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Spherically symmetric non-statical solutions of Einstein's equations

by

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ABSTRACT. — A group of metrics describing a spherically symmetric mass in radial motion is investigated. If the mass is at rest at the initial moment, it will collapse or expand according as the radial coordinate of the boundary, r_b, is greater or less than a limiting value, r_a. During the expansion or contraction of the mass, the density is a function of the time alone while the pressure is a function of the time and the radial coordinate. In the case of collapse no oscillatory motions are found; the mass collapses into a singular state of infinite density and pressure of zero volume. The internal metrics are fitted in a preliminary fashion to an external Schwarzschild metric.

I. — INTRODUCTION

The object of this paper is to study in detail one of the sub-classes of motions of collapse or of expansion of a spherically symmetrical mass, the general theory of which has been given by one of us (McVittie, 1966). The following results were there established:

(i) With co-moving coordinates the orthogonal metric inside the material has the form

$$\left.\begin{aligned} ds^2 &= y^2dt^2 - R_0^2S^2e^{\eta}(dr^2 + f^2d\omega^2)/c^2, \\ y &= 1 - \eta_z/2, \\ d\omega^2 &= d\theta^2 + \sin^2\theta d\Phi^2. \end{aligned}\right\} \quad (1)$$

Here R_0 and c are constants, of which R_0 is a length and c a velocity, f is a dimensionless function of r, S is a function of t, y and η are dimensionless functions of the variable z which is given by

$$e^z = Q/S, \tag{2}$$

where Q is still another function of r.

(ii) The stress in the material is isotropic.
(iii) The cosmical constant in Einstein's field equations is zero.

From condition (ii) and the fact that r and $y(z)$ are independent variables, the following three ordinary second-order differential equations were obtained:

$$Q_{rr}/Q - Q_r f_r/fQ = a(Q_r/Q)^2, \tag{3}$$

$$f_{rr}/f - f_r^2/f^2 + 1/f^2 = b(Q_r/Q)^2, \tag{4}$$

$$y_{zz} + (a - 3 + y)y_z + y\{a + b - 2 - (a - 3)y - y^2\} = 0, \tag{5}$$

where a and b are constants. The integral of (3) is

$$f = AQ_r/Q^a, \tag{6}$$

where A is the constant of integration. A new radial coordinate q may be introduced by

$$\left.\begin{aligned} q &= A(1 - a)^{-1}Q^{1-a}, \quad a \neq 1 \\ &= A \ln Q, \quad a = 1 \end{aligned}\right\} \tag{7}$$

whence

$$f = \frac{dq}{dr}. \tag{8}$$

Equation (4) becomes

$$\left.\begin{aligned} f_{qq} + 1/f^3 &= b(1 - a)^{-2}f/q^2, \quad a \neq 1 \\ &= (b/A^2)f, \quad a = 1 \end{aligned}\right\}. \tag{9}$$

This completes the summary of the general theory.

II. — THE SUB-CLASS OF MOTIONS

The sub-class of motions is defined by specifying the constants a and b in the following way

$$a \neq 1, \qquad a \neq 3 \qquad \text{and} \qquad b = 2 - a.$$

It is then easily verified that a particular solution of equation (9) is given by

$$\left.\begin{aligned} f^2 &= q/b_1 \\ b_1^2 &= \frac{1}{4} + \frac{b}{(1-a)^2} \end{aligned}\right\} \tag{10}$$

and it has been shown (McVittie, 1966) that the metric of the 3-space—which is of variable curvature—may then be put in the form

$$d\sigma^2 = 4b_1^2 df^2 + f^2 d\omega^2. \tag{11}$$

The functions y, η, f and Q have now to be determined. From the first of equations (7) and equation (10) it is found that

$$f = (q/b_1)^{1/2} = \left\{ \frac{A}{(1-a)b_1} \right\}^{1/2} Q^{(1-a)/2}. \tag{12}$$

This expression for f combined with equation (6), yields

$$\{ Ab_1(1-a) \}^{-1/2} dr = Q^{-(1+a)/2} dQ$$

whose solution is

$$Q^{(1-a)/2} = \frac{1}{2} \left(\frac{1-a}{Ab_1} \right)^{1/2} r + \text{Const.} \tag{13}$$

But if q and r are to tend together to zero the constant must be zero and so

$$f^2 = r^2/(4b_1^2) \tag{14}$$

$$Q = \left(\frac{1-a}{4Ab_1} \right)^{1/(1-a)} r^{2/(1-a)}, \tag{15}$$

where

$$b_1 = \pm \frac{a-3}{2(1-a)}. \tag{16}$$

When $b = 2 - a$, there exists a particular first integral of equation (5) and it is this integral that will be used in the sequel. The particular first integral is

$$\frac{dy}{dz} = (3-a)y - y^2, \tag{17}$$

whose solution is

$$y = (3-a) \{ e^{(a-3)z} + 1 \}^{-1}. \tag{18}$$

The constant of integration is here taken equal to -1. From the second of equations (1) it follows that

$$\frac{d\eta}{dz} = 2 - 2y,$$

the solution of which is

$$\eta = 2z + 2 \ln \left\{ \frac{e^{(a-3)z}}{e^{(a-3)z} + 1} \right\}.$$

Hence

$$y^2 = (3-a)^2 \{ e^{(a-3)z} + 1 \}^{-2}, \tag{19}$$

$$e^{\eta} = e^{2z} \left\{ \frac{e^{(a-3)z}}{e^{(a-3)z} + 1} \right\}^2. \tag{20}$$

If the components of the energy-tensor are written

$$T_4^4 = \rho \ , \ T_1^1 = T_2^2 = T_3^3 = -p/c^2,$$

where ρ and p are defined to be the density and isotropic pressure, respectively, then the Einstein field equations reduce to the two equations (McVittie, 1966 *a*)

$$\begin{aligned} 8\pi G\rho = 3(S_t/S)^2 + \frac{c^2 e^{-\eta}}{R_0^2 S^2} [3(1 - f_r^2)f^{-2} - 6(1-y)Q_r f_r/Qf \\ - \{ 2b - 2y_z + (1-y)(2a - 1 - y) \} (Q_r/Q)^2], \end{aligned} \tag{21}$$

$$\begin{aligned} 8\pi G p/c^2 = \frac{1}{y} [- 2S_{tt}/S - (3y - 2)(S_t/S)^2 - \frac{c^2 e^{-\eta}}{R_0^2 S^2} \{ y(1 - f_r^2)f^{-2} \\ + 2(y^2 - y - y_z) f_r Q_r/Qf + (1-y)(y^2 - y - 2y_z)(Q_r/Q)^2 \}]. \end{aligned} \tag{22}$$

After inserting the expressions for the known functions on the right hand sides of equations (21) and (22), we get after some calculation

$$\frac{8\pi G\rho}{3} = \left(\frac{S_t}{S}\right)^2 + \frac{4c^2}{R_0^2 r^2} \left(\frac{a-3}{1-a}\right)^2 \frac{1}{e^{(a-1)z} S^2}, \tag{23}$$

and

$$\begin{aligned} \frac{8\pi G p}{c^2} = 2 \frac{e^{(a-3)z} + 1}{a-3} \frac{S_{tt}}{S} - \frac{2e^{(a-3)z} + 3a - 7}{a-3} \left(\frac{S_t}{S}\right)^2 \\ + \frac{4c^2}{R_0^2 r^2 e^{(a-1)z}} \left(\frac{a-3}{a-1}\right)^2 (e^{(a-3)z} - 2) \frac{1}{S^2}. \end{aligned} \tag{24}$$

The se equations describe a spherically symmetric mass of material which is in radial motion and which may be considered to have a boundary at

$r = r_b$. The boundary condition selected will be $p = 0$ at $r = r_b$; then from equation (24),

$$2\frac{S_{tt}}{S} - \frac{2B + (3a-7)S^{a-3}}{B + S^{a-3}}\left(\frac{S_t}{S}\right)^2 + C\frac{B - 2S^{a-3}}{B + S^{a-3}}S^{a-3} = 0. \quad (25)$$

Here

$$B = Q_b^{a-3} = \left\{\pm\frac{(a-1)^2}{2A(a-3)}r_b^2\right\}^{(a-3)/(1-a)}, \quad (26)$$

$$C = \frac{c^2}{R_0^2}\frac{(a-3)^3}{(1-a)}\frac{1}{Ab_1} = \pm\frac{2c^2}{R_0^2A}(a-3)^2. \quad (27)$$

The first integral of equation (25) is

$$S_t^2 = KS^2(B + S^{a-3})^3 - C_1S^{a-1} \quad (28)$$

where

$$C_1 = \frac{C}{a-3} = \pm\frac{2c^2}{R_0^2A}(a-3), \quad (29)$$

and K is the constant of integration. The constant K is determined from the initial conditions at $t = 0$, namely, $S_0 = 1$ and $(S_t^2)_0 = \beta^2$. It is

$$K = \frac{\beta^2 + C_1}{(1+B)^3}. \quad (30)$$

Inserting the expressions for S_t^2 and S_{tt} into equations (23) and (24) we obtain, after some calculation,

$$(8\pi G/3)\rho = K(B + S^{a-3})^3 \quad (31)$$

$$(8\pi G/3)p/c^2 = BK(\bar{r}^h - 1)(B + S^{a-3})^2, \quad (32)$$

where $\bar{r}^2 = r^2/r_b^2$ and $h = 2(a-3)/(1-a)$. In order to discover if the requirements that $p \geq 0$, $\rho > 0$ are fullfilled, it is necessary to examine the values of the constants B and K. The ambiguity of sign in (26), (27) and (29) arises from the expression (16) for b_1. From equation (15) it follows that Q is real if the factor

$$\left(\frac{1-a}{4Ab_1}\right)^{1/(1-a)} = \left\{\pm\frac{(1-a)^2}{2A(a-3)}\right\}^{1/(1-a)}, \quad (33)$$

is real. Therefore if $a > 3$, the positive sign must be chosen in this factor and therefore also in (16), (26), (27) and (29), whereas if $a < 3$, the negative sign must be selected. This means that the constants C_1, B and K are always positive, and hence the density is always positive and homogeneous. The density is not zero at the boundary $r = r_b$ but the pressure is, of course, always zero there. If the pressure is to be positive within the spherical mass ($r < r_b$), the exponent of $\bar{r}$ in (32) must be negative. This occurs for $a > 3$ and also for $a < 1$. In both cases there is a singularity at the centre ($r = 0$), the pressure p being infinite there.

Writing out the expressions for b_1 and B in the two cases $a < 1$ and $a > 3$ separately, we find:

Case $a < 1$

$$b_1 = \frac{3 - a}{2(1 - a)}, \tag{34}$$

$$B = \left\{ \frac{2A(3 - a)}{(a - 1)^2 r_b^2} \right\}^{(3-a)/(1-a)}. \tag{35}$$

Case $a > 3$

$$b = -\frac{a - 3}{2(a - 1)}, \tag{36}$$

$$B = \left\{ \frac{2A(a - 3)}{(a - 1)^2 r_b^2} \right\}^{(a-3)/(a-1)}. \tag{37}$$

In both cases, by equation (14) and (16)

$$f^2 = \left(\frac{a - 1}{a - 3}\right)^2 r^2 \tag{38}$$

and the expression for the metric is

$$ds^2 = \frac{(3 - a)^2 S^{2(a-3)}}{(Q^{a-3} + S^{a-3})^2} \left[dt^2 - \frac{R_0^2}{c^2} \frac{Q^{2(a-2)}}{(3 - a)^2} S^{2(3-a)} \{ dr^2 + f^2 d\omega^2 \} \right], \tag{39}$$

where Q is given by (15). With the aid of equation (26) the metric can be written:

$$ds^2 = \frac{(3 - a)^2}{(BF\bar{r}^h + 1)^2} \left[dt^2 - \frac{R_0^2 r_b^2 B^{(h-2)/h}}{c^2 (3 - a)^2} F^2 \bar{r}^{h-2} \{ d\bar{r}^2 + \bar{f}^2 d\omega^2 \} \right]. \tag{40}$$

Here $F = S^{3-a}$, $\bar{r} = r/r_b$ and $\bar{f}^2 = f^2/r_b^2 = \left(\frac{a - 1}{a - 3}\right)^2 \bar{r}^2$. A simpler form for the metric is obtained by the coordinate transformation

$$\bar{r} = \bar{u}^{\frac{a-1}{a-3}}, \tag{41}$$

where $\bar{u} = u/u_b$ and $u = u_b$ means $r = r_b$. After some calculation we obtain

$$ds^2 = \frac{(3-a)^2}{(\mathrm{BF}+\bar{u}^2)^2}\left[\bar{u}^4 dt^2 - \frac{\mathrm{D}^2}{(3-a)^2 c^2}\,\mathrm{F}^2\left\{d\bar{u}^2 + \bar{u}^2 d\omega^2\right\}\right], \tag{42}$$

where the constant D is given by

$$\mathrm{D}^2 = \left(\frac{a-1}{a-3}\right)^2 \mathrm{R}_0^2 r_b^2 \mathrm{B}^{(h-2)/h}. \tag{43}$$

It is to be noticed that $(\bar{u}, \theta, \varphi)$ are isotropic spatial coordinates. It also follows that, when these coordinates are employed, the time dependence of the metric is given by the function $\mathrm{F}(t)$ rather then by $\mathrm{S}(t)$. The function also therefore controls the motion of the material. In terms of $\bar{u}$ and F, the equations (31) and (32) are

$$(8\pi \mathrm{G}/3)\rho = \mathrm{K}(\mathrm{B} + 1/\mathrm{F})^3 \tag{44}$$

$$(8\pi \mathrm{G}/3)p/c^2 = \mathrm{BK}(1/\bar{u}^2 - 1)(\mathrm{B} + 1/\mathrm{F})^2 \tag{45}$$

From (44) and (45) it follows that, whether $a < 1$ or $a > 3$, the pressure and the density increase when $\mathrm{F}(t)$ decreases from its initial value $\mathrm{F} = 1$. This is the case of collapse. But clearly the density and pressure decrease from their initial values in the case of expansion when F increases from its initial value. In either case the density is uniform at each instant of time and

$$p = c^2\mathrm{B}\left(\frac{8\pi \mathrm{G}}{3\mathrm{K}}\right)^{-1/3}\left(\frac{1}{\bar{u}^2} - 1\right)\rho^{2/3}.$$

But $\bar{u}$, like r, is a co-moving coordinate. Hence the pressure is proportional to the two-thirds power of the density throughout the motion at any fixed value of $\bar{u}$. But the constant of proportionality varies from one internal point to another.

The metric (42) could therefore be used to describe an « expanding universe » in which there was a singular point at $\bar{u} = 0$. However, it is perhaps of more interest to consider the gravitational collapse problem in which F decreases from its initial value and in which the collapsing material is not coextensive with the whole universe. Collapse from rest is the simplest case and it will be assumed that, at the initial instant, $\mathrm{F}_0 = \mathrm{S}_0 = 1$, $(\mathrm{F}_t)_0 = (\mathrm{S}_t)_0 = \beta = 0$. Since $\mathrm{F} = \mathrm{S}^{3-a}$, it follows from (25) that the initial value of F_{tt} is

$$(\mathrm{F}_{tt})_0 = (3-a)(\mathrm{S}_{tt})_0 = \frac{1}{2}(a-3)\mathrm{C}(\mathrm{B}-2)(\mathrm{B}+1)^{-1},$$

where C is given by equation (27). For collapse to occur, it must be the case that $(F_{tt})_0 < 0$. Since the positive sign in (27) must be chosen when $a > 3$, and the negative sign when $a < 1$, it follows in both cases that $(a - 3)C$ is positive. Therefore collapse occurs if

$$B < 2.$$

From (35) and (37) it follows that

$$\left. \begin{aligned} r_b^2 &> \frac{(3-a)A}{(a-1)^2} 2^{2/(3-a)} = r_a^2 \quad \text{if} \quad a < 1 \\ r_b^2 &> \frac{(a-3)A}{(a-1)^2} 2^{2/(3-a)} = r_a^2 \quad \text{if} \quad a > 3 \end{aligned} \right\} . \tag{46}$$

It is therefore always possible to select one or more values of r_b so that the inequalities (46) shall be satisfied.

Further light on the nature of the collapsing motion is obtained from the formula for F_t which by (28) and (30) is

$$F_t^2 = \frac{(3-a)^2 C_1 F^2}{(1+B)^3} \left\{ \left(B + \frac{1}{F}\right)^3 - (1+B)^3 \frac{1}{F} \right\}. \tag{47}$$

Obviously if $F = 0$ can be attained, the value of F_t will then be infinite. However, F_t might vanish for some value of F in $0 < F < 1$ and this might lead to a reversal of the motion. Let $x = B + 1/F$, then the vanishing of F_t would occur at a root of the equation

$$x^3 - (1+B)^3(x - B) = 0.$$

Clearly $x = 1 + B$ is the root which corresponds to the initial instant. The other roots are given by

$$x = \frac{1}{2}(1+B)\{-1 \pm (1+4B)^{1/2}\}. \tag{48}$$

Thus if $F_t = 0$ in $0 < F < 1$, it must be the case that one of the two roots given by (48) is finite and also gıeater than $(1 + B)$. But it is easy to show that this would mean $B > 2$, whereas it has already been shown that $B < 2$. Therefore there is no value of F in $0 < F < 1$ at which the motion of collapse could be reversed so that oscillations could occur. This result is in accordance with an argument given by Hoyle *et al.* (1965) for masses greater than about one solar mass.

III. — THE INTERNAL AND THE EXTERNAL SOLUTIONS

Outside the spherical distribution of matter there is a vacuum in which the gravitational field is given by the Schwarzschild space-time whose metric may be written, in terms of a radial coordinate ξ, as

$$ds^2 = \left(\frac{1 - m/2\xi}{1 + m/2\xi}\right)^2 dT^2 - \frac{1}{c^2}\left(1 + \frac{m}{2\xi}\right)^4 (d\xi^2 + \xi^2 d\omega^2). \tag{49}$$

The Schwarzschild limit in this coordinate system is

$$\xi = m/2.$$

The physical boundary of the spherical mass is ξ_b. If

$$\xi = \xi_b \bar{\xi},$$

then $\bar{\xi} = 1$ at the boundary and the metric (49) takes the form

$$ds^2 = \left(\frac{1 - \alpha/\bar{\xi}}{1 + \alpha/\bar{\xi}}\right)^2 dT^2 - \xi_b^2 (1 + \alpha/\bar{\xi})^4 (d\bar{\xi}^2 + \bar{\xi}^2 d\omega^2)/c^2, \tag{50}$$

where

$$\alpha = \frac{1}{2} m/\xi_b. \tag{51}$$

The problem of fitting the internal solution (42) to the external solution (50) is facilitated by the introduction into (42) of a new time coordinate, τ, where

$$\tau = (3 - a)t, \quad \text{if} \quad a < 1,$$
$$\tau = (a - 3)t, \quad \text{if} \quad a > 3.$$

Then (42) reads

$$ds^2 = \frac{1}{(\mathrm{BF} + \bar{u}^2)^2}\left[\bar{u}^4 d\tau^2 - \frac{\mathrm{D}^2}{c^2}\mathrm{F}^2(\tau)\left\{d\bar{u}^2 + \bar{u}^2 d\omega^2\right\}\right], \tag{52}$$

where now F is regarded as a function of τ. The pressure vanishes at the boundary $\bar{u} = 1$ and is, of course, zero everywhere in the field (50). Thus the continuity of pressure is assured at the boundary. However the complete solution of the problem of fitting (50) to (52) would require the conversion of the metric (50) to co-moving coordinates and the subsequent establishment of « admissible coordinates », Synge (1960), in both space-times. A weaker solution of the problem might be arrived at as follows: Consider a case in which the motion begins from rest so that at the initial instant $(F_\tau)_0 = 0$ and $(F)_0 = 1$. At this initial instant the internal solution (52) describes a momentarily statical distribution of matter. The coefficients of the metrics (50) and (52) at the boundary might then be equated each to each provided that this equality is regarded as valid only for $\bar{\xi} = 1$ in (50) and for $F = 1$, $\bar{u} = 1$ in (52). However, it is not *a priori* evident that T and τ are times proceeding at the same rate, nor is ξ_b necessarily equal to the parameter with the dimension of length found in the definition (43) of D, namely $R_0 r_b$. Suppose therefore that

$$T = \mu\tau \quad , \quad \xi_b = \lambda\left(\frac{a-1}{a-3}\right)R_0 r_b, \tag{53}$$

where μ, λ are positive constants of proportionality. If the coefficients of the metrics (50) and (52) at the boundary are now equated it follows that

$$\mu^2\left(\frac{1-\alpha}{1+\alpha}\right)^2 = \frac{1}{(1+B)^2}, \tag{54}$$

$$\lambda^2(1+\alpha)^4 = \frac{B^{2(a-2)/(a-3)}}{(1+B)^2}, \tag{55}$$

where the definition (43) of D has also been used. The equation (54) yields

$$\alpha = \frac{\mu(1+B)-1}{\mu(1+B)+1},$$

and it is therefore clearly advantagous to write

$$\mu = \frac{1+\nu}{1+B}. \tag{56}$$

Since $0 < B < 2$, it follows that for a fixed value of ν,

$$\frac{1}{3}(1+\nu) < \mu < 1+\nu. \tag{57}$$

Also

$$\alpha = \frac{\nu}{2+\nu}, \qquad (\nu > 0) \tag{58}$$

and

$$\xi_b = \frac{1}{2}\, m/\alpha. \tag{59}$$

It is now evident from (56) and (57) that it is always possible to choose $\mu = 1$ and thus to have the internal time τ proceeding at the same rate as the external time T. There would thus be no discontinuity in the coordinate time rates on crossing the boundary. The condition for this is $\nu = B$ so that

$$\alpha = B/(2 + B), \tag{60}$$

and then by (55)

$$\lambda = \frac{(1 + B/2)^2}{(1 + B)^3} B^{(a-2)/(a-3)}.$$

In general therefore the value of λ cannot also be pre-assigned if $\tau \equiv T$. Very small initial configurations are possible: for example, if $B = 1$, then for any permissible a,

$$\xi_b = 3(m/2), \qquad \lambda = 9/32,$$

and if a happens to be equal to $3\,\frac{18}{23}$, it also follows from equation (53) that $\xi_b = R_0 r_b$. The initial radius—from the point of view of the external Schwarzschild space-time—is thus only three times the Schwarzschild limit. But the condition $B < 2$ leads, by (60), to $\frac{1}{\alpha} > 2$ and therefore the initial radius ξ_b cannot be as small as twice the Schwarzschild limit. Clearly if ξ_b is to be very large compared with $m/2$, as would be expected in any normal physical situation, $1/\alpha$ must be large, and therefore B must be small, compared with unity.

SUMMARY

The radial motions of a spherically symmetric mass which have been worked out correspond to the similarity flows of classical gas-dynamics in which the scale-factor is a function of the time given by equation (47). It has not been possible to find cases of oscillatory motions in this class of solutions of Einstein's equations; only motions of expansion or of collapse to a singular state of zero volume are possible. These depend on the boundary value of the radial coordinate, r_b, and collapse occurs if r_b satisfies the inequalities (46). The density is a function of time alone as in uniform models of the universe. Howevei the pressure varies both with time and with the radial coordinate measured from the center of the configuration, in the manner shown in equation (45). At a given instant of time, the

pressure varies as the inverse square of the radial coordinate. The pressure is zero at the outer boundary of the configuration and is infinite at the center.

The internal metric can be thrown into a number of forms such as those shown in equations (39), (40) and (52), of which the third is probably the simplest. In the case of collapse from an initial state of rest, the internal metric is fitted to an external Schwarzschild metric. This fitting is valid at the initial moment only but it serves to throw some light on the value of the mass-constant in the external Schwarzschild field relative to the constants involved in the internal metric.

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