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# Fluctuating density fields

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## 1. — INTRODUCTION

The concept of a fluctuating driving force in its very general form is imbedded in many a physical phenomenon as the motive cause behind them. The motion of a Brownian particle suffering random changes in its accelerations provides an example of such a process. The theoretical implications of such a motion and its impact on physics in general was not realised to an appreciable extent until 1905, when Einstein [*1*] published his investigations on the statistical properties of the displacements experienced by such a particle. Detailed study of such a fluctuating force $F(t)$ and the resultant Brownian motion was carried out by Uhlenbeck and Ornstein [*2*], Chandrasekhar [*3*] and Wang and Uhlenbeck [*4*]. In fact the particles executing Brownian oscillations obey the equation

$$m\frac{du}{dt} = -fu + F(t), \tag{1.1}$$

where $u$ is the velocity of the particle. The influence of the surrounding medium has been visualised to consist essentially of two parts: (1) A deterministic part $-fu$ which causes friction (2) A stochastic part F($t$) characterised by the following two properties:

($i$) The mean of F($t$) at a given $t$, over an ensemble of particles is zero

$$\overline{F(t)} = 0. \tag{1.2}$$

($ii$) The values of F($t$) at two different times $t_1$ and $t_2$ are not correlated at all except for extremely small intervals of $t$ i. e. for small values of $| t_1 - t_2 |$. More precisely

$$\overline{F(t_1)F(t_2)} = \Phi(t_1 - t_2) \tag{1.3}$$

where $\Phi(x)$ is a function with a very sharp maximum at $x = 0$. If we superpose on the above the following restrictions on the higher order correlations of F($t$),

$$(i) \qquad \overline{F(t_1)F(t_2) \ldots F(t_{2n+1})} = 0$$

$$(ii) \qquad \overline{F(t_1)F(t_2) \ldots F(t_{2n})} = \sum_{\text{all pairs}} \overline{F(t_i)F(t_j)}\,\overline{F(t_k)F(t_l)} \ldots \tag{1.4}$$

we obtain a complet description for $u$ and such a process is known as the Uhlenbeck-Ornstein process.

However it may be worthwhile to explore the possibility of generating the Uhlenbeck-Ornstein process from a force field F($t$) which need not necessarily be of Gaussian character. It is relevant in this context to visualise certain density fields introduced by Ambarzumian [5] and Chandrasekhar [6] who have studied the distribution of interstellar matter in the galectic region. A particular model of a fluctuating density field has been investigated by Ramakrishnan [7] in detail.

In this paper we propose to investigate, the possibility of representing a general density field by a Markovian process $\rho(t)$ evolving with $t$ and depending on a large parameter $a$. The parameter $a$ can be related in a way to the distance (measured along $t$) within which the correlations persist. It is shown that there exist a wide class of distributions for F($t$) which will give rise to physical phenomena like Brownian Motion.

Section 2 of the paper contains a short discussion on the Fokker-Planck equation and its solution with special reference to Markovian features of F($t$) and $u(t)$. Since the authors have not come across any proof of the Markovian character of $u(t)$ in the literature, a short discussion of this aspect has been considered to be not entirely out of context. The model of the density field $\rho(t)$ is introduced in section 3 which also contains an explicit demonstration of the approximate equivalence of the solution with the Uhlenbeck-Ornstein process. Physical applications of the model are discussed in the final section.

## 2. — GAUSSIAN PROCESS AND FOKKER-PLANCK EQUATIONS

As mentioned in section 1, (1.1) has been the starting point of many an investigation in this field. Our object is to solve (1.1) using the given random characteristics of $F(t)$. By solution we mean the probability frequency function of $u$ at different times. In solving for the distribution function of $u$, it is tacitly assumed that $u(t)$ constitutes a Markov process (see for example Chandrasekhar [*7a*], p. 32 remarks under equation (218)). The markovian assumption implies that $P(u \mid u_0, t)$ (where $P(u \mid u_0, t)\, du$ denotes the probability that $u(t)$ has a value between $u$ and $u + du$ given $u(t)$ had a value $u_0$ at $t = 0$) is given by (1) Smoluchowski equation

$$P(u \mid u_0\,;\, t + \Delta t) = \int dv P(v \mid u_0\,;\, t) P(u \mid v\,;\, \Delta t). \tag{2.1}$$

If we consider

$$\int du \Phi(u) \frac{\partial P(u \mid u_0\,;\, t)}{\partial t} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int du \Phi(u) . [P(u \mid u_0\,;\, t + \Delta t) - P(u \mid u_0\,;\, t)] \tag{2.2}$$

where $\Phi(u)$ is an arbitrary smooth function of $u$ tending to zero as $u \to \pm\, \infty$ and feed (2.1) into (2.2), we obtain after some manipulation,

$$\begin{aligned} \frac{\partial P(u \mid u_0\,;\, t)}{\partial t} = &- \frac{\partial}{\partial u} [P(u \mid u_0\,;\, t) a_1] \\ &+ \frac{1}{2} \frac{\partial^2}{\partial u^2} [P(u \mid u_0\,;\, t) a_2] \end{aligned} \tag{2.3}$$

where we have assumed

$$\begin{aligned} &\int P(u' \mid u\,;\, \Delta t)(u' - u)\; du' = a_1(u) \Delta t \\ &\int P(u' \mid u\,;\, \Delta t)(u' - u)^2 du' = a_2(u) \Delta t \\ &\int P(u' \mid u\,;\, \Delta t)(u' - u)^n dn' = 0(\Delta t). \end{aligned} \tag{2.4}$$

(1) We have also assumed that the process is homogeneous.

An application of this method to $P(u \mid u_0, t)$ where $u(t)$ satisfies the Langevin equation (1.1) leads to the familiar equation

$$\frac{\partial P(u \mid u_0\, ;\, t)}{\partial t} = \beta \frac{\partial (uP)}{\partial u} + D \frac{\partial^2 P}{\partial u^2} \tag{2.5}$$

if $\beta = f/m$

where it has been tacitly assumed that $u(t)$ is markovian and also that

$$\overline{F(t_1)F(t_2)} = 2D\delta(t_1 - t_2). \tag{2.6}$$

It is interesting to note that (2.3) is not a good approximation, if every one of the moments of $P(u' \mid u, \Delta t)$ is proportional to $\Delta t$ (see for example Lax [*8*]). In fact there exist a variety of phenomena wherein $P(u' \mid u; \Delta t)$ is given by

$$P(u' \mid u; \Delta t) = R(u' \mid u)\Delta t + \delta(u - u')\left\{ 1 - \Delta t \int R(u' \mid u)du' \right\} \tag{2.7}$$

and in such a case we obtain the generalized Fokker-Planck equation

$$\frac{\partial P(u \mid u_0\, ;\, t)}{\partial t} = \sum_{m=1}^{\infty} \frac{1}{m\,!}\left(-\frac{\partial}{\partial u}\right)^m [P(u \mid u_0\, ;\, t)a_m(u)] \tag{2.8}$$

where

$$a_m(u) = \int (u' - u)^m R(u' \mid u)du'. \tag{2.9}$$

(2.8) is still based on the markovian character of $u(t)$. Though the markovian character is quite plausible from intuitive physical argument (see Chandrasekhar [*7a*] and Moyal [*9*]), it is worthwhile to investigate whether the Langevin equation (1.1) together with the conditions imposed on $F(t)$ do imply such a property of $u(t)$. To this end we first notice that the vector $(u, F)$ constitutes a Markov process and hence we have

$$\pi(u, F \mid u_0, F_0\, ;\, t + \Delta t) = \int\int \pi(u', F' \mid u_0, F_0\, ;\, t) . \pi(u, F \mid u', F'\, ;\Delta t)du'dF' \tag{2.10}$$

Let us consider

$$\begin{aligned} \int\int \Phi(u, F)[\pi(u, F \mid u_0, F_0, t + \Delta) - \pi(u, F \mid u_0, F_0, t)]dudF \\ = \int\int \Phi(u, F)\pi(u, F \mid u_0, F_0\, ;\, t + \Delta)dudF \\ - \int\int \Phi(u, F)\pi(u, F \mid u_0, F_0\, ;\, t)dudF, \end{aligned} \tag{2.11}$$

where $\Phi(u, F)$ is an arbitrary smooth function of $u$ and F vanishing at $\pm\infty$. Using (2.10) in the first integral and expanding $\Phi(u, F)$ about $(u', F')$, we obtain

$$\iint \Phi(u, F)[\pi(u, F \mid u_0, F_0 ; t + \Delta t) - \pi(u, F \mid u_0, F_0 ; t)] du dF$$

$$= \iint \pi(u', F' \mid u_0, F_0 ; t)\left[\sum_{n=1}^{\infty} \frac{1}{n!} \sum_{r=0}^{n} \binom{n}{r} \alpha'_{n-r,r} \frac{\partial^n \Phi(u', F')}{\partial u'^{n-r} \partial F'^{r}}\right] du' dF' \quad (2.12)$$

where

$$\alpha'_{m,n} = \iint (u' - u)^m (F' - F)^n \pi(u, F \mid u', F' ; \Delta t) du dF. \quad (2.13)$$

It is easy to note that

$$\int \pi(u', F' \mid u, F ; \Delta t) du' = P(F' \mid F ; \Delta t) \quad (2.14)$$

and P in turn due to the delta correlated nature of the process F, can be assumed to be of the form

$$P(F' \mid F ; \Delta t) = \frac{1}{(4\pi \Delta t)^{\frac{1}{2}}} \exp\left[-\frac{(F - F')^2}{4D\Delta t}\right] \quad (2.15)$$

In view of the Langevin equation (1.1) and the nature of assumptions on $F(t)$, it follows that

$$\alpha'_{m,n} = \alpha_{m,n}\Delta t \qquad \text{for} \qquad m, n \leqslant 2$$

and

$$= 0(\Delta t) \qquad \text{for} \qquad m, n > 2.$$

Also $\alpha_{01} = 0$ and adopting the same technique as that used in obtaining (2.3), we arrive at a Fockker-Planck equation for $\pi(u, F \mid u_0, F_0; t)$ involving $\alpha_{20}, \alpha_{11}, \alpha_{02}, \alpha_{01}$ and $\alpha_{10}$.

$$\frac{\partial \pi(u, F \mid u_0, F_0 ; t)}{\partial t} = -\frac{\partial \pi(u, F \mid u_0, F_0 ; t)}{\partial u}\alpha_{10} - \frac{\partial \pi(u \mid u_0, F_0, t)}{\partial F}\alpha_{01}$$
$$+ \frac{1}{2}\left[\frac{\partial^2}{\partial u^2}\alpha_{20}\pi + 2\frac{\partial^2}{\partial u \partial F}\alpha_{11}\pi + \frac{\partial^2}{\partial F^2}\alpha_{02}\pi\right] + 0(\Delta t). \quad (2.16)$$

If we integrate both sides of such an equation over the entire domain of F, remembering that both $\pi$ and $\frac{\partial \pi}{\partial F}$, are well behaved at $\pm\infty$, we regain the usual Fokker-Planck equation, corresponding to the Langevin equation without starting with the usual markovian assumption for the $u(t)$ process

$$\frac{\partial \pi(u \mid u_0 ; t)}{\partial t} = \beta \frac{\partial \pi(u \mid u_0 ; t) u}{\partial u} + \mathrm{D} \frac{\partial^2 \pi(u \mid u_0 ; t)}{\partial u^2}. \qquad (2.17)$$

## 3. — A PROBABILISTIC MODEL OF DENSITY FIELD

The object of the present section is to demonstrate explicitly the possibility of arriving at a Gaussian distribution as an approximation. We emphasise on the word approximation since the deviation from Gaussian law can be readily deduced on more detailed considerations. Towards this end let us assume that $\rho(t)$ describes a process evolving with respect to $t$. $t$ can stand for time or spacial coordinate and we shall take $\rho$ to be a density field and hence assumes positive values for $\rho$. Thus with every $t$ we can associate a function $\pi(\rho \mid \rho_0; t, t_0)$ where $\pi(\rho \mid \rho_0; t, t_0)$ denotes the probability that $\rho(t)$ takes a value between $\rho$ and $\rho + d\rho$ at the parametric value $t$, given that $\rho$ had assumed a value $\rho_0$ at $t_0$. If the process is homogeneous than $\pi(\rho \mid \rho_0; t, t_0)$ is a function only of $(t - t_0)$ and we denote $\pi(\rho \mid \rho_0; t - t_0)$ by $\pi(\rho \mid \rho_0, t)$, setting $t_0 = 0$. We next make use of the markovian nature of $\rho(t)$ to write

$$\pi(\rho \mid \rho_0 ; t + \Delta t) = \int_{\rho'} \pi(\rho' \mid \rho_0, t) \pi(\rho \mid \rho', \Delta t) d\rho' \qquad (3.1)$$

In the derivation of Fokker-Planck equation it is usually assumed that $\pi(\rho \mid \rho'; \Delta t)$ is of such a nature that

$$\int \pi(\rho \mid \rho', \Delta t)(\rho - \rho') d\rho' \qquad \text{and} \qquad \int \pi(\rho \mid \rho', \Delta t)(\rho - \rho')^2 d\rho'$$

are proportional to $\Delta t$ while the higher moments of $(\rho - \rho')$ are of smaller order of magnitude than $\Delta t$. However it is worthwhile to examine whether the following conditions, imposed on $\pi(\rho \mid \rho'; \Delta t)$ can lead to physically meaningful results:

(*i*) The probability that in the interval $(t, t + \Delta t)$, $\rho$ jumps to a value different from the one that it has assumed at $t$, is proportional to $\Delta t$.

(*ii*) The probability that $\rho$ continues to take the same value as it had at $t$, in the interval $(t,\ t+\Delta t)$ is approximately equal to 1. Stated precisely the asymptotic form for $\pi$ for small $\Delta t$ is given by

$$\pi(\rho \mid \rho' ; \Delta t) = \mathrm{R}(\rho \mid \rho')\Delta t + \delta(\rho - \rho')\left[1 - \Delta \int \mathrm{R}(\rho \mid \rho')d\rho\right] \tag{3.2}$$

Thus $\pi(\rho \mid \rho', t)$ satisfies the Kolmogorov forward equation

$$\begin{aligned}\frac{\partial \pi(\rho \mid \rho_0, t)}{\partial t} = &- \int \mathrm{R}(\rho' \mid \rho)d\rho'\pi(\rho \mid \rho_0 ; t)\\ &+ \int \pi(\rho' \mid \rho_0, t)\mathrm{R}(\rho \mid \rho')d\rho'.\end{aligned} \tag{3.3}$$

Next we assume $\mathrm{R}(\rho' \mid \rho)d\rho'$ (see Ramakrishnan [7]) is function only of $\rho'$.

$$\mathrm{R}(\rho' \mid \rho)d\rho' = \mathrm{R}(\rho')d\rho' \tag{3.4}$$

This yields

$$\frac{\partial \pi}{\partial t} = - a\pi(\rho \mid \rho_0, t) + \mathrm{R}(\rho) \tag{3.5}$$

where

$$a = \int \mathrm{R}(\rho)d\rho. \tag{3.6}$$

This equation can be solved using the initial condition

$$\pi(\rho \mid \rho_0, 0) = \delta(\rho - \rho_0). \tag{3.7}$$

Thus

$$\pi(\rho \mid \rho_0, t) = \frac{\mathrm{R}(\rho)}{a}(1 - e^{-at}) + \delta(\rho - \rho_0)e^{-at}. \tag{3.8}$$

Notice (3.3) admits a sationary solution:

$$\lim_{t\to\infty} \pi(\rho \mid \rho_0, t) = \psi(\rho) = \frac{\mathrm{R}(\rho)}{a} \tag{3.9}$$

where $\psi(\rho)$ is not necessarily a Gaussian distribution.

It is interesting to compare the differential equation (3.5) with the corresponding Fokker-Planck equation for $\pi$ itself. The coefficients $a_1$, $a_2, \ldots, a_n$, can be calculated in terms of the moments of $\mathrm{R}(\rho)$.

$$\begin{aligned}a_n(\rho) &= \frac{1}{a}\int (\rho' - \rho)^n \mathrm{R}(\rho')d\rho'\\ &= \frac{1}{a}\sum \overline{\rho^i}\,(-)^i \rho^{n-i}\binom{n}{i}\end{aligned} \tag{3.10}$$

We notice $a_n(\rho)$ is a polynomial, in $\rho$, of degree utmost equal to $n$. However, we can arrive at the usual Fokker-Planck equation if $a_n(\rho) \ll 1$ for $n > 2$. We shall show that this is exactly the case if we make a suitable choice of the distribution $\psi(\rho)$.

Apart from these general comments, let us ask a definite question: Is the density correlation at two points the same as that expected from a Gaussian law? This can be answered by merely stipulating that $a$ be very large. In fact larger the $a$, more wild is the fluctuation for $a$ denotes the total probability per unit $t$ of a change in $\rho$.

If $t \gg \frac{1}{a}$ then $e^{-at}$ is vanishingly small and the second term can be neglected. Thus probability distribution of the density at $t$ is independent of that at $t = 0$. On the other hand if $t = \frac{1}{a}$, then the densities are correlated. The statement $a$ is very large implies the substitution rule

$$e^{-at} \to \frac{1}{a}\,\delta(t) \tag{3.11}$$

where $\delta$ is the Dirac delta function.

To get a clear picture of the distribution, we calculate the correlation functions of different orders. The second order correlation is given by

$$\begin{aligned}\varepsilon\{\rho(t_1)\rho(t_2)\} &= \int\int \pi(\rho_2, \rho_1\,;\,t_2, t_1)\rho_1\rho_2 d\rho_1 d\rho_2 \\ &= \int\int \rho_1\rho_2\psi(\rho_1)[\psi(\rho_2)(1 - e^{-a(t_2-t_1)}) \\ &\qquad\qquad + \delta(\rho_2 - \rho_1)e^{-a(t_2-t_1)}]d\rho_1 d\rho_2 \\ &= [\varepsilon\{\rho\}]^2[1 - e^{-a(t_2-t_1)}] + \varepsilon\{\rho^2\}e^{-a(t_2-t_1)}\end{aligned} \tag{3.12}$$

where

$$\varepsilon\{\rho^n\} = \int \rho^n\psi(\rho)d\rho. \tag{3.13}$$

If we wish to calculate the higher order moments then we need the joint probability frequency function $\pi(\rho_1, \rho_2, \ldots, \rho_n, t_1, t_2, \ldots, t_n)$. This can be calculated by the markovian nature of the problem. There are many approaches to the problem. In the work of Ramakrishnan, all moments of $\rho$ are assumed to be of order unity

$$\varepsilon\{\rho^n\} = 0(1). \tag{3.14}$$

We will take the role played by $a$ more explicitly by noting

$$\psi(\rho, a) = \frac{R(\rho)}{a} \tag{3.15}$$

where $a$ is very large and

$$\int \psi(\rho, a) d\rho = 1. \tag{3.16}$$

We can expect $\int \rho^n \psi(\rho, a) d\rho$ to be dependent on $a$. We can think of many models, and some of these are listed in the table below, along with the expressions for the moments.

| | $\psi(\rho)$ | $\bar{\rho}$ | $\bar{\rho}^2$ | $\bar{\rho}^n$ |
|---|---|---|---|---|
| I | $\frac{1}{a} e^{-\rho/a}$ | $a$ | $2a^2$ | $a^n n!$ |
| II | $e^{-\rho/a}\rho/a^2$ | $a$ | $2a^2$ | $a^n n!$ |
| III | $e^{-\rho/\sqrt{a}}/\sqrt{a}$ | $a^{1/2}$ | $2a$ | $a^{n/2} n!$ |
| IV | $e^{-\rho/\sqrt{a}}\rho/a$ | $a^{1/2}$ | $2a$ | $a^{n/2} n!$ |

The joint probability frequency-function $\pi(\rho_1, \ldots, \rho_n, t_1, \ldots, t_n)$ for large values of $t_1, t_2, \ldots, t_n$ is given by

$$\begin{aligned} \pi = \psi(\rho_1) & \left\{ \psi(\rho_2)\left[1 + \frac{1}{a}\delta(t_2 - t_1)\right] + \frac{1}{a}\delta(\rho_2 - \rho_1)\delta(t_2 - t_1) \right\} \\ & + \left\{ \psi(\rho_3)\left[1 + \frac{1}{a}\delta(t_3 - t_2)\right] + \frac{1}{a}\delta(\rho_3 - \rho_1)\delta(t_2 - t_3) \right\} \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{etc.} \end{aligned} \tag{3.17}$$

We can easily calculate the correlations of the variable

$$x = \rho(t) - \bar{\rho} \tag{3.18}$$

$$\varepsilon\{x(t_1)x(t_2)\} = \frac{1}{a}\varepsilon\{x^2\}\delta(t_2 - t_1) \tag{3.19}$$

$$\varepsilon\{x(t_1)x(t_2)x(t_3)\} = \frac{1}{a^2}\varepsilon\{x^3\}\delta(t_2 - t_1)\delta(t_3 - t_2) \tag{3.20}$$

$$\varepsilon\{x(t_1)x(t_2)x(t_3)x(t_4)\} = \frac{1}{a^2}[\varepsilon(x^2)]^2\delta(t_3 - t_4)\delta(t_2 - t_1) \tag{3.21}$$

Thus the only nonvanishing term in the $m^{\text{th}}$ order correlation has a factor $\left(\frac{1}{a}\right)^{I\left(\frac{m+1}{2}\right)}$ where $I\left(\frac{m+1}{2}\right)$ is the integral part of $(m+1)/2$. Let us find the mean and moments of the integral of $\rho$ over $t$ defined by

$$M(t) = \int_0^t \rho(t')dt' \qquad \text{and} \qquad x = (\rho - \bar{\rho}) \tag{3.22}$$

If we take the model

$$\psi(\rho) = \frac{\rho}{\sqrt{a}} e^{-|\rho|/\sqrt{a}} \tag{3.23}$$

we have

$$\varepsilon\{\rho\} = \sqrt{a} \quad , \quad \varepsilon\{\rho^2\} \sim a \tag{3.24}$$

$$\varepsilon\{\rho^3\} \sim a^{3/2}$$

$$\begin{aligned}
\varepsilon\{(M-\bar{M})^{2n}\} &= \int\int \varepsilon(x_1, x_2, \ldots, x_n)dt_1 \ldots dt_n \\
&= 2n! \int_0^t dt_1 \int_{t_1}^t dt_2 \int_{t_2}^t dt_3 \ldots \\
&= 2n! \int \psi(\rho_1)\rho_1^2 d\rho_1 \int \psi(\rho_3)\rho_3^2 d\rho_3 \ldots \int dt_1 \delta(t-t_2) \ldots \\
&= \left[\int_0^t dt_1 \int_{t_1}^t dt_2 e^{-a|t_2-t_1|}\right]\left[\int dt_3 \int dt_4 \delta(t_4-t_3)\right] \ldots \left[\frac{\varepsilon(x^2)}{a}\right]^n .
\end{aligned} \tag{3.25}$$

Removing the time ordering within each bracket gives $\frac{1}{2^n}$ and removing the ordering $0, t_2, t_4, \ldots, t_{2n-2}$ of the brackets relative to each other we obtain another $n!$. Finally we arrive at

$$\varepsilon\{(M-\bar{M})^{2n}\} = \frac{2n!}{2^n n!}\left[\frac{\varepsilon(x^2)t}{a}\right]^n \tag{3.26}$$

neglecting lower order terms. If $\varepsilon(x^2)$ is of order less than $a$, for a given distribution, we are led to the result that skewness and flatness factor vanish to order $\frac{1}{a}$.

All the old moments lead to terms of order $0\left(\frac{1}{a^{\varepsilon}}\right)$, $\varepsilon > 0$. Thus in this approximation we get the usual moments of the Gaussian distribution. However, if $\bar{\rho} = 0$ and $\rho$ is distributed over the interval $-\infty$ to $+\infty$ as $\frac{1}{2}\frac{e^{-|\rho|/\sqrt{a}}}{\sqrt{a}}$ the odd moments of $\rho$ are automatically zero. The probability frequency function is therefore the well-known one for the Gaussian process. We can obtain this result in a more straightforward manner if we construct the generalised Fokker-Planck equation from the forward equations. Before going to this, let us find the moments of the stochastic variable $e^{-M(t)}$ in the above approximation

$$\begin{aligned}\varepsilon\{e^{-n[M-\overline{M}]}\} &= \varepsilon\{e^{-n\widetilde{M}(t)}\}\\ &= \varepsilon\left\{1 - n\widetilde{M} + \frac{n^2}{2}[\widetilde{M}(t)]^2 + \dots\right\}\\ &= \varepsilon\left\{1 + \frac{n^2}{2!}[\widetilde{M}(t)]^2 + \dots + \frac{n^{2r}}{2r!}[\widetilde{M}(t)^r] + \dots\right\} \end{aligned} \tag{3.27}$$

since all the odd moments of M vanish. Thus

$$\begin{aligned}\varepsilon[e^{-nM(t)}] &= \sum\left[\frac{n^2}{2}\frac{\varepsilon[x^2]}{a}t\right]^r\frac{1}{r!}\\ &= e^{n\overline{M}(t)}e^{\left[n^2\frac{\varepsilon(x^2)}{2a}t\right]}.\end{aligned} \tag{3.28}$$

(3.28) is useful in the determination of the fluctuations in brightness of Milky way considered in reference [7] and we shall illustrate this in the final section of this paper.

## 4. — LANGEVIN EQUATIONS

For convenience we shall rewrite (1.1) as

$$\frac{du}{dt} + \beta u = \rho(t) \tag{4.1}$$

where

$$\beta = \frac{f}{m} \quad ; \quad \rho(t) = \frac{F(t)}{m}. \tag{4.2}$$

Let us assume that $\rho(t)$ is a fluctuating density field of the type described in section 3. Since $\rho(t)$ is markovian, it is clear that the vector process $(u, \rho)$ is again markovian and hence we can obtain a partial differential equation for $\pi(u, \rho, t)$ the joint probability frequency function of $u$ and $\rho$ by the forward differential equation technique of Kolomogorov (see Feller, 1951). Thus if we increase $t$ by $\Delta$ we notice that there are two mutually exclusive events according as whether or not the random variable $\rho$ makes a transition to a different value in the infinitesimal interval. Using elementary probability arguments and taking into account the changes in $u$ as governed by (4.1), we notice

$$\begin{aligned}\pi(u, \rho, t+\Delta) &= \pi(u + \beta u\Delta - \rho\Delta, \rho, t)(1 - a\Delta)(1 + \beta\Delta)\\ &\quad + a\Delta\int\pi(u, \rho', t)\psi(\rho)d\rho'.\end{aligned} \tag{4.3}$$

Proceeding to the limit as $\Delta \to 0$, we obtain

$$\frac{\partial \pi(u, \rho, t)}{\partial t} + (\rho - \beta u) \frac{\partial \pi(u, \rho, t)}{\partial u} + a\pi(u, \rho, t) + \beta\pi(u, \rho, t) = a\psi(\rho)\pi(u, t) \text{ (}^2\text{)} \tag{4.4}$$

The above deviation can be suitably modified to lead to the usual Fokker-Planck equation if we take note of the fact that $\rho$ changes very rapidly, in comparison with the variations in $u$, in the time $\Delta t$. To incorporate this we replace (4.3) by the following:

$$\pi(u, \rho, t + \Delta t) = \int \pi(u', \rho', t)\pi(u, \rho \mid u', \rho', \Delta t)du'd\rho' \tag{4.5}$$

which can be rewritten in view of the Langevin equation as

$$= \int \pi(u', \rho', t)\delta\{u' - [u + \beta u\Delta - \mathrm{M}(\Delta)]\}\pi(\rho \mid \rho'; \Delta t)du'd\rho'$$

$$= \int \pi(u + \beta u\Delta - \mathrm{M}(\Delta), \rho', t)\pi(\rho \mid \rho'; \Delta t)(1 + \beta\Delta)d\rho' \tag{4.6}$$

where $\mathrm{M}(\Delta) = \int_t^{t+\Delta} \rho(t')dt'$. Expanding right hand side in Taylor series, we obtain after integrating over $d\rho$ on both sides, in the limit $\Delta \to 0$, the following:

$$\begin{aligned}\pi(u, \rho, t + \Delta) = &\int \pi(u, \rho, t)\pi(\rho \mid \rho'; \Delta t)d\rho' \\ &+ \int_{\rho'} \left\{ \frac{\partial \pi}{\partial u} [\beta u\Delta - \mathrm{M}(\Delta)] + \beta \frac{\partial \pi}{\partial u} \right\} \pi(\rho \mid \rho'; \Delta)d\rho' \\ &+ \int_{\rho'} \frac{\partial \pi}{\partial \rho} (\rho - \rho')\pi(\rho \mid \rho'; \Delta)d\rho' \\ &+ \frac{1}{2}\left\{ \int_{\rho'} 2 \frac{\partial^2 \pi}{\partial u \partial \rho} \{\beta u\Delta - \mathrm{M}(\Delta)\}(\rho - \rho')\pi(\rho \mid \rho'; \Delta)d\rho' \right. \\ &+ \int \frac{\partial^2 \pi}{\partial u^2} [\beta u\Delta - \mathrm{M}(\Delta)]^2\pi(\rho \mid \rho'; \Delta)d\rho' \\ &\left. + \int_{\rho'} \frac{\partial^2 \pi}{\partial \rho^2} (\rho - \rho')^2\pi(\rho \mid \rho'; \Delta)d\rho' \right\}. \end{aligned} \tag{4.7}$$

If $\pi(\rho \mid \rho'; \Delta t)$ is of such a type, that $\varepsilon\{(\rho - \rho')^2\}$ or $\varepsilon\{\mathrm{M}(\Delta^2)\}$ is of order $\Delta t$ and higher moments of order $0(\Delta t)$, $\varepsilon(\rho - \rho')$ and $\varepsilon\{\mathrm{M}(\Delta)\}$ are equal to zero, when we take the limit $\Delta \to 0$ and when we integrate both

(2) We use the same symbol $\pi$ to denote the probability frequency function of $u(t)$.

sides of (4.6) with respect to $\rho$, we are left with the usual Focker Planck equation

$$\frac{\partial \pi(u, t)}{\partial t} = \beta \frac{\partial}{\partial u} [\pi(u, t)u] + \frac{1}{2} \frac{\partial^2}{\partial u^2} \left\{ \pi(u, t) . \operatorname*{Lim}_{\Delta \to 0} \frac{\varepsilon \{ M^2(\Delta) \}}{\Delta} \right\} . \quad (4.8)$$

In equation (4.7) it is to be pointed out that the mixed term $\frac{\partial^2 \pi}{\partial u \partial \rho}$ which may be multiplied by $\operatorname*{Lim}_{\Delta \to 0} \frac{\varepsilon \{ M^2(\Delta) \}}{\Delta}$ does not survive since when integrated over $\rho$, $\frac{\partial \pi(u, \rho, t)}{\partial u}$ is well behaved enough to go to zero at the limits of one of its arguments $\rho$.

Now we can examine whether we can obtain a Fokker-Planck equation for $\pi(u, t)$. Using the technique employed in section 2, we obtain

$$\frac{\partial \pi(u, \rho, t)}{\partial t} = \sum_{m=1}^{\infty} \sum_{n=0}^{m} \frac{(-)^{m+n}}{m!\, n!} \frac{\partial^{m+n}}{\partial u^m \partial \rho^n} [\alpha'_{m,n} \pi(u, \rho, t)] \quad (4.9)$$

where

$$\alpha'_{m,n} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \iint (u - u')^m (\rho - \rho')^n \pi(u', \rho' \mid u, \rho ; \Delta t) du' d\rho' . \quad (4.10)$$

We shall assume that the mean value of $\rho$ is zero. It is easy to note

$$\alpha'_{10} = - \beta u \quad , \quad \alpha'_{m,n} \sim 0 \quad ; \quad n > 2 \quad \text{as} \quad \Delta t \to 0. \quad (4.11)$$

The rest of the $\alpha'_{m,n}$s are evaluated by making use of the magnitude of the parameter $a$. Using (3.19), we find

$$\alpha'_{20} = \frac{1}{a} \varepsilon \{ \rho^2 \} = 1 \qquad \text{or} \qquad 2D \quad (4.12)$$

in some suitable units.

Next we calculate the coefficients $\alpha'_{0m}$s:

$$\begin{aligned} \alpha'_{0m} &= \lim \frac{1}{\Delta t} \iint (\rho' - \rho)^m \pi(u', \rho' \mid u, \rho ; \Delta t) du' d\rho' \\ &= \lim \frac{1}{\Delta t} \int (\rho' - \rho)^m \pi(\rho' \mid \rho ; \Delta t) d\rho' . \end{aligned} \quad (4.13)$$

Using the form (3.8) for the p. f. f. of $\rho(t)$ and form IV for $\psi(\rho)$ we find

$$\alpha_{0m} = \text{is a polynomial in } \rho, \text{ of highest power } m = f^m(\rho). \quad (4.14)$$

Thus the Fokker-Planck equation for $\pi(u, \rho, t)$ is given by

$$\frac{\partial \pi(u, \rho, t)}{\partial t} = \frac{\partial}{\partial u} \beta u \pi(u, \rho, t) + \frac{1}{2} \frac{\partial^2}{\partial u^2} \pi(u, \rho, t) \frac{\varepsilon \{ M^2(\Delta) \}}{\Delta} + \sum_{m=1}^{\infty} \frac{(-)^m}{m!} \left( \frac{\partial}{\partial \rho} \right)^m [f^m(\rho) \pi(u, \rho, t)]. \quad (4.15)$$

Integrating over the entire domain of $\rho$ and making use of the smoothness property of $\pi(u, \rho, t)$ we obtain

$$\frac{\partial \pi(u, t)}{\partial t} = \frac{\partial}{\partial u} \beta u \pi(u, t) + \frac{1}{2} \frac{\partial^2}{\partial u^2} \pi(u, t) \frac{\varepsilon \{ M^2(\Delta) \}}{\Delta} \quad (4.16)$$

which is identical with the corresponding equation if $\rho$ were to be taken a Gaussian random process.

## 5. — APPLICATION TO AN ASTROPHYSICAL PROBLEM

We shall next consider how Chandrasekhar's theory of fluctuations in brightness of Milky Way can be described in terms of our fluctuating density field. Following the ref. [7] we will attempt to solve the problem of moments of the intensity distribution by assuming the density of the interstellar matter to vary in a continuous but fluctuating albeit widly. The problem is treated essentially as a one dimensional problem and the observer at the origin $t = 0$ measures the intensity received at the origin due to the stars which are uniformly distributed along the line of sight. If $k$ is the absorption coefficient and $\rho$ the density of interstellar matter, then $M(t) = k \int_0^t \rho d\tau$ is the optical thickness of matter corresponding to the distance $t$. K can be put equal to $1/\varepsilon \{ \rho \}$ without loss of generality by suitably choosing the unit of $t$. Thus unit intensity, on passing through matter of extension $t$, is cut down to an intensity $e^{-M(t)}$ where

$$M(t) = \int_0^t \rho' d\tau \quad , \quad \rho' = \rho / \varepsilon \{ \rho \}. \quad (5.1)$$

Assuming the production of intensity to be $\beta d\tau$ in any interval $d\tau$, and taking $\beta = 1$, without loss of generality the cumulative net intensity observed at $t = 0$ is given by

$$I(t) = \int_0^t e^{-M(\tau)} d\tau \quad (5.2)$$

and it is the variable $I(t)$ whose probability frequency function we seek. Thus the $n$th moment of $I(t)$ is given by

$$\varepsilon\{ I^n(t) \} = \int_0^t \int_0^t \dots \int_0^t \varepsilon\{ Y(t_1)Y(t_2) \dots Y(t_n) \} \, dt_1 \dots dt_n \quad (5.3)$$

where

$$Y(t) = e^{-M(t)}. \quad (5.4)$$

We next order the variables $t_1, t_2, \dots, t_n$ and evaluate $\varepsilon\{ Y(t_1)Y(t_2) \dots Y(t_n) \}$ by taking over equation (3.28). The $\varepsilon\{ Y(t_1) \dots Y(t_n) \}$ can be written down by visualising and adding up the intensities allowed to pass through from the intervals, $t$ to $t_{n-1}$, $t_{n-1}$ to $t_{n-2}$, $\dots$, $t_1 = 0$ where

$$t_1 < t_2 < \dots < t_n.$$

If we write $\dfrac{\varepsilon\{ x^2 \}}{2a} = B$ we obtain for the $n$th order moment (as medium extends up to $\infty$)

$$\varepsilon\{ I^n \} = \frac{1}{[\varepsilon(\rho)]^n} \left[ 1 + \frac{n(n+1)}{2} \frac{B}{\varepsilon(\rho)} \right]. \quad (5.5)$$

Taking $\varepsilon(\rho)$ to be equal to 1, according to the normalization in equation (5.1) and substituting for $\alpha^2 = \dfrac{\varepsilon(\rho^2) - [\varepsilon(\rho)]^2}{[\varepsilon(\rho)]^2}$ and $\tau_0 = \dfrac{\varepsilon\{ \rho \}}{a}$ we obtain

$$\varepsilon\{ I^n \} = \left[ 1 + \frac{n(n+1)}{2} \alpha^2 \frac{\tau_0}{2} \right]. \quad (5.6)$$

The second term in the right hand side of expression (5.6) differ from the expressions obtained in reference [7] by some factors.

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## REFERENCES

[1] A. Einstein, *Ann. d. Physik*, t. 17, 1905, p. 549; t. 19, 1906, p. 371.
[2] G. E. Uhlenbeck and L. S. Ornstein, *Phys. Rev.*, t. 36, 1930, p. 823.
[3] S. Chandrasekhar, *Rev. Mod. Phys.*, t. 15, 1943, p. 1.

[4] M. C. Wang and G. E. Uhlenbeck, *Rev. Mod. Phys.*, t. 17, 1945, p. 323.
[5] W. A. Ambarzumian, *Math. Rev.*, t. 14, 1953, p. 192.
[6] S. Chandrasekhar and G. Munch, *A. P. J.*, t. 115, 1952, p. 103.
[7] A. Ramakrishnan, *A. P. J.*, t. 119, 1954, p. 443.
[7a] S. Chandrasekhar, Article selected papers on *Noise and Stochastic Processes*, edited by Nelson Wax. Dover Publications, 1954.
[8] M. Lax, *Rev. Mod. Phys.*, t. 32, 1960, p. 25.
[9] J. E. Moyal, *J. Roy. Stat. Soc.*, B 11, 1949, p. 150.

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