nu_0 \rho}(z_1, z_2) = N_{jm} (J^-)^{j-m} f_{jj}^{\nu_0 \rho}(z_1, z_2), \quad (4.4)$$

where

$$N_{j,m} = \left\{ \frac{(j+m)!}{(2j)! (j-m)!} \right\}^{1/2} \quad (4.5)$$

$$J^- = z_2 \frac{\partial}{\partial z_1} - z_1^{-1} \frac{\partial}{\partial z_2^{-2}} \quad (4.6)$$

From (4.2)-(4.6) we get the final expression for $f_{jm}^{\nu_0 \rho}(z_1, z_2)$:

$$f_{jm}^{\nu_0 \rho}(z_1, z_2) = \frac{1}{\sqrt{\pi}} \left\{ (2j+1)(j+m)! (j-m)! (j+\nu_0)! (j-\nu_0)! \right\}^{1/2} (z_1 z_1^{-1} + z_2 z_2^{-2})^{\frac{i\rho}{2} - 1 - j} \\ \cdot \sum_d (-1)^d \frac{1}{d! (j-m-d)! (\nu_0+m+d)! (j-\nu_0-d)!} (z_1)^{\nu_0+m+d} (z_2)^{j-m-d} (z_1^{-1})^d (z_2^{-2})^{j-\nu_0-d} \quad (4.7)$$

Having the explicit expression for the canonical basis $| \nu_0 \rho; jm \rangle$ we can find the matrix elements $D_{jm; j'm'}^{\nu_0 \rho}(g)$. It is well known that every matrix g can be represented in the form

$$g = u_1 \epsilon u_2$$

where u_1 and u_2 are the unitary unimodular matrices which corresponding to the space rotation;

$$\epsilon = \begin{pmatrix} \epsilon^{-1} & 0 \\ 0 & \epsilon \end{pmatrix} \quad (\epsilon \text{—real number})$$

and corresponds to the pure Lorentz transformation in the plane (x_3, x_4) . Thus without losing the generality we can consider only the matrix element $D_{jm; j'm'}^{\nu_0 \rho}(\epsilon)$.

From (4.1) and from the orthonormalization relations

$$\frac{i}{2} \int f_{jm}^{\nu_0 \rho}(z) \overline{f_{j'm'}^{\nu_0 \rho}(z)} dz d\bar{z} = \delta_{jj'} \cdot \delta_{mm'}$$

we have

$$D_{jm; j'm'}^{\nu_0 \rho}(\epsilon) = \frac{i}{2} \int T_\epsilon f_{jm}^{\nu_0 \rho}(z) \cdot \overline{f_{j'm'}^{\nu_0 \rho}(z)} dz d\bar{z}. \quad (4.8)$$

From (2.1) and (4.7) we get:

$$\begin{aligned}
 & D_{jm;j'm'}^{\nu_0\rho}(\varepsilon) = \\
 & \delta_{mm'} \cdot \frac{1}{\pi} \left\{ (2j+1)(2j'+1) \cdot (j+m)! (j-m)! (j+\nu_0)! (j-\nu_0)! \right. \\
 & \qquad \qquad \qquad \left. (j'+m')! (j'-m)! (j'+\nu_0)! (j'-\nu_0)! \right\}^{1/2} \\
 & \cdot \sum_{d,d'} (-1)^{d+d'} \left\{ d! d'! (j-m-d)! (j'-m-d')! (\nu_0+m+d)! (\nu_0+m+d')! \right. \\
 & \qquad \qquad \qquad \left. (j-\nu_0-d)! (j'-\nu_0-d')! \right\}^{-1} \\
 & \cdot \varepsilon^{-2\left(2d+m+\nu_0+1-\frac{i\rho}{2}\right)} \cdot \frac{i}{2} \int dz d\bar{z} |z|^{2(d+d'+m+\nu_0)} (1+|z|^2)^{-\frac{i\rho}{2}-1-j'} (1+\varepsilon^{-4}|z|^2)^{\frac{i\rho}{2}-1-j}, \\
 & \qquad \qquad \qquad (4.9)
 \end{aligned}$$

where d and d' can take any integer number which does not make each factor under the factorial to become a negative number. By putting $z = \sqrt{v}e^{i\varphi}$ ($0 \leq v < \infty$; $0 \leq \varphi \leq 2\pi$) the integral in (4.9) can be rewritten in the form

$$\begin{aligned}
 J &= \pi \int_0^\infty dv \cdot v^{d+d'+m+\nu_0} (1+v)^{-\frac{i\rho}{2}-1-j'} (1+\varepsilon^{-4}v)^{\frac{i\rho}{2}-1-j} \\
 &= \pi \cdot \varepsilon^{4(d+d'+m+\nu_0+1)} \frac{(d+d'+m+\nu_0)! (j+j'-d-d'-m-\nu_0)!}{(j+j'+1)!} \\
 & \quad F\left(j'+1+\frac{i\rho}{2}, d+d'+m+\nu_0+1; j+j'+2; 1-\varepsilon^4\right), \quad (4.10)
 \end{aligned}$$

where $F(\alpha; \beta; \gamma; z)$ is the hypergeometric function. By setting (4.10) into (4.9) we get the final result:

$$\begin{aligned}
 & D_{jm;j'm'}^{\nu_0\rho}(\varepsilon) = \frac{\delta_{mm'}}{(j+j'+1)!} \\
 & \left\{ (2j+1)(2j'+1)(j+m)! (j-m)! (j+\nu_0)! (j-\nu_0)! (j'+m)! (j'-m)! (j'+\nu_0)! (j'-\nu_0)! \right\}^{1/2} \\
 & \cdot \sum_{d,d'} (-1)^{d+d'} \frac{(d+d'+m+\nu_0)! (j+j'-d-d'-m-\nu_0)!}{d! d'! (j-m-d)! (j'-m-d')! (\nu_0+m+d)! (\nu_0+m+d')! (j-\nu_0-d)! (j'-\nu_0-d')!} \\
 & \cdot \varepsilon^{2\left(2d'+m+\nu_0+1+\frac{i\rho}{2}\right)} F\left(j'+1+\frac{i\rho}{2}, d+d'+m+\nu_0+1; j+j'+2; 1-\varepsilon^4\right). \quad (4.11)
 \end{aligned}$$

This result exactly coincides with the result obtained earlier by the authors in another way [15].

Now we note some simple properties of $D_{jm;j'm'}^{\nu_0\rho}(\varepsilon)$.

1. Putting in (4.11) $\varepsilon = 1$ and taking into account $F(\alpha, \beta; \gamma; 0) = 1$ we have:

$$D_{jm;j'm'}^{\nu_0\rho}(1) = \delta_{jj'} \cdot \delta_{mm'} \quad (4.12)$$

This is a trivial relation. It means that we deal here with the identity transformation.

2. Making the permutation of jm and $j'm'$ in (4.11) and using the properties of the homogeneous functions

$$F(\alpha, \beta; \gamma; z) = (1-z)^{-\beta} F\left(\gamma - \alpha, \beta; \gamma; \frac{z}{z-1}\right) = (1-z)^{-\alpha} F\left(\alpha, \gamma - \beta; \gamma; \frac{z}{z-1}\right), \quad (4.13)$$

we obtain

$$D_{j'm';jm}^{\nu_0\rho}(\varepsilon) = \overline{D_{jm;j'm'}^{\nu_0\rho}(\varepsilon^{-1})} \quad (4.14)$$

3. Making the substitution $m \rightarrow -m$, $m' \rightarrow -m'$ in (4.14), putting $i - \nu_0 - d \equiv d_1$, $j' - \nu_0 - d' \equiv d'_1$ and using (4.13) we can prove that

$$D_{j'm';jm}^{\nu_0\rho}(\varepsilon) = (-1)^{j+j'-2\nu_0} D_{j,-m;j',-m'}(\varepsilon^{-1}). \quad (4.15)$$

4. From (4.12) (4.14) and from group property of $D_{jm;j'm'}$ we get:

$$\sum_{jm} D_{jm;j'm'}^{\nu_0\rho}(\varepsilon) \overline{D_{jm;j''m''}^{\nu_0\rho}(\varepsilon)} = \delta_{j'j''} \cdot \delta_{m'm''}$$

This last relation means the unitarity condition of the representation.

§ 5. GENERALIZED TENSORS

From the canonical basis which was given in (4.7) we go to the other basis called generalized tensors of the $SU(2, C)$ group. They are constructed in the following manner:

$$f_{a_1 a_2 \dots a_{j+\nu_0}}^{b_1 b_2 \dots b_{j-\nu_0}}(z_1 z_2) = (z_c z)^{\frac{i\rho}{2} - 1 - j} \Phi_{a_1 a_2 \dots a_{j+\nu_0}}^{b_1 b_2 \dots b_{j-\nu_0}}, \quad (a, b = 1, 2) \quad (5.1)$$

where

$$\Phi_{a_1 a_2 \dots a_{t+k}}^{b_1 b_2 \dots b_t} = \sum_{s=0}^t \alpha(t, s, k) (z_c z)^s S \delta_{a_1}^{b_1} \delta_{a_2}^{b_2} \dots \delta_{a_s}^{b_s} z_{a_{s+1}} \dots z_{a_{t+k}} z^{-b_{s+1}} \dots z^{-b_t} \quad (5.2)$$

$$\alpha(t, s, k) = (-1)^s \frac{t! (t+k)! (2t+k-s)!}{s! (t-s)! (t+k-s)! (2t+k)!}$$

S denotes the symmetrization in the upper and lower indices a and b separately:

$$S(\mathbb{T}_{a_1 a_2 \dots a_i}^{b_1 b_2 \dots b_j}) \equiv \frac{1}{i! j!} \sum_{\mathbb{P}(a; b)} \mathbb{T}_{a_1 a_2 \dots a_i}^{b_1 b_2 \dots b_j}$$

$\left(\sum_{\mathbb{P}(a; b)} \right)$ stands for summation over all permutations of a and all permutations of b .

The tensors $\Phi_{a_1 a_2 \dots}^{b_1 b_2 \dots}$ are symmetrical in upper and in lower indices and are traceless with respect to the contraction of any upper index with any lower index. They are irreducible under the $SU(2)$ subgroup. Putting in (5.1), e. g., $a_1 = a_2 = \dots = a_{j+v_0} = 1$, $b_1 = b_2 = \dots = b_{j-v_0} = 2$ and using (5.1) (5.3) and (4.2) we get:

$$\underbrace{f_{\substack{11 \dots 1 \\ j+v_0 \text{ times}}}^{22 \dots 2}}_{\substack{j-v_0 \text{ times}}} = \sqrt{\pi} \left\{ \frac{(j+v_0)! (j-v_0)!}{(2j+1)!} \right\}^{1/2} f_{jj}^{v_0}$$

The inverse expansion is

$$z_{a_1} z_{a_2} \dots z_{a_{t+k}} z^{-b_1} z^{-b_2} \dots z^{-b_t} = \sum_{s=0}^t \beta(t, s, k) (z_c z)^{-s} S \Phi_{a_1 a_2 \dots a_{s+k}}^{b_1 b_2 \dots b_s} \delta_{a_{s+k+1}}^{b_{s+1}} \dots \delta_{a_{t+k}}^{b_t} \quad (5.4)$$

where

$$\beta(t, s, k) = \frac{t! (t+k)! (2s+k+1)!}{s! (s+k)! (t-s)! (t+k+s+1)!} \quad (5.5)$$

Under the transformation g the tensor $f_{a_1 a_2 \dots a_{j+v_0}}^{b_1 b_2 \dots b_{j-v_0}}$ is transformed as

$$\begin{aligned} f_{a_1 a_2 \dots a_{j+v_0}}^{b_1 b_2 \dots b_{j-v_0}}(z_1 z_2) &\rightarrow T_g f_{a_1 a_2 \dots a_{j+v_0}}^{b_1 b_2 \dots b_{j-v_0}}(z_1 z_2) \\ &= \sum_{j'} D_{a_1 a_2 \dots a_{j+v_0}; d_1 d_2 \dots d_{j'-v_0}}^{b_1 b_2 \dots b_{j-v_0}; c_1 c_2 \dots c_{j'+v_0}}(g) f_{c_1 c_2 \dots c_{j'+v_0}}^{d_1 d_2 \dots d_{j'-v_0}}(z_1, z_2), \end{aligned} \quad (5.6)$$

where the matrix elements $D_{a_1 a_2 \dots; d_1 d_2 \dots}^{b_1 b_2 \dots; c_1 c_2 \dots}(g)$ are the generalization of $D_{jm; j'm'}(g)$. The explicit expressions of these matrix elements will be given in the next section.

Any homogeneous function $\varphi(z_1, z_2)$ from the space $D_{\left(v_0 + \frac{ip}{2} - 1, -v_0 + \frac{ip}{2} - 1\right)}$ can be represented in the form:

$$\varphi(z_1, z_2) = \sum_{j'} \varphi_{d_1 d_2 \dots d_{j'-v_0}; c_1 c_2 \dots c_{j'+v_0}}^{c_1 c_2 \dots c_{j'+v_0}; d_1 d_2 \dots d_{j'-v_0}}(z_1, z_2). \quad (5.7)$$

The components $\varphi_{d_1 d_2 \dots}^{c_1 c_2 \dots}$ are also symmetrical in upper and lower indices and traceless. These quantities will be called the generalized tensors. Under g function $\varphi(z_1, z_2)$ transforms into $T_g \varphi(z_1, z_2)$ which can be also represented in the form of (5.7):

$$T_g \varphi(z_1, z_2) = \sum_j T_g \varphi_{b_1 b_2 \dots b_{j-v_0}}^{a_1 a_2 \dots a_{j+v_0}} \cdot f_{a_1 a_2 \dots a_{j+v_0}}^{b_1 b_2 \dots b_{j-v_0}}(z_1, z_2). \quad (5.8)$$

On the other hand, from (5.6) and (5.7) it follows that

$$T_g \varphi(z_1, z_2) = \sum_{j' j} \varphi_{d_1 d_2 \dots d_{j'-v_0}}^{c_1 c_2 \dots c_{j'+v_0}} D_{c_1 c_2 \dots c_{j'+v_0}; b_1 b_2 \dots b_{j-v_0}}^{d_1 d_2 \dots d_{j'-v_0}; a_1 a_2 \dots a_{j+v_0}}(g) f_{a_1 a_2 \dots a_{j+v_0}}^{b_1 b_2 \dots b_{j-v_0}} \quad (5.9)$$

Comparing (5.8) and (5.9) we find immediately the transformation law for tensors $\varphi_{b_1 b_2 \dots}^{a_1 a_2 \dots}$:

$$\varphi_{b_1 b_2 \dots b_{j-v_0}}^{a_1 a_2 \dots a_{j+v_0}} \rightarrow T_g \varphi_{b_1 b_2 \dots b_{j-v_0}}^{a_1 a_2 \dots a_{j+v_0}} = \sum_{j'} D_{c_1 c_2 \dots c_{j'+v_0}; b_1 b_2 \dots b_{j-v_0}}^{d_1 d_2 \dots d_{j'-v_0}; a_1 a_2 \dots a_{j+v_0}}(g) \varphi_{d_1 d_2 \dots d_{j'-v_0}}^{c_1 c_2 \dots c_{j'+v_0}} \quad (5.10)$$

§ 6. MATRIX ELEMENTS FOR GENERALIZED TENSORS

Now we determine the matrix elements $D_{a_1 a_2 \dots, d_1 d_2 \dots}^{b_1 b_2 \dots, c_1 c_2 \dots}(g)$ defined in (5.6). At first we rewrite (5.1) and (5.2) in the form

$$\begin{aligned} & f_{a_1 a_2 \dots a_{j+v_0}}^{b_1 b_2 \dots b_{j-v_0}}(z_1, z_2) \\ &= \sum_{s=0}^{j-v_0} \alpha(j-v_0, s, 2v_0) (z_c \bar{z}^c)^{\frac{ig}{2} - 1 - j + s} S \delta_{a_1}^{b_1} \delta_{a_2}^{b_2} \dots \delta_{a_s}^{b_s} z_{a_{s+1}}^{b_s} \dots z_{a_{j+v_0}}^{b_{s+1}} \dots z_{a_{j+v_0}}^{-b_{s+1}} \dots z_{a_{j+v_0}}^{-b_{j-v_0}} \end{aligned} \quad (6.1)$$

Since under the transformation g

$$\begin{aligned} z_a &\rightarrow z'_a = z_b g_{ba} \equiv z_b g_a^b \\ z_a &\rightarrow z^{\bar{a}} = z^{\bar{b}} g^{\bar{b}a} \equiv z^{\bar{b}} g_b^{\bar{a}} \end{aligned}$$

$$f(z_1, z_2) \rightarrow T_g f(z_1, z_2) = f(z'_1, z'_2)$$

then the tensor $f_{a_1 a_2 \dots}^{b_1 b_2 \dots}$ is transformed in the following manner:

$$\begin{aligned}
 f_{a_1 a_2 \dots a_{j+\nu_0}}^{b_1 b_2 \dots b_{j-\nu_0}} &\rightarrow \mathbf{T}_g f_{a_1 a_2 \dots a_{j+\nu_0}}^{b_1 b_2 \dots b_{j-\nu_0}} \\
 &= \sum_{s=0}^{j-\nu_0} \alpha(j-\nu_0, s, 2\nu_0) \left\{ z_p \bar{z}^q (gg)_q^p \right\}^{\frac{i\rho}{2}-1-j+s} z_{c_{s+1}} z_{c_{s+2}} \dots z_{c_{j+\nu_0}} \bar{z}^{-d_{s+1}} \bar{z}^{-d_{s+2}} \dots \bar{z}^{-d_{j-\nu_0}} \\
 &\cdot \mathbf{S}_{(a;b)}^{b_1 b_2 \dots b_s \ c_{s+1} \ c_{s+2} \ \dots \ c_{j+\nu_0} \ +b_{s+1} \ +b_{s+2} \ \dots \ +b_{j-\nu_0}} \\
 &\quad \delta_{a_1}^{b_1} \delta_{a_2}^{b_2} \dots \delta_{a_s}^{b_s} g_{a_{s+1}}^{c_{s+1}} g_{a_{s+2}}^{c_{s+2}} \dots g_{a_{j+\nu_0}}^{c_{j+\nu_0}} g_{d_{s+1}}^{+b_{s+1}} g_{d_{s+2}}^{+b_{s+2}} \dots g_{d_{j-\nu_0}}^{+b_{j-\nu_0}}
 \end{aligned} \quad (6.2)$$

We represent the 2×2 matrix gg in terms of the Pauli matrices:

$$gg = \alpha_0 \sigma_0 + \hat{\alpha} = \alpha_0 (1 + \hat{\beta}); \quad \hat{\beta} \equiv \frac{1}{\alpha_0} \hat{\alpha} = \frac{1}{\alpha_0} \vec{\alpha} \cdot \vec{\sigma} \quad (6.3)$$

Putting this expression for gg into $\left\{ z_p \bar{z}^q (gg)_q^p \right\}^{\frac{i\rho}{2}-1-j+s}$ and performing some elementary expansion

$$\begin{aligned}
 \left\{ z_p \bar{z}^q (gg)_q^p \right\}^{\frac{i\rho}{2}-1-j+s} &= \alpha_0^{\frac{i\rho}{2}-1-j+s} \left\{ z_c \bar{z}^c + z_p \hat{\beta}_q^p \bar{z}^q \right\}^{\frac{i\rho}{2}-1-j+s} \\
 &= \alpha_0^{\frac{i\rho}{2}-1-j+s} \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{i\rho}{2} - j + s\right)}{k! \Gamma\left(\frac{i\rho}{2} - j + s - k\right)} \left(z_c \bar{z}^c \right)^{\frac{i\rho}{2}-1-j+s-k} \left(z_p \hat{\beta}_q^p \bar{z}^q \right)^k
 \end{aligned} \quad (6.4)$$

we can rewrite (6.2) in the form

$$\begin{aligned}
 \mathbf{T}_g f_{a_1 a_2 \dots a_{j+\nu_0}}^{b_1 b_2 \dots b_{j-\nu_0}} &= \sum_{s=0}^{j-\nu_0} \sum_{k=0}^{\infty} \alpha(j-\nu_0, s, 2\nu_0) \frac{\Gamma\left(\frac{i\rho}{2} - j + s\right)}{k! \Gamma\left(\frac{i\rho}{2} - j + s - k\right)} \alpha_0^{\frac{i\rho}{2}-1-j+s} \left(z_c \bar{z}^c \right)^{\frac{i\rho}{2}-1-j+s-k} \\
 &\cdot \hat{\beta}_{q_1}^{p_1} \hat{\beta}_{q_2}^{p_2} \dots \hat{\beta}_{q_k}^{p_k} z_{p_1} z_{p_2} \dots z_{p_k} z_{p_{k+1}} z_{p_{k+2}} \dots z_{p_{k+j+\nu_0-s}} \bar{z}^{-q_1} \bar{z}^{-q_2} \dots \bar{z}^{-q_k} \bar{z}^{-q_{k+1}} \bar{z}^{-q_{k+2}} \dots \bar{z}^{-q_{k+j-\nu_0-s}} \\
 &\cdot \mathbf{S}_{(a;b)}^{b_1 b_2 \dots b_s \ p_{k+1} \ p_{k+2} \ \dots \ p_{k+j+\nu_0-s} \ +b_{s+1} \ +b_{s+2} \ \dots \ +b_{j-\nu_0}} \\
 &\quad \delta_{a_1}^{b_1} \delta_{a_2}^{b_2} \dots \delta_{a_s}^{b_s} g_{a_{s+1}}^{p_{k+1}} g_{a_{s+2}}^{p_{k+2}} \dots g_{a_{j+\nu_0}}^{p_{k+j+\nu_0-s}} g_{q_{k+1}}^{+b_{s+1}} g_{q_{k+2}}^{+b_{s+2}} \dots g_{q_{k+j-\nu_0-s}}^{+b_{j-\nu_0}}
 \end{aligned} \quad (6.5)$$

Here for convenience we had made some changes in the notations of the summation indices:

$$\begin{array}{ll}
 c_{s+1} \rightarrow p_{k+1} & d_{s+1} \rightarrow q_{k+1} \\
 c_{s+2} \rightarrow p_{k+2} & d_{s+2} \rightarrow q_{k+2} \\
 \dots & \dots \\
 c_{j+\nu_0} \rightarrow p_{k+j+\nu_0-s} & d_{j-\nu_0} \rightarrow q_{k+j-\nu_0-s}
 \end{array}$$

Now we express the product $z_{p_1} z_{p_2} \dots z_{p_{k+j+\nu_0-s}} \bar{z}^{q_1} \bar{z}^{q_2} \dots \bar{z}^{q_{k+j-\nu_0-s}}$ in terms of $\Phi_{p_1 p_2 \dots}$ by means of (5.4) and we put these expressions into (6.5). We get:

$$\begin{aligned} & T_g f_{a_1 a_2 \dots a_{j+\nu_0}}^{b_1 b_2 \dots b_{j-\nu_0}} \\ &= \sum_{s=0}^{j-\nu_0} \sum_{k=0}^{\infty} \sum_{r=0}^{k+j-\nu_0-s} \alpha(j-\nu_0, s, 2\nu_0) \cdot \beta(k+j-\nu_0-s, r, 2\nu_0) \frac{\Gamma\left(\frac{i\rho}{2} - j+s\right)}{k! \Gamma\left(\frac{i\rho}{2} - j+s-k\right)} \alpha_0^{\frac{i\rho}{2} - 1 - j+s} \\ & \quad \cdot (z_c z')^{\frac{i\rho}{2} - 1 - \nu_0 - r} \widehat{\beta}_{q_1} \widehat{\beta}_{q_2} \dots \widehat{\beta}_{q_k} \\ & \quad \cdot S_{(a;b)} \delta_{a_1}^{b_1} \delta_{a_2}^{b_2} \dots \delta_{a_s}^{b_s} g_{a_{s+1}}^{p_{k+1}} g_{a_{s+2}}^{p_{k+2}} \dots g_{a_{j+\nu_0}}^{p_{k+j+\nu_0-s}} g_{q_{k+2}}^{+b_{s+1} + b_{s+2}} \dots g_{q_{k+j-\nu_0-s}}^{+b_{j-\nu_0}} \\ & \quad \cdot S_{(p;q)} \Phi_{p_1 p_2 \dots p_{r+2\nu_0}}^{q_1 q_2 \dots q_r} \delta_{p_{r+2\nu_0+1}}^{q_{r+1}} \delta_{p_{r+2\nu_0+2}}^{q_{r+2}} \dots \delta_{p_{k+j+\nu_0-s}}^{q_{k+j-\nu_0-s}} \end{aligned} \quad (6.6)$$

Note that the product

$$\left\{ \widehat{\beta}_{q_1}^{p_1} \dots \widehat{\beta}_{q_k}^{p_k} g_{a_{s+1}}^{p_{k+1}} g_{a_{s+2}}^{p_{k+2}} \dots g_{a_{j+\nu_0}}^{p_{k+j+\nu_0-s}} g_{q_{k+1}}^{+b_{s+1} + b_{s+2}} \dots g_{q_{k+j-\nu_0-s}}^{+b_{j-\nu_0}} \right\} \cdot \sum_{P(p;q)} \left\{ \Phi_{p_1 p_2 \dots p_{r+2\nu_0}}^{q_1 q_2 \dots q_r} \delta_{p_{r+2\nu_0+1}}^{q_{r+1}} \delta_{p_{r+2\nu_0+2}}^{q_{r+2}} \dots \delta_{p_{k+j+\nu_0-s}}^{q_{k+j-\nu_0-s}} \right\}$$

can be represented also in the following manner

$$\begin{aligned} & \Phi_{p_1 p_2 \dots p_{r+2\nu_0}}^{q_1 q_2 \dots q_r} \delta_{p_{r+2\nu_0+1}}^{q_{r+1}} \delta_{p_{r+2\nu_0+2}}^{q_{r+2}} \dots \delta_{p_{k+j+\nu_0-s}}^{q_{k+j-\nu_0-s}} \\ & \quad \cdot \sum_{P(p;q)} \widehat{\beta}_{q_1}^{p_1} \dots \widehat{\beta}_{q_k}^{p_k} g_{a_{s+1}}^{p_{k+1}} g_{a_{s+2}}^{p_{k+2}} \dots g_{a_{j+\nu_0}}^{p_{k+j+\nu_0-s}} g_{q_{k+1}}^{+b_{s+1} + b_{s+2}} \dots g_{q_{k+j-\nu_0-s}}^{+b_{j-\nu_0}} \end{aligned}$$

Therefore we can rewrite (6.6) in a more convenient form

$$\begin{aligned} & T_g f_{a_1 a_2 \dots a_{j+\nu_0}}^{b_1 b_2 \dots b_{j-\nu_0}} = \sum_{s=0}^{j-\nu_0} \sum_{k=0}^{\infty} \sum_{j'=\nu_0}^{k+j-s} (z_c z')^{\frac{i\rho}{2} - 1 - j'} \Phi_{p_1 p_2 \dots p_{j'+\nu_0}}^{q_1 q_2 \dots q_{j'-\nu_0}} \alpha_0^{\frac{i\rho}{2} - 1 - j+s} \\ & \quad \cdot \alpha(j-\nu_0, s, 2\nu_0) \beta(k+j-\nu_0-s, j'-\nu_0, 2\nu_0) \frac{\Gamma\left(\frac{i\rho}{2} - j+s\right)}{k! \Gamma\left(\frac{i\rho}{2} - j+s-k\right)} \delta_{p_{j'+\nu_0+1}}^{q_{j'-\nu_0+1}} \delta_{p_{j'+\nu_0-2}}^{q_{j'-\nu_0-2}} \\ & \quad \quad \quad \dots \delta_{p_{k+j+\nu_0-s}}^{q_{k+j-\nu_0-s}} \\ & \quad \cdot \frac{1}{(j-\nu_0)! (j+\nu_0)! (k+j-\nu_0-s)! (k+j+\nu_0-s)!} \sum_{P(a;b;p;q)} \widehat{\beta}_{q_1}^{p_1} \dots \widehat{\beta}_{q_k}^{p_k} \delta_{a_1}^{b_1} \dots \delta_{a_s}^{b_s} g_{a_{s+1}}^{p_{k+1}} \\ & \quad \quad \quad \dots g_{a_{j+\nu_0}}^{p_{k+j+\nu_0-s}} g_{q_{k+1}}^{+b_{s+1}} \dots g_{q_{k+j-\nu_0-s}}^{+b_{j-\nu_0}} \end{aligned} \quad (6.7)$$

where we put $r + \nu_0 \equiv j'$. From (6.7) and from definitions (5.1) and (5.6) we get the following expression for $D_{a_1 a_2 \dots; d_1 d_2 \dots}^{b_1 b_2 \dots; c_1 c_2 \dots}(g)$:

$$\begin{aligned}
 & D_{a_1 a_2 \dots; d_1 d_2 \dots}^{b_1 b_2 \dots; c_1 c_2 \dots}(g) \\
 &= \sum_{k=\max(0, j'-j)}^{\infty} \sum_{s=0}^{\min(j-\nu_0, j+k-j')} \alpha_0^{\frac{i\rho}{2}-1-j+s} \alpha(j-\nu_0, s, 2\nu_0) \\
 & \quad \beta(k+j-\nu_0-s, j'-\nu_0, 2\nu_0) \frac{\Gamma\left(\frac{i\rho}{2}-j+s\right)}{k! \Gamma\left(\frac{i\rho}{2}-j+s-k\right)} \\
 & \quad \cdot \frac{1}{(j-\nu_0)!(j+\nu_0)!(k+j-\nu_0-s)!(k+j+\nu_0-s)!} \delta_{c'_{j'+\nu_0+1}}^{d_{j'-\nu_0+1}} \delta_{c'_{j'+\nu_0+2}}^{d_{j'-\nu_0+2}} \dots \delta_{c_{k+j+\nu_0-s}}^{d_{k+j+\nu_0-s}} \\
 & \quad \cdot \sum_{P(a;b;c;d)} \widehat{\beta}_{d_1}^{c_1} \dots \widehat{\beta}_{d_k}^{c_k} \delta_{a_1}^{b_1} \dots \delta_{a_s}^{b_s} g_{a_{s+1}}^{c_{s+1}} \dots g_{a_{j+\nu_0}}^{c_{j+\nu_0}} g_{d_{k+1}}^{+b_{s+1}} \dots g_{d_{k+j-\nu_0-s}}^{+b_{j-\nu_0}} \quad (6.8)
 \end{aligned}$$

Putting here the explicit expressions (5.3) and (5.5) for α and β we obtain the final result:

$$\begin{aligned}
 & D_{a_1 a_2 \dots; d_1 d_2 \dots}^{b_1 b_2 \dots; c_1 c_2 \dots}(g) \\
 &= \sum_{k=\max(0, j'-j)}^{\infty} \sum_{s=0}^{\min(j-\nu_0, j+k-j')} (-1)^s \frac{(2j-s)!(2j'+1)!}{s! k! (2j)!(j'-\nu_0)!(j'+\nu_0)!(j-\nu_0-s)!(j+\nu_0-s)!} \\
 & \quad \frac{\Gamma\left(\frac{i\rho}{2}-j+s\right)}{\Gamma\left(\frac{i\rho}{2}-j+s-k\right)} \alpha_0^{\frac{i\rho}{2}-1-j+s} \delta_{c'_{j'+\nu_0+1}}^{d_{j'-\nu_0+1}} \delta_{c'_{j'+\nu_0+2}}^{d_{j'-\nu_0+2}} \dots \delta_{c_{k+j+\nu_0-s}}^{d_{k+j+\nu_0-s}} \\
 & \quad \cdot \sum_{P(a;b;c;d)} \widehat{\beta}_{d_1}^{c_1} \widehat{\beta}_{d_2}^{c_2} \dots \widehat{\beta}_{d_k}^{c_k} \delta_{a_1}^{b_1} \dots \delta_{a_s}^{b_s} g_{a_{s+1}}^{c_{s+1}} \dots g_{a_{j+\nu_0}}^{c_{j+\nu_0}} g_{d_{k+1}}^{+b_{s+1}} \dots g_{d_{k+j-\nu_0-s}}^{+b_{j-\nu_0}} \quad (6.9)
 \end{aligned}$$

Consider now some particular cases:

1. If g is a pure Lorentz transformation in the (x_3, x_4) plane, i. e.

$$g = \varepsilon = \begin{pmatrix} \varepsilon^{-1} & 0 \\ 0 & \varepsilon \end{pmatrix}$$

then

$$gg = \begin{pmatrix} \varepsilon^{-2} & 0 \\ 0 & \varepsilon^2 \end{pmatrix} = \frac{1}{2}(\varepsilon^{-2} + \varepsilon^2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}(\varepsilon^{-2} - \varepsilon^2) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and therefore

$$\alpha_0 = \frac{1}{2}(\varepsilon^{-2} + \varepsilon^2); \quad \widehat{\beta} = \frac{\varepsilon^{-2} - \varepsilon^2}{\varepsilon^{-2} + \varepsilon^2} \sigma_3 \quad (6.10)$$

Putting (6.10) into (6.9) we get:

$$\begin{aligned} & D_{a_1 a_2 \dots a_{j+v_0}; c_1 c_2 \dots c_{j'+v_0}}^{b_1 b_2 \dots b_{j-v_0}; d_1 d_2 \dots d_{j'-v_0}}(\varepsilon) \\ &= \sum_{k=\max(0, j'-j)}^{\infty} \sum_{s=0}^{\min(j-v_0+k-j')} \sum_{m=0}^{\infty} (-1)^{s+m} \frac{(2j-s)!(2j'+1)!}{m!s!k!(2j)!(j'-v_0)!(j'+v_0)!(j-v_0-s)!} \\ & \quad \frac{1}{2^{k+m}} \cdot \frac{\Gamma\left(\frac{i\rho}{2} - j+s\right)}{\Gamma\left(\frac{i\rho}{2} - j+s-k-m\right)} \varepsilon^{2(j-s+1-\frac{i\rho}{2})} (1-\varepsilon)^{4m+k} \delta_{c_{j'+v_0+1}^{d_{j'-v_0+1}}} \delta_{c_{j'+v_0+2}^{d_{j'-v_0+2}}} \dots \delta_{c_{k+j+v_0-s}^{d_{k+j+v_0-s}}} \\ & \cdot \sum_{P(a;b;c;d)} (\sigma_3)_{d_1}^{c_1} \dots (\sigma_3)_{d_k}^{c_k} \delta_{a_1}^{b_1} \dots \delta_{a_s}^{b_s} \varepsilon_{a_{s+1}}^{c_{s+1}} \dots \varepsilon_{a_{j+v_0}}^{c_{k+j+v_0-s}} \varepsilon_{d_{k+1}}^{b_{s+1}} \dots \varepsilon_{d_{k+j-v_0-s}}^{b_{j-v_0}} \quad (6.11) \end{aligned}$$

If we put $j = j'$ in Eq. (6.11) and

$$\begin{aligned} a_1 = a_2 = \dots = a_{j+v_0} = c_1 = c_2 = \dots = c_{j+v_0} = 1 \\ b_1 = b_2 = \dots = b_{j-v_0} = d_1 = d_2 = \dots = d_{j-v_0} = 2, \end{aligned}$$

then we have

$$\begin{aligned} D_{ij;ij}^{v_0\rho}(\varepsilon) &\equiv D_{\substack{22\dots 2; 11\dots 1 \\ 11\dots 1; 22\dots 2}}^{\substack{j-v_0 \text{ times} & j+v_0 \text{ times} \\ j+v_0 \text{ times} & j-v_0 \text{ times}}}(\varepsilon) \\ &= \varepsilon^{2(j-v_0+1-\frac{i\rho}{2})} F\left(j+1-\frac{i\rho}{2}, j-v_0+1; 2j+2; 1-\varepsilon^4\right) \end{aligned}$$

This result can be obtained also from (4.11).

2. If g is a Lorentz transformation from the rest frame of a particle with mass m_0 to the frame in which this particle has momentum $p_\mu = (\vec{p}, iE)$ then

$$g = \frac{E + m_0 - \vec{\sigma} \cdot \vec{p}}{\sqrt{2m_0(E + m_0)}}; \quad gg^+ = \frac{E - \vec{\sigma} \cdot \vec{p}}{m_0}$$

and therefore

$$\begin{aligned} \alpha_0 &= \frac{E}{m_0} \\ \widehat{\beta} &= -\frac{1}{E} \vec{\sigma} \cdot \vec{p} \end{aligned}$$

3. In the case $j = 0$ ($\nu_0 = 0$) formula (6.9) becomes

$$D_{d_1 d_2 \dots d_{j'}}^{c_1 c_2 \dots c_{j'}}(g) = \alpha_0^{\frac{i\rho}{2} - 1} \cdot \sum_{k=j'}^{\infty} \frac{(2j' + 1)!}{(j'!)^2 (k - j')! (k + j' + 1)!} \cdot \frac{\Gamma\left(\frac{i\rho}{2}\right)}{\Gamma\left(\frac{i\rho}{2} - k\right)} \cdot \delta_{c_{j'+1}}^{d_{j'+1}} \delta_{c_{j'+2}}^{d_{j'+2}} \dots \delta_{c_k}^{d_k} \sum_{P(c)} \widehat{\beta}_{d_1}^{c_1} \widehat{\beta}_{d_2}^{c_2} \dots \widehat{\beta}_{d_k}^{c_k}$$

§ 7. MOST DEGENERATE PRINCIPAL SERIES OF $SL(n, C)$

For the most degenerate principal series of $SL(n, c)$ we can immediately use the method developed in § 5, 6, for the group $SL(2, C)$ with some light changes. Thus, instead of Eq. (5.1) we have

$$f_{a_1 a_2 \dots a_{j+\nu_0}}^{b_1 b_2 \dots b_{j-\nu_0}}(z_1, \dots, z_n) = (z_c \bar{z}^c)^{\frac{i\rho}{2} - \frac{n}{2} - j} \Phi_{a_1 a_2 \dots a_{j+\nu_0}}^{b_1 b_2 \dots b_{j-\nu_0}}, \quad |a, b = 1, 2, \dots, n|$$

where j, ν_0 are also integer or half-integer, however, j does not still denote the spin. The formulæ (5.2) and (5.4) remain if instead of the expressions (5.3) and (5.5) for α and β we take

$$\alpha(t, s, k) = (-1)^s \frac{t! (t+k)! (2t+k+n-2-s)!}{s! (t-s)! (t+k-s)! (2t+k+n-2)!}$$

$$\beta(t, s, k) = \frac{t! (t+k)! (2s+k+n-1)!}{s! (s+k)! (t-s)! (t+k+n+s-1)!}$$

In order to get the formula for matrix elements

$$D_{a_1 a_2 \dots a_{j+\nu_0}}^{b_1 b_2 \dots b_{j-\nu_0}; c_1 c_2 \dots c_{j'+\nu_0}}(g)$$

corresponding to the transformation $z_a \rightarrow z'_a = z_b g_{ba}$ we make the analogous procedure as in § 6. The only difference is that we must now expand g in terms of the matrices generators λ_i of the subgroup $SU(n)$. The results is:

$$D_{a_1 a_2 \dots a_{j+\nu_0}}^{b_1 b_2 \dots b_{j-\nu_0}; c_1 c_2 \dots c_{j'+\nu_0}}(g) - \sum_{k=\max(0, j'-j)}^{\infty} \sum_{s=0}^{\min(j-\nu_0, j+k-j')} (-1)^s \frac{(2j+n-2-s)! (2j'+n-1)!}{s! k! (j-\nu_0-s)! (j+\nu_0-s)! (2j+n-2)! (j'-\nu_0)! (j'+\nu_0)! (k+j-j'-s)! (k+j+j'-s+n-1)!}$$

$$\begin{aligned} & \frac{\Gamma\left(\frac{i\rho}{2} - \frac{n}{2} - j + s + 1\right)}{\Gamma\left(\frac{i\rho}{2} - \frac{n}{2} - j + s + 1 - k\right)} \alpha_0^{2 - \frac{n}{2} - j + s} \cdot \delta_{c_j' + \nu_0 + 1}^{d_j' - \nu_0 + 1} \delta_{c_j' + \nu_0 + 2}^{d_j' - \nu_0 + 2} \cdots \delta_{c_{k+j} + \nu_0 - s}^{d_{k+j} - \nu_0 - s} \\ & \cdot \sum_{P(a;b;c;d)} \widehat{\beta}_{d_1}^{c_1} \widehat{\beta}_{d_2}^{c_2} \cdots \widehat{\beta}_{d_k}^{c_k} \delta_{a_1}^{b_1} \delta_{a_2}^{b_2} \cdots \delta_{a_s}^{b_s} g_{a_{s+1}}^{c_{k+1}} \cdots g_{a_j + \nu_0}^{c_{k+j} + \nu_0 - s} g_{d_{k+1}}^{+b_{s+1}} \cdots g_{d_{k+j} - \nu_0 - s}^{+b_j - \nu_0} \end{aligned}$$

where

$$gg = \alpha_0 \mathbf{I} + \sum_{i=1}^{n^2-1} \alpha_i \lambda_i = \alpha_0 (1 + \widehat{\beta}); \quad \widehat{\beta} \equiv \frac{1}{\alpha_0} \sum_{i=1}^{n^2-1} \alpha_i \lambda_i$$

§ 8. SPACE REFLECTION FOR THE GROUP SL(2, C)

Now we identify the group SL(2, C) with the homogeneous proper Lorentz group and consider the space reflection P. There exist the following relations between P and the generators of the SL(2, C) group:

$$\begin{aligned} P M_j^i P^{-1} &= M_j^i \\ P N_j^i P^{-1} &= -N_j^i \\ P^2 &= 1, \end{aligned} \tag{8.1}$$

where M_j^i and N_j^i are compact and non-compact generators, respectively.

As it was known, the commutation relations for M_j^i and N_j^i are of the form:

$$\begin{aligned} [M_j^i, M_l^k] &= \delta_l^i M_j^k - \delta_j^k M_l^i \\ [N_j^i, N_l^k] &= -\delta_l^i M_j^k + \delta_j^k M_l^i \\ [M_j^i, N_l^k] &= \delta_l^i N_j^k - \delta_j^k N_l^i \end{aligned} \tag{8.2}$$

It is easy to see that in the space of homogeneous functions with canonical basis (4.7) these generators are ⁽¹⁾:

$$\begin{aligned} M_j^i &= z \frac{\partial}{\partial z_i} - \bar{z}^i \frac{\partial}{\partial \bar{z}^j} - \frac{1}{2} \delta_j^i \left(z_a \frac{\partial}{\partial z_a} - \bar{z}^a \frac{\partial}{\partial \bar{z}^a} \right) \\ i N_j^i &= z_j \frac{\partial}{\partial z_i} + \bar{z}^i \frac{\partial}{\partial \bar{z}^j} - \frac{1}{2} \delta_j^i \left(z_a \frac{\partial}{\partial z_a} + \bar{z}^a \frac{\partial}{\partial \bar{z}^a} \right) \end{aligned} \tag{8.3}$$

⁽¹⁾ The correspondence between our M_j^i, N_j^i and H, F in reference [12] is following:

$$\begin{aligned} M_2^1 &= H_- \quad , \quad M_1^2 = H_+ \quad , \quad M_1^1 = -M_2^2 = H_3 \\ N_2^1 &= F_- \quad , \quad N_1^2 = F_+ \quad , \quad N_1^1 = -N_2^2 = F_3 \end{aligned}$$

From eqs. (8.1) and (8.3) it follows that the operator P acts in such a way that

$$\begin{aligned} z_a &\rightarrow \varepsilon_{ab} \bar{z}^b \\ \bar{z}^a &\rightarrow z_b \varepsilon^{ba} \\ \varepsilon_{ab} = \varepsilon^{ab} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{aligned}$$

and therefore

$$Pf(z_1, z_2; \bar{z}^1, \bar{z}^2) = f(\bar{z}^2, -\bar{z}^1; -z_2, z_1) \quad (8.4)$$

From eqs. (8.4) and (8.5) after some simple calculations we get:

$$Pf_{jm}^{\nu_0 \rho}(z_1, z_2) = (-1)^{\frac{i\rho}{2} - 1 - j} f_{jm}^{-\nu_0 \rho}(z_1, z_2) \quad (8.5)$$

Thus, under P the basis elements of representation $\mathfrak{C}_{\nu_0 \rho}$ transform into the basis elements of $\mathfrak{C}_{-\nu_0 \rho}$ which in its turn is equivalent to $\mathfrak{C}_{\nu_0, -\rho}$. This means that under P only the space $D\left(\frac{i\rho}{2} - 1, \frac{i\rho}{2} - 1\right)$ ($\nu_0 = 0$) or the space $D_{(\nu_0 - 1, -\nu_0 - 1)}(\rho = 0)$ transforms into itself. Moreover, it is seen from (8.5) that the parity of the basis vectors f_{jm} differs one from another by a factor $(-1)^j$.

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