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**Reduction of the most degenerate
unitary irreducible representations
of $SO_0(m, n)$ when restricted
to a non-compact rotation subgroup**

par

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ABSTRACT. — The reduction of the most degenerate unitary irreducible representations (of principal series) of an arbitrary $SO_0(m, n)$ group when restricted to the subgroup $SO_0(k, l)$ $m + n > k + l \geq 3$, $m \geq k$, $n \geq l$ is given.

1. INTRODUCTION

Recently unitary irreducible representations (UIR) of non-compact groups have been used in elementary particle physics and in quantum mechanics in various approaches (relativistic $SU(6)$, non-invariance or spectrum generating groups [1], generalization of a partial wave analysis [2], etc.). These different trends lead to some common problems. For example, in order to label the states in a given UIR of a non-compact group or in order to facilitate the calculation of Clebsch-Gordan coefficients as well as to know which $SU(6)$ multiplets are contained in a given UIR of higher symmetry group or in order to be able to investigate a connection between complex angular momentum and IR of the Poincaré group, and

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son on, we have to solve a problem of reducing a UIR of a considered group when restricted to its subgroup.

If the group or at least its subgroup is compact the reduction problem is, in principal, completely solved (see [3]). However, if the group and its subgroup is non-compact the problem is more complicated (mainly due to infinite dimensionality of their UIR and, in general, discrete and continuous spectra of their generators) and is studied only in special cases [4] [5]. In our work we have solved the reduction problem for the most degenerate UIR's (of principal series) of an arbitrary $SO_0(m, n)$ group when restricted to the subgroup $SO_0(k, l)$ $m + n > k + l \geq 3, m \geq k, n \geq l$.

In Section 2 we shall briefly explain our notation and summarize the most degenerate UIR's of $SO_0(m, n)$ derived in [6] [8]. Sections 3 and 4 are devoted to reducing these representations when restricted to the subgroup $SO_0(m, n - 1)$ or $SO_0(m - 1, n)$. In particular, in Section 3 we reduce UIR of the pseudorotation group when restricted to the subgroup by choosing a complete set of common eigenfunctions of all invariant operators of the group and of the subgroup. Since this method can be applied to some of our representations only, a modified method has been developed for reducing the other representations in Section 4. The main results of our work are contained in theorems (3.1) (3.2) and (4.2).

2. PRELIMINARIES

Let $\Gamma_{m,n}^{k,l}$ $k \leq m, l \leq n$ be a $(m + n) \times (m + n)$ matrix non-vanishing elements of which are only $(\Gamma_{m,n}^{k,l})_{ii} = 1, i = 1, 2, \dots, k$ and $(\Gamma_{m,n}^{k,l})_{jj} = -1, j = m + 1, m + 2, \dots, m + l$. We denote a matrix group whose elements g are all real $(m + n) \times (m + n)$ matrices satisfying the equation $g^T(\Gamma_{m,n}^{m,n})g = (\Gamma_{m,n}^{m,n})$ by $SO_0(m, n)$, ($m + n \geq 3$), where g^T is the transpose matrix of g . In the present article only the component of the identity $SO_0(m, n)$ of the group $SO(m, n)$ is considered.

We use three homogeneous spaces:

$$SO_0(m - n)/SO_0(m - 1, n), \quad SO_0(m, n)/SO_0(m, n - 1),$$

$$SO_0(m, n)/\Gamma^{m+n-2} \square SO_0(m - 1, n - 1),$$

where the closed subgroup $SO_0(k, l)$, $k \leq m, l \leq n$ is a component of the identity of that subgroup $SO(k, l) \in SO(m, n)$ elements of which are all matrices $g \in SO(m, n)$ satisfying relations

$$g^T(\Gamma_{m,n}^{k,l})g = (\Gamma_{m,n}^{k,l}) \quad \text{and} \quad g(\Gamma_{m,n}^{m,n} - \Gamma_{m,n}^{k,l}) = (\Gamma_{m,n}^{m,n} - \Gamma_{m,n}^{k,l})$$

The group T^{m+n-2} is a $(m + n - 2)$ -dimensional Abelian subgroup of $SO_0(m, n)$ for which $SO_0(m - 1, n - 1)$ is the group of automorphisms so that their semidirect product can be defined (for details see Appendix). Models of our three homogeneous spaces may be taken as three submanifolds of the pseudo-euclidean space R_{m+n}^n , in particular, as hyperboloids H_{m+n}^n and H_{m+n}^m determined by

$$H_{m+n}^n := \{ x \in R_{m+n}^n \mid [x, x]_{m,n} = 1 \}, \quad m \geq n, \quad (2.1)$$

$$H_{m+n}^m := \{ x \in R_{m+n}^n \mid [x, x]_{m,n} = -1 \}, \quad m > n > 1, \quad (2.2)$$

and as the cone C_{m+n}^n defined by

$$C_{m+n}^n := \left\{ x \in R_{m+n}^n \mid [x, x]_{m,n} = 0, \sum_{i=1}^{m+n} x_i^2 \neq 0 \right\}, \quad m \geq n > 1 \quad (2.3)$$

respectively. Here,

$$[x, x]_{r,s} = x_1^2 + x_2^2 + \dots + x_r^2 - x_{r+1}^2 - x_{r+2}^2 - \dots - x_{r+s}^2.$$

The hyperboloid H_{1+n}^n and the cone C_{1+n}^1 are defined as in (2.2) and (2.3), respectively, with an additional condition $x_{1+n} > 0$.

Since the group $SO_0(m, n)$ and its three closed subgroups $SO_0(m - 1, n)$, $SO_0(m, n - 1)$ and $T^{m+n-2} \square SO_0(m - 1, n - 1)$ are unimodular [9], there is a $SO_0(m, n)$ -left invariant measure on three defined homogeneous spaces [10]. Let us denote this measure by μ , any of the homogeneous spaces (2.1)-(2.3) (or any of their parts) by M and a point of M by p . The unitary representations of the $SO_0(m, n)$ group induced by the identity representation of the subgroup (i. e. quasiregular representation) for every considered homogeneous space can be easily defined and decomposed into unitary irreducible representations of $SO_0(m, n)$ [6] [7] [8]. The quasiregular representations of $SO_0(m, n) \times \mathfrak{H}(M) \ni (g, f) \rightarrow U(g)f \in \mathfrak{H}(M)$, where $[U(g)f](p) = f(g^{-1} \cdot p)$, induces the representation of the Lie algebra $\mathfrak{sv}(m, n)$ on a certain linear manifold $\mathfrak{D}(M)$ [7] dense in the Hilbert space $\mathfrak{H}(M)$. $\mathfrak{D}(M)$ is invariant with respect to the representation of the universal enveloping algebra $\mathfrak{U} \times \mathfrak{D}(M) \ni (A, f) \rightarrow \rho(A)f \in \mathfrak{D}(M)$, where $\rho(X) = dU(X)$ for $X \in \mathfrak{sv}(m, n)$. The commutative subalgebra of \mathfrak{U} generated by the invariants of the Lie algebra $\mathfrak{sv}(m, n)$ is represented on $\mathfrak{D}(M)$ into the algebra generated by only one operator $Q_G = \rho(C_2)$ [12]. Because of this, the considered representations are called the most degenerate representations of the group $SO_0(m, n)$. The operator Q_G has the

continuous spectrum $\Lambda^2 + \left(\frac{m+n-2}{2}\right)^2$, $\Lambda \in (0, \infty)$ and the discrete spectrum [13] – $L(L+m+n-2)$;

$$L = - \left\{ \frac{m+n-4}{2} \right\}, \left\{ \frac{m+n-4}{2} \right\} + 1, \dots$$

However, it happens that there are more irreducible representations corresponding to the same value of L or Λ . In order to distinguish such representations we have introduced operators P and T with eigenvalues $\{\pm 1\}$, $\{0, \pm 1\}$ and $\{0, \pm 1\}$ respectively [7] [8].

Since some unitary irreducible representations related to three homogeneous spaces M are equivalent [14] we shall summarize in the following only the inequivalent ones denoted by D together with the structure of their carrier spaces denoted by \mathfrak{S} . (For precise definitions of representations D and of Hilbert spaces \mathfrak{S} see [8].)

A. Continuous principal series.

i) *The group $SO_0(m, n)$, $m \geq n \geq 2$.*

For every $\Lambda \in (0, \infty)$ there are two UIR's of the group $SO_0(m, n)$, $m \geq n \geq 2$

$$D_{m,n}^{\Lambda,+} ; \mathfrak{S}_{m,n}^{\Lambda,+} = \sum_{l+\tilde{l}=\text{even}} \oplus \mathfrak{S}^{l,\tilde{l}}, \tag{2.4}$$

$$D_{m,n}^{\Lambda,-} ; \mathfrak{S}_{m,n}^{\Lambda,-} = \sum_{l+\tilde{l}=\text{odd}} \oplus \mathfrak{S}^{l,\tilde{l}} \tag{2.5}$$

where $\mathfrak{S}^{l,\tilde{l}}$ are the carrier spaces of UIR's of the maximal compact subgroup $SO(m) \otimes SO(n)$ classified by non-negative integers l, \tilde{l} determining the eigenvalues – $\{l(l+m-2) + \tilde{l}(\tilde{l}+n-2)\}$ of the second-order Casimir operator of the group $SO(m) \otimes SO(n)$.

ii) *The group $SO_0(m, 1)$.*

For every $\Lambda \in (0, \infty)$ the UIR of $SO_0(m, 1)$ has the form

$$D_{m,1}^{\Lambda} ; \mathfrak{S}_{m,1}^{\Lambda} = \sum_{l=0}^{\infty} \oplus \mathfrak{S}^l, \tag{2.6}$$

where \mathfrak{H}^l are the carrier spaces of the UIR's of the group $SO(m)$ classified by a non-negative integer l which determines the eigenvalue $-l(l + m - 2)$ of the second order Casimir operator of $SO(m)$.

B. Discrete principal series.

i) *The group $SO_0(m, n)$, $m \geq n \geq 3$.*

For every $L = -\left\{\frac{m+n-4}{2}\right\}, -\left\{\frac{m+n-4}{2}\right\} + 1, \dots$, there are two UIR's of $SO_0(m, n)$, $m \geq n \geq 3$ (they are equivalent only for $m = n$):

$$D^L(H_{m+n}^n) : \mathfrak{H}^L(H_{m+n}^n) = \sum_{\substack{\tilde{l}-l-L-n=2r \\ r=0,1,2,\dots}}^{\infty} \oplus \mathfrak{H}^{l,\tilde{l}}, \tag{2.7}$$

$$D^L(H_{m+n}^m) : \mathfrak{H}^L(H_{m+n}^m) = \sum_{\substack{\tilde{l}-l-L-m=2r \\ r=0,1,2,\dots}}^{\infty} \oplus \mathfrak{H}^{l,\tilde{l}}. \tag{2.8}$$

ii) *The group $SO_0(m, 2)$, $m \geq 2$.*

There are three UIR's of the group $SO_0(m, 2)$, $m > 2$:

$$D^L(H_{m+2}^2) : \mathfrak{H}^L(H_{m+2}^2) = \sum_{\substack{\tilde{l}-l-L-2=2r \\ r=0,1,2,\dots}}^{\infty} \oplus \mathfrak{H}^{l,\tilde{l}}, \quad |\tilde{l}| = 0, 1, 2, \dots \tag{2.9}$$

$$D_{\pm}^L(H_{m+2}^m) : \mathfrak{H}_{\pm}^L(H_{m+2}^m) = \sum_{\substack{|\tilde{l}|-l-L-m=2r \\ r=0,1,2,\dots}}^{\infty} \oplus \mathfrak{H}^{l,\pm|\tilde{l}|}, \tag{2.10}$$

and for $SO_0(2, 2)$ there are only two inequivalent representations described by (2.10).

iii) *The group $SO_0(m, 1)$, $m \geq 2$.*

Finally, for every $L = -\left\{\frac{m-3}{4}\right\}, -\left\{\frac{m-3}{4}\right\} + 1, \dots$ we have one UIR of the group $SO_0(m, 1)$, $m \geq 3$

$$D^L(H_{m+1}^1) : \mathfrak{H}^L(H_{m+1}^1) = \sum_{l=L+1}^{\infty} \oplus \mathfrak{H}^l \tag{2.11}$$

and two UIR's of the group $SO_0(2, 1)$

$$D_{\pm}^L(H_3^1) : \mathfrak{H}_{\pm}^L(H_3^1) = \sum_{|\tilde{l}|=L+1} \oplus \mathfrak{H}^{\pm|\tilde{l}|}. \tag{2.12}$$

3. REDUCTION OF REPRESENTATIONS $D_{m,n}^{\Lambda,\pm}$ AND $D_{m,1}^{\Lambda}$ WHEN RESTRICTED TO $SO_0(m, n-1)$ AND $SO_0(m-1, n)$

The representations $D_{m,1}^{\Lambda,\pm}(D_{m,1}^{\Lambda})$ have been found in the decomposition of the quasiregular representation of $SO_0(m, n)(SO_0(m, 1))$ induced on any of three homogeneous spaces (2.1)-(2.3). The generators of the group are represented by the first order differential operators on $\mathfrak{D}(M)$ and therefore the invariant operators of the Lie algebras of the group and the subgroup are represented by higher-order differential operators on $\mathfrak{D}(M)$. The reduction of an irreducible representation of the group when restricted to the subgroup can be solved by choosing a complete set of common eigenfunctions of all the invariant operators of the group and the subgroup. This method will be used in the present section.

LEMMA 3.1. — The representations $D_{m,n}^{\Lambda,\pm}(D_{m,1}^{\Lambda})$ of the group $SO_0(m, n)$, ($SO_0(m, 1)$) can decompose into a direct integral of only those unitary irreducible representations of $SO_0(m', n')$, $m' \leq m$, $n' \leq n$, which are classified by (2.4)-(2.12).

Proof. — Let $G = SO_0(m, n)$ and $H = SO_0(m, n - 1)$ or $SO_0(m - 1, n)$ and let us consider the mapping $H \times M \ni (h, p) \rightarrow h.p \in M$ as a restriction of the mapping $G \times M \ni (g, p) \rightarrow g.p \in M$ to the subgroup H . The orbits $H.p := \{ h.p \in M \mid h \in H \}$ are submanifolds of M and are analytic transitive manifolds with respect to H by the topology induced from the analytic manifold M . We show that for every subgroup H such a G -transitive manifold M can be chosen, that all orbits $H.p$ are homeomorphic to one of three H -transitive manifolds described by (2.1)-(2.3). In other words, the set M can be divided into subsets M_{α} , $\alpha \in A$, where $M_{\alpha} := \{ p \in M \mid H.p \text{ are homeomorphic to each other} \}$.

There are three manifolds M at our disposal: H_{m+n}^n , H_{m+n}^m and C_{m+n}^n . We choose the one which gives the simplest division with respect to the orbits $H.p$. For $H = SO_0(m, n - 1)$ the most convenient manifold is the hyperboloid H_{m+n}^n for the following reasons. For a fixed, but arbi-

trary point $p \in H_{m+n}^n, p = (x_1, x_2, \dots, x_{m+n})$, the orbit $H.p$ is defined by

$$H.p := \{ q = (y_1, y_2, \dots, y_{m+n-1}, x_{m+n}) \in H_{m+n}^n \mid [y, y]_{m, n-1} = 1 + |x_{m+n}|^2 \}.$$

As the group $SO_0(m, n - 1)$ is transitive on H_{m+n-1}^{n-1} , the obtained orbit $H.p$ is homeomorphic to H_{m+n-1}^{n-1} . In this way we have proved that all submanifolds $SO_0(m, n - 1).p, p \in H_{m+n}^n$ are homeomorphic to H_{m+n-1}^{n-1} . By a similar argument we conclude that the manifold H_{m+n-1}^{m-1} is the most convenient when we deal with the subgroup $SO_0(m - 1, n)$. In this case all orbits $SO_0(m - 1, n).p, p \in H_{m+n}^n$ are homeomorphic to the manifolds H_{m+n-1}^{m-1} .

The representation of the group $G \times \mathfrak{H}(M) \ni (g, f) \rightarrow U(g)f \in \mathfrak{H}(M)$ induces the representation of the subgroup H on $\mathfrak{H}(M)$. Let us show which unitary irreducible representations of the subgroup H can be found in the reduction of the representation

$$H \times \mathfrak{H}(M) \ni (h, f) \rightarrow U(h)f \in \mathfrak{H}(M), [U(h)f](p) = f(h^{-1}.p).$$

A parametrization for H_{m+n}^n convenient for our purpose is given by $m + n - 1$ relations [4]:

$$\begin{aligned} x_1 &= \text{ch } \theta \text{ ch } \eta x'_1, & x_{m+1} &= \text{ch } \theta \text{ sh } \eta \tilde{x}_{m+1}, \\ \vdots & & \vdots & \\ x_{m-1} &= \text{ch } \theta \text{ ch } \eta x'_{m-1}, & x_{m+n-1} &= \text{ch } \theta \text{ sh } \eta \tilde{x}_{m+n-1}, & \theta &\in (-\infty, \infty) \\ & & & & \eta &\in [0, \infty) \text{ for } n \geq 3 \\ x_m &= \text{ch } \theta \text{ ch } \eta x'_m, & x_{m+n} &= \text{sh } \theta, & \eta &\in (-\infty, \infty) \text{ for } n = 2, \end{aligned} \tag{3.1}$$

where x'_1, \dots, x'_m are expressed in terms of $m - 1$ real parameters which parametrize the sphere S^m , and similarly $x'_{m+1}, \dots, x'_{m+n-1}$ are expressed in $n - 2$ real parameters which parametrize the sphere S^{n-1} . These two sets of parameters of the spheres S^m and S^{n-1} will be denoted by ω_m and $\tilde{\omega}_{n-1}$, respectively. Quite analogous parametrization can be defined for the hyperboloid H_{m+n}^m .

The linear manifold $\mathfrak{D}(H_{k+l}^l)$ is determined in [8] by vectors

$$f(\theta, \eta, \omega_m, \tilde{\omega}_{n-1}) \in \mathfrak{H}(H_{k+l}^l),$$

which have the form $P(x_1, x_2, \dots, x_{k+l}) \exp \{ -\sum_{i=1}^{k+l} x_i^2 \}$, where $P(x_1, x_2, \dots, x_{k+l})$ is a polynomial in x_1, x_2, \dots, x_{k+l} , expressed in $\theta, \eta, \omega_m, \tilde{\omega}_{n-1}$ according to (3.1).

Let us construct one definite decomposition of $\mathfrak{H}(H_{m+n}^n)$ with respect to the selfadjoint operator $Q_H^{s.a.}$, which is the representation of the second-order Casimir operator of the group $H = SO_0(m, n - 1)$, knowing from [7] and [8] the decomposition of $\mathfrak{H}(H_{m+n-1}^{n-1})$ into a direct integral

$$\widehat{\mathfrak{H}}_{m,n-1} = \int_{Sp(Q_H^{s.a.})} \widehat{\mathfrak{H}}_{m,n-1}^{\sigma_H} d\rho_H(\sigma_H).$$

Here, $Sp(Q_H^{s.a.})$ is the spectrum of the operator $Q_H^{s.a.}$, σ_H are the points of the spectrum $Sp(Q_H^{s.a.})$ and ρ_H is a Lebesgue-Stieltjes measure on $Sp(Q_H^{s.a.})$. Thus, let $\mathfrak{D}(R)$ be the linear manifold determined by all functions $f(\theta)$ continuous on the interval $(-\infty, \infty)$ for which

$$\int_{-\infty}^{+\infty} d\theta \operatorname{ch}^4 \theta \left| \frac{d^s}{d\theta^s} f(\theta) \right|^2 \operatorname{ch}^{m+n-2} \theta < \infty, \quad s = 0, 1, 2,$$

and let $\mathfrak{H}(R) = [\mathfrak{D}(R)]^\sim$ be the closure of the linear manifold $\mathfrak{D}(R)$ with respect to the norm introduced by the scalar product in

$$\mathfrak{D}(R) : (f, g) = \int_{-\infty}^{+\infty} d\theta \operatorname{ch}^{m+n-2} \theta \overline{f(\theta)} g(\theta).$$

The tensor product $\mathfrak{D}(R) \otimes \mathfrak{D}(H_{m+n-1}^{n-1})$ is dense in $\mathfrak{H}(H_{m+n}^n)$. If the decomposition of

$$\mathfrak{D}(H_{m+n-1}^{n-1}) \rightarrow \int_{Sp(Q_H^{s.a.})} \widehat{\mathfrak{D}}_{m,n-1}^{\sigma_H} d\rho_H(\sigma_H)$$

is induced by the above-mentioned decomposition of $\mathfrak{H}(H_{m+n-1}^{n-1})$, then the decomposition of $\mathfrak{H}(H_{m+n}^n)$ with respect to $Q_H^{s.a.}$ is of the form

$$\mathfrak{H}(H_{m+n}^n) \rightarrow \widehat{\mathfrak{H}}_{m,n} = \int_{Sp(Q_H^{s.a.})} \widehat{\mathfrak{H}}_{m,n}^{\sigma_H} d\rho_H(\sigma_H), \quad \widehat{\mathfrak{H}}_{m,n}^{\sigma_H} = [\mathfrak{D}(R) \otimes \widehat{\mathfrak{D}}_{m,n-1}^{\sigma_H}]^\sim. \quad (3.2)$$

From the expression (3.2) we may conclude that only those representations of the subgroup H can appear in the reduction of the representation $H \times \mathfrak{H}(M) \ni (h, f) \rightarrow U(h)f \in \mathfrak{H}(M)$ which are classified by (2.4)-(2.12). Of course, every irreducible representation appears in a denumerable multiplet as is easily seen from the structure of $\widehat{\mathfrak{H}}_{m,n}^{\sigma_H}$.

In this way we have proved the lemma for the subgroup $SO_0(m, n - 1)$. Since quite an analogous proof holds for the subgroup $SO_0(m - 1, n)$, the statement of the lemma follows by induction.

Let Q_G be the representation of the second-order Casimir operator on $\mathfrak{D}(M)$ (see Section 2). The operator Q_G is defined on $\mathfrak{D}(H_{m+n}^n)$ as the following differential operator [4]:

$$Q_G = -\frac{1}{\text{ch}^{m+n-2}} \frac{\partial}{\partial \theta} \text{ch}^{m+n-2} \theta \frac{\partial}{\partial \theta} + \frac{1}{\text{ch}^2 \theta} \Delta(H_{m+n-1}^{n-1}), \quad (3.3)$$

where $\Delta(H_{m+n-1}^{n-1})$ is the Laplace-Beltrami operator related to the manifold H_{m+n-1}^{n-1} . The operator Q_G in (3.3) is essentially selfadjoint on $\mathfrak{D}(H_{m+n}^n)$ [8]. The operator Q_G on $\mathfrak{D}(H_{m+n}^n)$ can be extended to the operator \tilde{Q}_G on $\mathfrak{D}(\mathbb{R}) \otimes \mathfrak{D}(H_{m+n-1}^{n-1})$:

$$\tilde{Q}_G = -\frac{1}{\text{ch}^{m+n-2}} \frac{\partial}{\partial \theta} \text{ch}^{m+n-2} \theta \frac{\partial}{\partial \theta} \otimes I + \frac{1}{\text{ch}^2 \theta} \otimes \Delta(H_{m+n-1}^{n-1}). \quad (3.4)$$

\tilde{Q}_G is essentially selfadjoint on $\mathfrak{D}(\mathbb{R}) \otimes \mathfrak{D}(H_{m+n-1}^{n-1})$ as

$$\mathfrak{D}(H_{m+n}^n) \subset \mathfrak{D}(\mathbb{R}) \otimes \mathfrak{D}(H_{m+n-1}^{n-1}).$$

Hence the operator \tilde{Q}_G strongly commutes with the operator $Q_H^{\text{s.a.}}$ so that \tilde{Q}_G is decomposable in the direct integral $\int Sp(Q_H^{\text{s.a.}}) Q_G^{\sigma_H} d\rho_{\sigma_H}$ on the space $\widehat{\mathfrak{H}}_{m,n}$ defined in (3.2). The operators $Q_G^{\sigma_H}$ have a simple differential expression on $\mathfrak{D}(\mathbb{R}) \otimes \widehat{\mathfrak{D}}_{m,n-1}^{\sigma_H} \subset \widehat{\mathfrak{H}}_{m,n}^{\sigma_H}$:

$$Q_G^{\sigma_H} = \left(-\frac{1}{\text{ch}^{m+n-2}} \frac{d}{d\theta} \text{ch}^{m+n-2} \theta \frac{d}{d\theta} + \frac{\sigma_H}{\text{ch}^2 \theta} \right) \otimes I. \quad (3.5)$$

LEMMA 3.2. — For every $\sigma_H \in Sp(Q_H^{\text{s.a.}})$, the operator $Q_G^{\sigma_H}$ is essentially selfadjoint on $\mathfrak{D}(\mathbb{R}) \otimes \widehat{\mathfrak{D}}_{m,n-1}^{\sigma_H}$. For $\sigma_H = \lambda^2 + \left(\frac{m+n-3}{2}\right)^2$ the spectrum $Sp(Q_G^{\sigma_H, \text{s.a.}})$ of the selfadjoint extension of $Q_G^{\sigma_H}$ is purely continuous $\sigma_G^{\sigma_H} = \Lambda^2 + \left(\frac{m+n-2}{2}\right)^2$, $\Lambda \in [0, \infty)$. For $\sigma_H = -l(l+m+n-3)$ the operator $Q_G^{\sigma_H, \text{s.a.}}$ has the continuous spectrum as in the previous case and a discrete spectrum [13] $\sigma_G^{\sigma_H} = -L(L+m+n-2)$,

$$L = -\left\{ \frac{m+n-4}{2} \right\}, -\left\{ \frac{m+n-4}{2} \right\} + 1, \dots, l-2, l-1.$$

Every point of the continuous spectrum of the operator $Q_G^{\sigma_H, \text{s.a.}}$ has multiplicity two.

Proof. — The operator (3.5) on $\mathfrak{D}(\mathbb{R}) \otimes \widehat{\mathfrak{D}}_{m,n-1}^{\sigma_H}$ is equivalent to the following differential operator [7]

$$A = -\frac{d^2}{d\theta^2} + \frac{\sigma_H - \left(\frac{m+n-3}{2}\right)^2 + \frac{1}{4}}{\text{ch}^2 \theta} + \left(\frac{m+n-2}{2}\right)^2 \quad (3.6)$$

on a certain domain $\mathfrak{D}(A)$ described in [8]. It is easy to see that A is essentially selfadjoint on $\mathfrak{D}(A)$. The eigenfunction expansion associated with the second-order differential operator (3.6) is calculated in (4.19) of [15]. Written as the eigenfunction expansion associated with the differential operator (3.5), the expansions of [15] have the forme: For

$$\sigma_H = \lambda^2 + \left(\frac{m+n-3}{2}\right)^2 \in \text{CSp}(Q_H^{\text{s.a.}})$$

($\text{CSp}(Q)$ is the continuous spectrum of the operator Q and $\text{DSp}(Q)$ is the discrete spectrum of Q)

$$\left\{ f(\theta) - \sum_{\alpha=1,2} \int_0^\infty d\Lambda_\alpha Y^{\lambda,\Lambda}(\theta) \int_{-\infty}^{+\infty} d\theta' \text{ch}^{m+n-2} \theta'_\alpha \overline{Y^{\lambda,\Lambda}(\theta')} f(\theta') \right\} \otimes I = 0, \quad f \in \mathfrak{D}(\mathbb{R}), \quad (3.7)$$

where

$${}_1 Y^{\lambda,\Lambda}(\theta) = \frac{1}{\sqrt{{}_1 N}} \text{ch} \theta^{i\Lambda - \frac{m+n-3}{2}} {}_2 F_1 \left(\frac{1}{2} \left(\frac{3}{2} + i \frac{\lambda + \Lambda}{2} \right), \frac{1}{2} \left(\frac{3}{2} + i \frac{\lambda - \Lambda}{2} \right); \frac{3}{2}; -\text{sh}^2 \theta \right) \quad (3.8)$$

$${}_2 Y^{\lambda,\Lambda}(\theta) = \frac{1}{\sqrt{{}_2 N}} \text{sh} \theta \text{ch} \theta^{i\Lambda - \frac{m+n-3}{2}} {}_2 F_1 \left(\frac{1}{2} \left(\frac{3}{2} + i \frac{\lambda + \Lambda}{2} \right), \frac{1}{2} \left(\frac{3}{2} + i \frac{\lambda - \Lambda}{2} \right); \frac{1}{2}, -\text{sh}^2 \theta \right)$$

and

$${}_1 N = -\frac{\text{sh}(\pi\Lambda)}{4\pi^3 \left| \Gamma\left(\frac{1}{4} + i \frac{\lambda + \Lambda}{2}\right) \Gamma\left(\frac{1}{4} + i \frac{\lambda - \Lambda}{2}\right) \right|^2}, \quad (3.9)$$

$${}_2 N = -\frac{\pi \text{sh}(\pi\Lambda)}{(\text{ch}^2 \pi\lambda + \text{sh}^2 \pi\Lambda) \left| \Gamma\left(\frac{3}{4} + i \frac{\lambda + \Lambda}{2}\right) \Gamma\left(\frac{3}{4} + i \frac{\lambda - \Lambda}{2}\right) \right|^2}.$$

Let us underline that the discrete spectrum is absent and the multiplicity

of the continuous spectrum is two, as follows from (3.7). It is easy to see that $\overline{{}_\alpha Y^{\lambda, \Lambda}(\theta)} = {}_\alpha Y^{\lambda, \Lambda}(\theta)$.

For $\sigma_H = -l(l + m + n - 3) \in \text{DSp}(Q_H^{\text{s.a.}})$:

$$\left\{ f(\theta) - \sum_{\alpha=1,2} \int_0^\infty d\Lambda {}_\alpha Y^{l, \Lambda}(\theta) \int_{-\infty}^{+\infty} d\theta' \text{ch}^{m+n-2} \theta' \overline{{}_\alpha Y^{l, \Lambda}(\theta')} f(\theta') \right. \\ \left. - \sum_{\substack{\alpha=1, L-l=\text{even} \\ \alpha=2, L-l=\text{odd}}} \sum_{L=-\left\{\frac{m+n-4}{2}\right\}}^{l-1} {}_\alpha Y^{l, L}(\theta) \int_{-\infty}^{+\infty} \overline{{}_\alpha Y^{l, L}(\theta')} f(\theta') \text{ch}^{m+n-2} \theta' d\theta' \right\} \otimes \mathbf{I} = 0, \\ f \in \mathfrak{D}(\mathbf{R}), \quad (3.10)$$

where the functions ${}_\alpha Y^{l, \Lambda}(\theta)$, ${}_\alpha Y^{l, L}(\theta)$ have the same expressions as the function ${}_\alpha V_{l(p/2)}^L(\theta)$ of [7] and [8] respectively for $p = m + n - 1$, $l_{\{p/2\}} = l$. The double multiplicity of the continuous spectrum of $Q_G^{\sigma_H, \text{s.a.}}$ follows again directly from (3.10). Q. E. D.

In this way we have found that the eigenfunction expansion of every vector $f \in \mathfrak{D}(\mathbf{R})$ and consequently of every vector $f \in \mathfrak{D}(\mathbf{R}) \otimes \mathfrak{D}(H_{m+n-1}^{n-1})$ is of the form:

$$f(\theta, \eta, \omega_m, \tilde{\omega}_{n-1}) = \sum_{\alpha=1,2} \sum_{\mathcal{N}^\lambda(H_{m+n-1}^{n-1})} \int_0^\infty d\Lambda \int_0^\infty d\lambda {}_\alpha Y^{\lambda, \Lambda}(\theta) Y_{m_{SO}(m, n-1)}^{l, SO(m, n-1)}(\Omega) \\ \cdot \int_{H_{m+n}^n} d\mu(\theta', \Omega') \overline{{}_\alpha Y^{\lambda, \Lambda}(\theta')} \overline{Y_{m_{SO}(m, n-1)}^{l, SO(m, n-1)}(\Omega')} f(\theta', \Omega') \\ + \sum_{\alpha=1,2} \sum_{\mathcal{N}^l(H_{m+n-1}^{n-1})} \sum_{l=-\left\{\frac{m+n-5}{2}\right\}}^\infty \int_0^\infty d\Lambda {}_\alpha Y^{l, \Lambda}(\theta) Y_{m_{SO}(m, n-1)}^{l, l, SO(m, n-1)}(\Omega) \\ \cdot \int_{H_{m+n}^n} d\mu(\theta', \Omega') \overline{{}_\alpha Y^{l, \Lambda}(\theta')} \overline{Y_{m_{SO}(m, n-1)}^{l, l, SO(m, n-1)}(\Omega')} f(\theta', \Omega') \\ + \sum_{\substack{\alpha=1, L-l=\text{even} \\ \alpha=2, L-l=\text{odd}}} \sum_{\mathcal{N}^l(H_{m+n-1}^{n-1})} \sum_{l=-\left\{\frac{m+n-5}{2}\right\}}^\infty \sum_{L=-\left\{\frac{m+n-4}{2}\right\}}^{l-1} {}_\alpha Y^{l, L}(\theta) Y_{m_{SO}(m, n-1)}^{l, l, SO(m, n-1)}(\Omega) \\ \cdot \int_{H_{m+n}^n} d\mu(\theta', \Omega') \overline{{}_\alpha Y^{l, L}(\theta')} \overline{Y_{m_{SO}(m, n-1)}^{l, l, SO(m, n-1)}(\Omega')} f(\theta', \Omega'), \quad \Omega = (\eta, \omega_m, \tilde{\omega}_{n-1}). \quad (3.11)$$

Here, the functions ${}_\alpha Y^{\lambda, \Lambda}(\theta)$, ${}_\alpha Y^{l, \Lambda}$ and ${}_\alpha Y^{l, L}(\theta)$ are defined as in (3.7) and

(3.10). The functions $Y_{m_{SO(m,n-1)}}^{v, l_{SO(m,n-1)}}(\eta, \omega_m, \tilde{\omega}_{n-1})$, $v = \lambda, L$, have the same expressions as the functions

$$Y_{m_1, \dots, m_{\lfloor p/2 \rfloor}, \tilde{m}_1, \dots, \tilde{m}_{\lfloor q/2 \rfloor}}^{v, l_2, \dots, l_{\lfloor p/2 \rfloor}, \tilde{l}_2, \dots, \tilde{l}_{\lfloor q/2 \rfloor}}(\eta, \omega_n, \tilde{\omega}_{n-1})$$

for $v = \Lambda, L$, $p = m$ and $q = n - 1$ in [7], [6], [8]. Hence, by the set $\{l_{SO(m,n-1)}, m_{SO(m,n-1)}\}$ we mean the set of indices $l_2, \dots, l_{\lfloor \frac{m}{2} \rfloor}, m_1, \dots, m_{\lfloor \frac{m}{2} \rfloor}, \tilde{l}_2, \dots, \tilde{m}_{\lfloor \frac{n-1}{2} \rfloor}$. By $\mathcal{N}_\lambda(H_{m+n-1}^{n-1})$ we denote the set of allowed values of indices $\{l_{SO(m,n-1)}, m_{SO(m,n-1)}\}$ in particular, if m is even, $m = 2r$, $r = 1, 2, \dots$, they have to satisfy

$$\begin{aligned} |m_2| + |m_1| &= l_2 - 2n_2 \\ |m_3| + l_2 &= l_3 - 2n_3 \\ \vdots & \\ |m_r| + l_{r-1} &= l_r - 2n_r \end{aligned} \quad n_k = 0, 1, \dots, \left\lfloor \frac{l_k}{2} \right\rfloor, \quad (3.12)$$

$$k = 2, 3, \dots, r,$$

and if m is odd, $m = 2r + 1$, $r = 1, 2, \dots$, they are restricted by (3.12) as well as by

$$l_r = l_{r+1} - n_{r+1}, \quad n_{r+1} = 0, 1, \dots, l_{r+1}.$$

An analogous set of conditions holds for the tilde indices. Finally

$$\begin{aligned} \mathcal{N}_l(H_{m+n-1}^{n-1}) &:= \mathcal{N}_\lambda(H_{m+n-1}^{n-1}) \\ &\cap \left\{ l_{\lfloor \frac{m}{2} \rfloor}, \tilde{l}_{\lfloor \frac{n-1}{2} \rfloor} \mid l_{\lfloor \frac{m}{2} \rfloor} - \tilde{l}_{\lfloor \frac{n-1}{2} \rfloor} - l - (n-1) = 2, 4, 6, \dots \right\}. \end{aligned} \quad (3.13)$$

The eigenfunction expansion of every $f \in \mathfrak{D}(\mathbb{R}) \otimes \mathfrak{D}(H_{m+n-1}^{n-1})$ has the form (3.11), where the functions $Y_{m_{SO(m,n-1)}}^{v, l_{SO(m,n-1)}}$, $v = \lambda, L$, are replaced by the functions $Y_{m_{SO(m,n-1)}}^{v, l_{SO(m,n-1)}}$ defined on H_{m+n-1}^{m-1} in [6] and [7], $SO_0(m, n)$ -left invariant measure on H_{m+n}^n by the corresponding measure on H_{m+n}^m , and the set $\mathcal{N}_\lambda(H_{m+n-1}^{n-1})$, $\mathcal{N}_l(H_{m+n-1}^{n-1})$ by the sets $\mathcal{N}_\lambda(H_{m+n-1}^{m-1})$, $\mathcal{N}_l(H_{m+n-1}^{m-1})$ respectively, where $\mathcal{N}_\lambda(H_{m+n-1}^{m-1}) = \mathcal{N}_\lambda(H_{m+n-1}^{n-1})$ but

$$\begin{aligned} \mathcal{N}_l(H_{m+n-1}^{m-1}) &= \mathcal{N}_\lambda(H_{m+n-1}^{m-1}) \\ &\cap \left\{ l_{\lfloor \frac{m-1}{2} \rfloor}, \tilde{l}_{\lfloor \frac{n}{2} \rfloor} \mid \tilde{l}_{\lfloor \frac{n}{2} \rfloor} - l_{\lfloor \frac{m-1}{2} \rfloor} - l - (m-1) = 2, 4, 6, \dots \right\}. \end{aligned} \quad (3.14)$$

From the expression (3.11) follows the existence of a Lebesgue-Stieltjes measure ρ_G on the spectrum $\text{Sp}(Q_G^{\text{s.a.}})$ and consequently the decomposition

of $\mathfrak{H}(H_{m+n}^n)$ into the direct integral $\int \text{Sp}(Q_G^{\text{s.a.}}) \mathfrak{H}_{m,n}^{\sigma_G} d\rho_G(\sigma_G)$. Then the spaces

$$\mathfrak{H}_{m,n}^\Lambda = \mathfrak{H}_{m,n}^{\Lambda^2 + \left(\frac{m+n-2}{2}\right)^2} \quad \text{and} \quad \mathfrak{H}_{m,n}^L(H_{m+n}^n) = \mathfrak{H}_{m,n}^{-L(L+m+n-2)}$$

up to unitary equivalence are defined by

$$\mathfrak{H}_{m,n}^\Lambda = \sum_{l=-\left\{\frac{m+n-5}{2}\right\}}^{\infty} \oplus \mathfrak{H}^{l,\Lambda} \oplus \int_0^\infty d\lambda \mathfrak{H}^{\lambda,\Lambda}, \tag{3.15}$$

where $\mathfrak{H}^{l,\Lambda}$ and $\mathfrak{H}^{\lambda,\Lambda}$ are Hilbert spaces of l^2 -type determined by vectors $\chi^{l,\Lambda}(f)$ and $\chi^{\lambda,\Lambda}(f)$ respectively, where

$$\chi^{v,\Lambda}(f) = \left\{ \chi_{mSO(m,n-1)}^{v,\Lambda,ISO(m,n-1)}(f) \mid f \in \mathfrak{D}(\mathbb{R}) \otimes \mathfrak{D}(H_{m+n-1}^{n-1}), \right. \\ \left. \{ISO(m,n-1), m_{SO(m,n-1)}\} \in \mathfrak{N}_v \right\}, \tag{3.16}$$

$$\chi_{mSO(m,n-1)}^{v,\Lambda,ISO(m,n-1)}(f) = \int_{H_{m+n}^n} d\mu(\theta, \eta, \omega_m, \tilde{\omega}_{n-1}) \overline{Y^{v,\Lambda}(\theta)} \\ \times Y_{mSO(m,n-1)}^{\lambda,ISO(m,n-1)}(n, \omega_m, \tilde{\omega}_{n-1}) f(\theta, \eta, \omega_m \tilde{\omega}_{n-1}). \tag{3.17}$$

The unitary representation of the group $SO_0(m, n)$ is defined by

$$U(g) \left(\sum_l \chi^{l,\Lambda}(f) = \int d\lambda \chi^{\lambda,\Lambda}(f) \right) = \sum_l \chi^{l,\Lambda}(U(g)f) + \int d\lambda \chi^{\lambda,\Lambda}(U(g)f), \\ [U(g)f](p) = f(g^{-1} \cdot p). \\ \mathfrak{H}^L(H_{m+n}^n) = \sum_{l=L+1}^{\infty} \mathfrak{H}_{m,n}^{l,L} \tag{3.18}$$

where $\mathfrak{H}_{m,n}^{l,L}$ are determined by vectors ${}_\alpha Y^{l,L}(\theta) Y_{mSO(m,n-1)}^{l,ISO(m,n-1)}(\eta, \omega_m, \tilde{\omega}_{n-1})$, $\alpha = 1$, for $L - l$ even and $\alpha = 2$ for $L - l$ odd. The unitary representation of the group is defined by the left translations $[U(g)f](p) = f(g^{-1} \cdot p)$.

For the pair H_{m+n-1}^{m-1} and $SO_0(m-1, n)$ the same expressions as (3.15) and (3.18) hold, only we have to replace H_{m+n-1}^{m-1} , $SO_0(m, n-1)$ everywhere by H_{m+n-1}^{m-1} , $SO_0(m-1, n)$.

As the decomposition of $\mathfrak{S}(M)$ into the direct integral

$$\int_{\text{Sp}(Q_G^{\text{s.a.}})} \mathfrak{S}_{m,n}^{\sigma_G} d\rho_G(\sigma_G)$$

is the central decomposition (although the operator $Q_G^{s.a.}$ is an unbounded operator on $\mathfrak{H}(M)$ we can still use the theorems from the representation of the associative algebras as we can work with the spectral family of projectors $E_G(\lambda)$ of the operator $Q_G^{s.a.}$) and the central decomposition is unique up to equivalence, we conclude that the realized decompositions of this section are equivalent to the corresponding decompositions of [8]. Hence, knowing irreducibility or reducibility of the spaces $\mathfrak{H}^L(M)$ from [8], we obtain the reduction of representations $D^L(H_{m+n}^n)$ and $D^L(H_{m+n}^n)$ when restricted to $SO_0(m, n - 1)$ and $SO_0(m - 1, n)$, respectively, as follows:

THEOREM 3.1. — The representations $D^L(H_{m+n}^n)$ when restricted to the representations of the subgroup $SO_0(m, n - 1)$ have the following reduction:

$$D^L(H_{m+n}^n) = \sum_{l=L+1}^{\infty} \oplus D^l(H_{m+n-1}^{n-1}). \tag{3.19}$$

The representations $D^L(H_{m+n}^m)$, $m \geq n \geq 3$ and $D_{\pm}^L(H_{m+2}^m)$, $m > 2$, when restricted to the representations of the subgroup $SO_0(m - 1, n)$ have the following reduction:

$$D^L(H_{m+n}^m) = \sum_{l=L+1}^{\infty} \oplus D^l(H_{m+n-1}^{m-1}), \quad n \geq 3, \tag{3.20}$$

$$D_{\pm}^L(H_{m+2}^m) = \sum_{l=L+1}^{\infty} \oplus D_{\pm}^l(H_{m+1}^{m-1}), \quad m > 2. \tag{3.21}$$

These representations and their reductions have been found in [4] by another method. The main result of this section is contained in

THEOREM 3.2. — The representations $D_{m,n}^{\Lambda, \pm}$, $m \geq n \geq 2$, when restricted to the representation of the subgroup $SO_0(m, n - 1)$, have the following reduction:

$$D_{m,n}^{\Lambda, \pm} = \int_0^{\infty} d\lambda D_{m,n-1}^{\lambda, +} \oplus \int_0^{\infty} d\lambda D_{m,n-1}^{\lambda, -} \oplus \sum_{l=-\left\{\frac{m+n-5}{2}\right\}}^{\infty} D^l(H_{m+n-1}^{n-1}). \tag{3.22}$$

The representations $D_{m,n}^{\Lambda, \pm}$, $m \geq n \geq 3$, $D_{m,2}^{\Lambda, \pm}$ and $D_{m,1}^{\Lambda, \pm}$, when restricted to

the representations of the subgroup $SO_0(m - 1, n)$, have the following reduction:

$$D_{m,n}^{\Lambda,\pm} = \int_0^\infty d\lambda D_{m-1,n}^{\lambda,+} \oplus \int_0^\infty d\lambda D_{m-1,n}^{\lambda,-} \oplus \sum_{l=-\left\{\frac{m+n-5}{2}\right\}}^\infty D^l(H_{m+n-1}^{m-1}), \quad (3.23)$$

$$D_{m,2}^{\Lambda,\pm} = \int_0^\infty d\lambda D_{m-1,2}^{\lambda,+} \oplus \int_0^\infty d\lambda D_{m-1,2}^{\lambda,-} \oplus \sum_{l=-\left\{\frac{m-3}{2}\right\}}^\infty (D_+^l(H_{m+1}^{m-1}) \oplus D_-^l(H_{m+1}^{m-1})), \quad (3.24)$$

$$D_{m,1}^\Lambda = 2 \int_0^\infty d\lambda D_{m-1,1}^\lambda. \quad (3.25)$$

Proof. — The representation $D_{m,1}^\Lambda$ is found in the decomposition of the quasiregular representation of the group $SO_0(m, 1)$ on $\mathfrak{H}(H_{m+1}^m)$. The representations $D_{m,1}^\Lambda$ are irreducible on $\mathfrak{H}_{m,1}^\Lambda$, where $\mathfrak{H}_{m,1}^\Lambda$ are spaces in the decomposition $\mathfrak{H}(H_{m+1}^m) \rightarrow \int_0^\infty d\lambda \mathfrak{H}_{m,1}^\Lambda$. Hence, because of the uniqueness of the central decomposition, the decomposition of $\mathfrak{H}_{m,1}^\Lambda$, when restricted to the subgroup $SO_0(m - 1, n)$, follows directly from the expansion (3.11) for the hyperboloid H_{m+n}^m . In formula (3.11) we have only the part which has a purely continuous spectrum. As α has two values, $\alpha = 1, 2$, there are two and only two orthogonal vectors $\chi^{\lambda,\Lambda}(f) \in \mathfrak{H}_{m-1,n}^\lambda$ which transform in the same way under the group $SO_0(m - 1, n)$. This tells us that the multiplicity of every representation $D_{m-1,\Lambda}^\lambda$ in the reduction of the representation $D_{m,1}^\Lambda$ is two.

In all other cases the representations $D_{m,n}^\Lambda$ were reducible on the spaces $\mathfrak{H}_{m,n}^\Lambda$, where $\mathfrak{H}_{m,n}^\Lambda$ are expressed by (3.15). It is shown that two subspaces $\mathfrak{H}_{m,n}^{\Lambda,\pm}$ of the space $\mathfrak{H}_{m,n}^\Lambda$ defined in formulae (2.4) and (2.5) transform irreducibly under $SO_0(m, n)$. The reflection operator P was found as the representation of the transformation $p = (x_1, x_2, \dots, x_{m+n}) \rightarrow p', p' = (-x_1, -x_2, \dots, -x_{m+n})$ on $\mathfrak{H}(M)$ (see [7]). The operator P commutes with $Q_G^{s.a.}$ and has the eigenvalues ± 1 in every $\mathfrak{H}_{m,n}^{\Lambda,\pm}$, respectively. Hence the decomposition of $\mathfrak{H}_{m,n}^\Lambda$ with respect to the operator P realizes a decomposition of $\mathfrak{H}_{m,n}^\Lambda$ into subspaces which transform irreducibly under $SO_0(m, n)$. The operator P can be expressed as $P_1 \otimes P_{m,n-1}$ on $\mathfrak{D}(R) \otimes \mathfrak{D}(H_{m+n-1}^{n-1})$ or $P_1 \otimes P_{m-1,n}$ on $\mathfrak{D}(R) \otimes \mathfrak{D}(H_{m+n-1}^{m-1})$, where $P_{m,n-1}$

and $P_{m-1,n}$ are the reflection operators on $\mathfrak{H}(H_{m+n-1}^{n-1})$ and $\mathfrak{H}(H_{m+n-1}^{m-1})$, respectively, and P_1 is defined by $(P_1 f)(\theta) = f(-\theta)$, $f \in \mathfrak{D}(\mathbb{R})$. The functions ${}_a Y^{\nu, \lambda}(\theta)$, $\nu = \lambda, l$ are even for $\alpha = 1$ and odd for $\alpha = 2$. Hence the decompositions (2.4), (2.5) follow as a simple consequence of this analysis. Q. E. D.

4. REDUCTION OF REPRESENTATIONS $D^L(H_{m+n}^n)$, $m > 2$, $D^L(H_3^1)$ AND $D^L(H_{m+2}^m)$, $m \geq 2$, WHEN RESTRICTED TO THE SUBGROUPS $SO_0(m, n-1)$ AND $SO_0(m-l, n)$

In the last section we have decomposed representations (2.4)-(2.6) when restricted to the subgroup $H = SO_0(m, n-1)$ or $H = SO_0(m-1, n)$. As these representations have been found in the decomposition of the quasiregular representation on any of three manifolds, it was sufficient to consider one of three homogeneous spaces (2.1)-(2.3) and this enables us to chose the most convenient one, namely that for which all the orbits $H.p$ have been homeomorphic one to the other. Since the reduction of the quasiregular representation on a definite manifold M contains some of the representation (2.7)-(2.12), we have partially solved the problem of the section. We could not solve the problem completely, as the representations described in (2.7)-(2.12) were unequivalent for different homogeneous spaces M . Now we shall solve the rest of the problem. It happens that the method used in the last section is not applicable on the whole here, so we must also use other properties of the representations.

LEMMA 4.1. — Any representation of the group $SO(m, n)$ which is described in (2.4)-(2.12) when restricted to any subgroup $SO_0(k, l)$ reduces into the direct integral of only those unitary irreducible representations of the subgroup $SO_0(k, l)$ which are classified in (2.4)-(2.12).

Proof. — A part of the proof is already contained in the proof of Lemma 3.1, so that we have only to consider representations $D^L(H_{m+n}^n)$ and $D^L(H_{m+n}^m)$ when restricted to the subgroups $SO_0(m-1, n)$ and $SO_0(m, n-1)$, respectively. We consider the representations $D^L(H_{m+n}^n)$ and the subgroup $SO_0(m-1, n)$ because the other case is completely analogous.

First we divide the manifold H_{m+n}^n into the parts according to the prescription in the Proof of Lemma 3.1. The division is

$$H_{m+n}^n = \bigcup_{\alpha=-3}^3 (H_{m+n}^n)_\alpha,$$

where

$$\begin{aligned} (H_{m+n}^n)_0 &:= \{ H.p \sim H_{m+n-1}^{n-1} \mid p = (x_1, \dots, x_{m+n}), |x_m| < 1 \}, \\ (H_{m+n}^n)_{\pm 1} &:= \{ H.p \sim H_{m+n-1}^{n-1} \mid p = (x_1, \dots, x_{m+n}), \pm x_m > 1 \}, \\ (H_{m+n}^n)_{\pm 2} &:= \left\{ H.p \sim C_{m+n-1}^m \mid p = (x_1, \dots, x_{m+n}), \sum_{i=1}^{m+n} x_i^2 \neq 0, x_m = \pm 1 \right\}, \\ (H_{m+n}^n)_{\pm 3} &:= \{ H.p = (x_1, x_2, \dots, x_m) \mid x_m = \pm 1, x_i = 0 \text{ for } i \neq m \}. \end{aligned}$$

As the sets $(H_{m+n}^n)_{\pm 2}$ and $(H_{m+n}^n)_{\pm 3}$ are of measure zero with respect to the $SO_0(m, n)$ —left invariant measure on H_{m+n}^n , we omit them. Thus we conclude by induction that almost all submanifolds $SO_0(k, l).p$ are homeomorphic to one of the manifolds H_{k+l}^l or H_{k+l}^k .

Parametrization of the submanifolds $(H_{m+n}^n)_\alpha$, $\alpha = -1, 0, 1$ is chosen to be of the form:

For the submanifolds $(H_{m+n}^n)_\alpha$, $\alpha = \pm 1$:

$$\begin{aligned} x_1 &= \alpha \operatorname{sh} \theta \operatorname{sh} \eta x'_1, & x_{m+1} &= \alpha \operatorname{sh} \theta \operatorname{ch} \eta \tilde{x}'_{m+1}, \\ \vdots & & \vdots & \\ x_{m-1} &= \alpha \operatorname{sh} \theta \operatorname{sh} \eta x'_{m-1}, & x_{m+n-1} &= \alpha \operatorname{sh} \theta \operatorname{ch} \eta \tilde{x}'_{m+n-1}, & \theta \in (0, \infty) \\ & & & & \eta \in [0, \infty), m > 2 \\ x_m &= \alpha \operatorname{ch} \theta, & x_{m+n} &= \alpha \operatorname{sh} \theta \operatorname{ch} \eta \tilde{x}'_{m+n}, & \eta \in (-\infty, \infty), m = 2, \end{aligned} \tag{4.1}$$

where $x'_1, x'_2, \dots, x'_{m-1}$ and $\tilde{x}'_{m+1}, \tilde{x}'_{m+2}, \dots, \tilde{x}'_{m+n}$ have the same role as in the expressions (3.1).

For the submanifolds $(H_{m+n}^n)_0$ we have

$$\begin{aligned} x_1 &= \sin \theta \operatorname{ch} \eta x'_1, & x_{m+1} &= \sin \theta \operatorname{sh} \eta \tilde{x}'_{m+1}, \\ \vdots & & \vdots & \\ x_{m-1} &= \sin \theta \operatorname{ch} \eta x'_{m-1}, & x_{m+n-1} &= \sin \theta \operatorname{sh} \eta \tilde{x}'_{m+n-1}, & \theta \in (0, \pi) \\ & & & & \eta \in [0, \infty), n > 2, \\ x_m &= \cos \theta, & x_{m+n} &= \sin \theta \operatorname{sh} \eta \tilde{x}'_{m+n}, & \eta \in (-\infty, \infty), n = 2, \end{aligned} \tag{4.2}$$

where $x'_1, x'_2, \dots, x'_{m-1}, \tilde{x}'_{m+1}, \tilde{x}'_{m+2}, \dots, \tilde{x}'_{m+n}$ are the same as in (4.1).

The Hilbert space $\mathfrak{H}(H_{m+n}^n)$ may be decomposed into sum

$$\mathfrak{H}(H_{m+n}^n) = \sum_{\alpha=-1}^1 \oplus \mathfrak{H}((H_{m+n}^n)_\alpha).$$

Let us construct three linear manifolds

$$\mathfrak{D}(R_\alpha), \quad R_{\pm 1} := \{ \theta \in \mathbb{R} \mid \pm \theta > 0 \}, \quad R_0 := \{ \theta \in \mathbb{R} \mid \theta \in (0, \pi) \}$$

determined by the functions $f \in C_0^\infty(R_\alpha)$ with the scalar product in $\mathfrak{D}(R_\alpha)$ introduced by

$$(f_\alpha, g_\alpha) = \int_{R_\alpha} d\theta \overline{f_\alpha(\theta)} g_\alpha(\theta) \{ \sin i^{|\alpha|} \theta \}^{m+n-2}.$$

The tensor product $\mathfrak{D}(R_\alpha) \otimes \mathfrak{D}(N_\alpha)$, where $N_{\pm 1} = H_{m+n-1}^{m-1}$ and $N_0 = H_{m+n-1}^n$, is dense in $\mathfrak{H}((H_{m+n}^n)_\alpha)$. As in the Proof of Lemma 3.1 the representation $H \times \mathfrak{H}((H_{m+n}^n)_\alpha) \ni (h, f) \rightarrow U(h)f \in \mathfrak{H}((H_{m+n}^n)_\alpha)$ decomposes into the direct integral of unitary irreducible representations (with denumerable multiplicity) which are equivalent to the representations classified by (2.4)-(2.12). Hence the statement of the Lemma follows by the induction.

The selfadjoint operator $Q_G^{s.a.}$ defined in [8] on $\mathfrak{D}(Q_G^{s.a.}) \subset \mathfrak{H}(H_{m+n}^n)$ is reduced by each of the manifolds $\mathfrak{D}(R_\alpha) \otimes \mathfrak{D}(N_\alpha)$ to the following differential operator:

$$Q_{G, \pm 1} = - \frac{1}{\text{sh}^{m+n-2}} \frac{\partial}{\partial \theta} \text{sh}^{m+n-2} \theta \frac{\partial}{\partial \theta} \otimes I - \frac{1}{\text{sh}^2 \theta} \otimes \Delta(H_{m+n-1}^{m-1})$$

on $\mathfrak{D}(R_{\pm 1}) \otimes \mathfrak{D}(H_{m+n-1}^{m-1}),$ (4.3)

$$Q_{G, 0} = \frac{1}{\sin^{m+n-2}} \frac{\partial}{\partial \theta} \sin^{m+n-2} \theta \frac{\partial}{\partial \theta} \otimes I + \frac{1}{\sin^2 \theta} \otimes \Delta(H_{m+n-1}^n)$$

on $\mathfrak{D}(R_0) \otimes \mathfrak{D}(H_{m+n-1}^n),$ (4.4)

where $\Delta(H_{k+i}^l)$ are Laplace-Beltrami operators related to the manifold H_{k+i}^l .

Unfortunately, the operator $Q_{G,\alpha}$ is not essentially selfadjoint on $\mathfrak{D}(R_\alpha) \otimes \mathfrak{D}(N_\alpha)$. It would be essentially selfadjoint if the projectors P_α with the domain $\mathfrak{H}(H_{m+n}^n)$ and the range $\mathfrak{H}((H_{m+n}^n)_\alpha)$ strongly commute with $Q_G^{s.a.}$, but this is not the case. This difficulty forces us to deviate from

the method of Section 3. But we do not abandon completely the approach of Section 3 as we can gain at least some partial knowledge from it. Although the operators $Q_{G,\alpha}$ are not essentially selfadjoint on $\mathfrak{D}(R_\alpha) \otimes \mathfrak{D}(N_\alpha)$ it happens that they are essentially selfadjoint on some subspaces of $\mathfrak{D}(R_\alpha) \otimes \mathfrak{D}(N_\alpha)$. In fact, the representations $D^l(H_{m+n-1}^n)$ and $D^l(H_{m+n-1}^{m-1})$ are inequivalent and give rise to the decomposition of the direct integral on the subspaces $\mathfrak{H}((H_{m+n}^n)_0)$ and $\mathfrak{H}((H_{m+n}^n)_{\pm 1})$ respectively. Therefore, the subspaces determined by the eigenvectors belonging to the discrete spectrum of $Q_H^{s.a.}$ are closures of some subspaces of $\mathfrak{D}(R_\alpha) \otimes \mathfrak{D}(N_\alpha)$. Let $E_{H,\alpha}(\lambda)$ be the projector from the spectral family of the projectors of the operator $Q_H^{s.a.}$, where $Q_{H,\pm 1}^{s.a.}$ is the selfadjoint extension of $\Delta(H_{m+n-1}^{m-1})$ on $\mathfrak{D}(H_{m+n-1}^{m-1})$ and $Q_{H,0}^{s.a.}$ of $\Delta(H_{m+n-1}^n)$ on $\mathfrak{D}(H_{m+n-1}^n)$. Then

$$E_\alpha^l = E_{H,\alpha}(-l(l+m+n+3)) - E_{H,\alpha}(-l(m+n-3) - \varepsilon), \quad 0 < \varepsilon < 1,$$

are projectors into the subspaces of the space $\mathfrak{H}(H_{m+n}^n)$ which are determined by the eigenvectors of the operator $Q_H^{s.a.}$ belonging to the eigenvalue $-l(l+m+n-3)$. Since $E_{H,\alpha}(\lambda)$ strongly commute with $Q_G^{s.a.}$, the spaces

$$E_\alpha^l \left(\sum_{\beta=-1}^1 \mathfrak{D}(R_\beta) \otimes \mathfrak{D}(N_\beta) \right) = E_\alpha^l(\mathfrak{D}(R_\alpha) \otimes \mathfrak{D}(N_\alpha))$$

reduce the operator $Q_G^{s.a.}$ to the operators

$$Q_{G,\pm 1}^l = \left(-\frac{1}{\operatorname{sh}^{m+n+2}} \frac{d}{d\theta} \operatorname{sh}^{m+n-2} \theta \frac{d}{d\theta} + \frac{l(l+m+n-3)}{\operatorname{sh}^2 \theta} \right) \otimes I$$

on $E_{\pm 1}^l(\mathfrak{D}(R_{\pm 1}) \otimes \mathfrak{D}(H_{m+n-1}^{n-1}))$, (4.5)

$$Q_{G,0}^l = \left(\frac{1}{\sin^{m+n-2}} \frac{d}{d\theta} \sin^{m+n-2} \theta \frac{d}{d\theta} - \frac{l(l+m+n-3)}{\sin^2 \theta} \right) \otimes I$$

on $E_0^l(\mathfrak{D}(R_0) \otimes \mathfrak{D}(H_{m+n-1}^n))$, (4.6)

LEMMA 4.2. — For every $l = -\left\{ \frac{m+n-3}{2} \right\}, -\left\{ \frac{m+n-3}{2} \right\} + 1, \dots$ the operators $Q_{G,\alpha}^l$ defined by (4.5) and (4.6) are essentially selfadjoint on $E_\alpha^l(\mathfrak{D}(R_\alpha) \otimes \mathfrak{D}(N_\alpha))$. The spectrum of $Q_{G,\pm 1}^{l,s.a.}$ is purely continuous :

$$\sigma_{G,\pm 1}^l = \Lambda^2 + \left(\frac{m+n-2}{2} \right)^2, \quad \Lambda \in [0, \infty).$$

The spectrum of $Q_{G,0}^{l,s.a.}$ is purely discrete : $\sigma_{G,0}^l = -L(L+m+n-2)$, $L = l, l+1, l+2, \dots$

Proof. — In Section 2 of [7] it has been shown that the spectrum of $Q'_{G,\pm 1}$ is purely continuous as described in the Lemma. The operator $Q'_{G,0}$ on $E'_0(\mathfrak{D}(\mathbb{R}_0) \otimes \mathfrak{D}(H_{m+n-1}^n))$ is equivalent to the following:

$$A_0 = \frac{d^2}{d\theta^2} - \frac{\left(l + \frac{m+n-3}{2}\right)^2 - \frac{1}{4}}{\sin^2 \theta} + \left(\frac{m+n-2}{2}\right)^2, \quad \theta \in (0, \pi) \quad (4.7)$$

on a certain domain $\mathfrak{D}(A_0)$. It is easy to see that the operator A_0 is essentially selfadjoint on any domain \mathfrak{D} dense in the space $L^2(0, \pi)$ vectors f of which are square integrable differentiable functions on the interval $(0, \pi)$ such that $(A_0 f)(\theta)$ are measurable functions with the property $\|A_0 f\| < \infty$. The eigenfunction expansion associated with the differential operator (4.7) or (4.6) is calculated in [16]:

$$f(\theta) = \sum_{L=l}^{\infty} Y^{l,L}(\theta) \int_0^{\pi} \overline{Y^{l,L}(\theta')} f(\theta') \sin^{m+n-2} \theta' d\theta', \quad (4.8)$$

where

$$Y^{l,L}(\theta) = \text{tg}^L \theta \cos^l \theta {}_2F_1\left(\frac{1}{2}(l-L), \frac{1}{2}(l-L+1); l + \frac{m+n-1}{2}; -\text{tg}^2 \theta\right). \quad (4.9)$$

Hence our proof is finished.

Remark. — There are common eigenfunctions of the selfadjoint operator $Q_G^{\text{s.a.}}$ and $Q_H^{\text{s.a.}}$ which are identically zero outside $(H_{m+n}^n)_0$ or outside $(H_{m+n}^n)_{+1} \cup (H_{m+n}^n)_{-1}$. These eigenfunctions belong to the discrete spectrum of the operator $Q_H^{\text{s.a.}}$.

COROLLARY. — Let $E_G(\lambda)$ be the projectors of the spectral family of projectors of the operator $Q_G^{\text{s.a.}}$. The two following equations then hold:

$$\left[E_G\left(\left(\frac{m+n-2}{2}\right)^2\right) - E_G(-\infty) \right] \left[E_{H,\pm 1}\left(\left(\frac{m+n-3}{2}\right)^2\right) - E_{H,\pm 1}(-\infty) \right] = 0 \quad \text{on } \mathfrak{S}(H_{m+n}^n), \quad (4.10)$$

$$\left[E_G\left(\left(\frac{m+n-2}{2}\right)^2\right) - E_G(-\infty) \right] \left[E_{H,0}\left(\left(\frac{m+n-3}{2}\right)^2\right) - E_{H,0}(-\infty) \right] = E \quad \text{on } \mathfrak{S}(H_{m+n}^n), \quad (4.11)$$

where E is a projector defined by

$$(Ef)(p) = 0 \quad \text{for} \quad p \in (H_{m+n}^n)_{\pm 1},$$

$$(Ef)(\theta, \eta, \omega_{m-1}, \tilde{\omega}_n) = \sum_{l=-\left\{\frac{m+n-5}{2}\right\}}^{\infty} \sum_{L=l}^{\infty} Y^{l,L}(\theta) Y_{m_{SO(m-1,n)}}^{l,L}(\eta, \omega_{m-1}, \tilde{\omega}_n) \cdot \int d\mu(\theta', \Omega') Y^{l,L}(\theta') Y_{m_{SO(m-1,n)}}^{l,L}(\Omega') f(\theta', \Omega'),$$

for $p = (\theta, \eta, \omega_{m-1}, \tilde{\omega}_n) = (\theta, \Omega) \in (H_{m+n}^n)_0. \quad (4.12)$

Here the Y -functions are related to the hyperboloid H_{m+n-1}^n and are described in Section 3.

The Lemmas 4.1 and 4.2 and corollary to Lemma 4.2 give us information useful for reducing the desired representation but incomplete since they concern the discrete spectrum of $Q_H^{s.a.}$ only. In order to solve our reduction problem completely we have to add some other information.

This is : a) the reduction of the representations $D^L(M)$ when restricted to the maximal compact subgroup $SO(m) \otimes SO(n)$ which is described in (2.7)-(2.12) and b) the explicit forms of the eigenfunctions of the operator $Q_G^{s.a.}$ determining the spaces $\mathfrak{H}^L(M)$ given in ref. [6].

Let us denote by $Y_{m_G}^{L,l_G}$ vectors of $\mathfrak{H}(H_{m+n}^n)$ which, expressed as functions on H_{m+n}^n , have the known form from [6] with the indices $\{l_G, m_G\}$ from the set $\mathcal{N}_L(H_{m+n}^n)$. We know that the vectors $Y_{m_G}^{L,l_G}$, together with the eigendistributions $Y_{m_G}^{\Lambda,l_G}$ on $\mathfrak{D}(H_{m+n}^n)$, $\Lambda \in [0, \infty)$, $\{l_G, m_G\} \in \mathcal{N}_1(H_{m+n}^n)$, make a complete set of eigendistributions associated with the operator $Q_G^{s.a.}$ [8]. The eigendistributions of the operator $Q_H^{s.a.}$ on $\mathfrak{D}(H_{m+n-1}^{m-1})$ for $\alpha = \pm 1$ and $\mathfrak{D}(H_{m+n-1}^n)$ for $\alpha = 0$ are denoted analogously by Y_{x,m_H}^{ν,l_H} , $\nu = l, \lambda$. To make a complete set of eigendistributions associated with the operator $Q_H^{s.a.}$ on some linear manifold of $\mathfrak{H}(H_{m+n}^n)$, we find an orthonormal complete sequence $\{g_{1,x}, g_{2,x}, \dots\} \subset \mathfrak{D}(R_x)$ of the space $[\mathfrak{D}(R_x)]^\sim$. Then the set of eigendistributions $g_{k,x} \otimes Y_{m_H}^{l,H}$ and $g_{k,x} \otimes Y_{m_H}^{\lambda,H}$ on $\Sigma_x^{+1} \mathfrak{D}(R_x) \otimes \mathfrak{D}(N_x)$ is a complete set of eigendistributions of the operator $Q_H^{s.a.}$. Let us calculate now the expansion of the vector $Y_{m_G}^{L,l_G}$ in terms of the eigendistributions $g_{k,x} \otimes Y_{m_H}^{\nu,l_H}$, $\nu = l, \lambda$. For this purpose we prove first the following Lemma:

LEMMA 4.3. — The integral

$$C_{\alpha, m_G}^{L, l_G}(\theta, \lambda) = \int_{M \subset (H_{m+n})_{\alpha}} d\mu(\Omega) Y_{\alpha, m_H}^{\lambda, l_H}(\Omega) Y_{m_G}^{L, l_G}(\theta, \Omega) \tag{4.13}$$

is a regular analytic function of λ in the strip $|m\lambda| < 3/2$ of the complex λ -plane, uniformly bounded with respect to θ of the interval $[0, \infty)$.

Proof. — Using the expressions of the functions $Y_{m_G}^{L, l_G}$ of ref. [6] we obtain the bound $|Y_{m_G}^{L, l_G}(\theta, \Omega)| < A(\text{sh } \theta \text{ ch } \eta)^{-(L+m+n-2)}$. Similarly, we estimate $|Y_{m_H}^{\lambda, l_H}(\eta, \omega_{m-1}, \tilde{\omega}_n)| < (\text{ch } \eta)^{-\frac{1}{2}(m+n-3)+|lm\lambda|}$ using the expressions of ref. [8]. As $d\mu(\Omega) = \text{ch}^{m-2} \eta \text{ sh}^{n-1} \eta \, d\eta d\mu(\omega_{m-1}) d\mu(\tilde{\omega}_n)$, where the last two factors are invariant measures on the spheres S^{m-1} and S^n , we conclude that the integrand can be estimated by $K(\text{ch } \eta)^{|lm\lambda| - (L + \frac{m+n-1}{2})}$ and the minimal possible value of L is $-\left\{ \frac{m+n-4}{2} \right\}$. The integrand is an analytic function in λ in the strip $|m\lambda| < 3/2$ for every $\theta \in [0, \infty)$. Hence the integral (4.13) has the analytical properties of the statement.

Let us consider the functions

$$C_{\alpha, m_H}^{L, l \left\{ \frac{m}{2} \right\}, l_H}(\theta, \lambda) = C_{\alpha, m_G}^{L, l_G}(\theta, \lambda),$$

where we divided the set $\{l_G, m_G\}$ into $\left\{ l \left\{ \frac{m}{2} \right\} \right\} \cup \{l_H, m_H\}$, and the

function $C_{\alpha, m_H}^{L, l \left\{ \frac{m}{2} \right\}, l_H}(\theta)$ to be defined by

$$C_{\alpha, m_G}^{L, l \left\{ \frac{m}{2} \right\}, l_G}(\theta) = \int_{M \subset (H_{m+n})_{\alpha}} d\mu(\Omega) Y_{\alpha, m_H}^{l, l_H}(\Omega) Y_{m_G}^{L, l_G}(\theta, \Omega). \tag{4.14}$$

We define distributions $\mathcal{Z}_{\alpha, m_H}^{L, l \left\{ \frac{m}{2} \right\}, l_H}$ on $\mathcal{D}(\mathbb{R}_{\alpha}) \otimes \mathcal{D}(\mathbb{N}_{\alpha})$ by

$$\begin{aligned} \mathcal{Z}_{\alpha, m_H}^{L, l \left\{ \frac{m}{2} \right\}, l_H}(\theta, \Omega) &= C_{\alpha, m_H}^{L, l \left\{ \frac{m}{2} \right\}, l_H}(\theta) Y_{\alpha, m_H}^{l, l_H}(\Omega), \\ \mathcal{Z}_{\alpha, m_H}^{L, \lambda, l \left\{ \frac{m}{2} \right\}, l_H}(\theta, \Omega) &= C_{\alpha, m_H}^{L, l \left\{ \frac{m}{2} \right\}, l_H}(\theta, \lambda) Y_{\alpha, m_H}^{\lambda, l_H}(\Omega). \end{aligned} \tag{4.15}$$

These distributions are eigendistributions of the operator $Q_H^{s.a.}$. Of course, the system (4.15) is not a complete system of eigendistributions of $Q_H^{s.a.}$ on

$\sum_{\alpha} \mathfrak{D}(R_{\alpha}) \otimes \mathfrak{D}(N_{\alpha})$. Let \mathfrak{K} be the closed subspace of the space $\mathfrak{H}(H_{m+n}^n)$ determined by the system (4.15) (we mean the closed subspace $\mathfrak{K} \subset \mathfrak{H}(H_{m+n}^n)$ which is determined by all the vectors

$$\mathfrak{Z}_{\alpha, mH}^{L, l, \left\{ \begin{smallmatrix} m \\ 2 \end{smallmatrix} \right\}, l_H} \in \mathfrak{H}(H_{m+n}^n) \quad \text{and} \quad \int_0^{\infty} d\lambda k(\lambda) \mathfrak{Z}_{\alpha, mH}^{L, \lambda, l, \left\{ \begin{smallmatrix} m \\ 2 \end{smallmatrix} \right\}, l_H}, \quad k(\lambda) \in C_0((0, \infty)).$$

We can complete the system (4.15) by the eigendistributions of $Q_H^{s.a.}$, which determine the close subspace $\mathfrak{H}(H_{m+n}^n) \ominus \mathfrak{K}$. We do not need this additional set of eigendistribution as the space \mathfrak{K} contains the closed subspace of $\mathfrak{H}(H_{m+n}^n)$ determined by the vectors Y_{mG}^{L, l_G} , which is clear from the construction.

THEOREM 4.1. — Every irreducible representation $D^L(H_{m+n}^n)$, $m > n \geq 2$, $D^L(H_{m+1}^1)$, when restricted to the subgroup $SO_0(m-1, n)$, has the reduction

$$D^L(H_{m+n}^n) = \sum_{l=-\left\{ \frac{m+n-5}{2} \right\}}^L D^l(H_{m+n-1}^n) \oplus \int_0^{\infty} d\lambda D_{m-1, n}^{\lambda, +} \oplus \int_0^{\infty} d\lambda D_{m-1, n}^{\lambda, -}, \quad m \geq 4, n \geq 2, \quad (4.16)$$

$$D^l(H_3^2) = \sum_{l=0}^L (D^l_+(H_4^2) \oplus D^l_-(H_4^2)) \oplus \int_0^{\infty} d\lambda D_{2, 2}^{\lambda, +} \oplus \int_0^{\infty} d\lambda D_{2, 2}^{\lambda, -}, \quad (4.17)$$

$$D^L(H_{m+1}^1) = \sum_{l=-\left\{ \frac{m-4}{2} \right\}}^L D^l(H_m^1) \oplus \int_0^{\infty} d\lambda D_{m-1, 1}^{\lambda}, \quad m > 3, \quad (4.18)$$

$$D^L(H_4^1) = \sum_{l=0}^L (D^l_+(H_3^1) \oplus D^l_-(H_3^1)) \oplus \int_0^{\infty} d\lambda D_{2, 1}^{\lambda}. \quad (4.19)$$

Every irreducible representation $D^L(H_{m+n}^m)$, $n \geq 3$; $D^L_{\pm}(H_{m+2}^m)$, $n > 2$, when restricted to the subgroup $SO_0(m, n-1)$, has the following reduction:

$$D^L(H_{m+n}^m) = \sum_{l=-\left\{ \frac{m+n-5}{2} \right\}}^L D^l(H_{m+n-1}^m) \oplus \int_0^{\infty} d\lambda D_{m, n-1}^{\lambda, +} \oplus \int_0^{\infty} d\lambda D_{m, n-1}^{\lambda, -}, \quad m \geq n \geq 3, \quad (4.20)$$

$$D^L(H_{m+3}^m) = \sum_{l=-\lfloor \frac{m-2}{2} \rfloor}^L (D^l_+(H_{m+2}^m) \oplus D^l_-(H_{m+2}^m)) \oplus \int_0^\infty d\lambda D_{m,2}^{\lambda,+} \oplus \int_0^\infty d\lambda D_{m,2}^{\lambda,-}, \quad m > 2, \quad (4.21)$$

$$D^\pm_L(H_{m+2}^m) = \int_0^\infty d\lambda D_{m,1}^\lambda. \quad (4.22)$$

Proof. — First we consider the representations $D^L(H_{m+n}^n)$. According to the corollary to the Lemma 4.2 we have, a) $C_{\pm 1, m_H}^{\left\{ \frac{m}{2} \right\}, l_H}(\theta) = 0$ and b) there must be some value of $l_{\left\{ \frac{m}{2} \right\}}$ for every fixed value of $L, l, l_H, m_H; L > l, l_H, m_H \in \mathcal{N}_L(H_{m+n-1}^n)$ such that the function $C_{0, m_H}^{\left\{ \frac{m}{2} \right\}, l_H}(\theta)$ is different from zero. The space $\mathfrak{S}^L(H_{m+n}^n)$ is invariant with respect to the representation $H \times \mathfrak{S}^L(H_{m+n}^n) \ni (h, f) \rightarrow U(h)f \in \mathfrak{S}^L(H_{m+n}^n)$. Hence the function $C_{0, m_H}^{\left\{ \frac{m}{2} \right\}, l_H}(\theta)$ for a fixed θ may be considered as a component of the Fourier transform $\mathfrak{S}^L(H_{m+n}^n) \supset \mathfrak{D}^L(H_{m+n}^n) \ni f \rightarrow \widehat{\chi}^{L, l}(f) \in \mathfrak{L}$, where the vectors $\widehat{\chi}^{L, l}(f)$ are defined by the sequences

$$\begin{aligned} \widehat{\chi}^{L, l}(f) &= \left\{ \chi_{m_H}^{\left\{ \frac{m}{2} \right\}, l_H} \left(f \right) \middle| l_{\left\{ \frac{m}{2} \right\}}, l_H, m_H \in \mathcal{N}_L(H_{m+n}^n) \right\}, \\ \chi_{m_H}^{\left\{ \frac{m}{2} \right\}, l_H} (f) &= \int d\mu(\Omega) \overline{Y_{m_H}^{l, l_H}(\Omega)} f(\theta, \Omega) \\ f &= \sum_{m_G, l_G} k_{l_G}^{m_G} Y_{m_G}^{L, l_G} \in \mathfrak{D}_L(H_{m+n}^n). \end{aligned}$$

The unitary representation of the group H on \mathfrak{L} is defined by

$$U(g)\widehat{\chi}(f) = \widehat{\chi}(U(g)f).$$

Using [6] it is easy to see that the representation $H \ni h \rightarrow \widehat{\chi}(U(h)f) \in \mathfrak{L}$ is irreducible for $m > 3$, and has two irreducible subspaces for $m = 3$.

Thus the vectors $Z_{0, m_H}^{\left\{ \frac{m}{2} \right\}, l_H}$ for fixed L determine a closed vector space equivalent to $\mathfrak{S}^L(H_{m+n-1}^n)$ for $m > 3$ or to $\mathfrak{S}^l_+(H_{n+2}^n) \oplus \mathfrak{S}^l_-(H_{n+2}^n)$ for $m = 3$.

In this way we have proved that the subspaces $\mathfrak{H}^L(H_{m+n}^n)$ contain each of the subspaces $\mathfrak{H}^l(H_{m+n-1}^n)$, $l = -\left\{\frac{m+n-5}{2}\right\}, -\left\{\frac{m+n-5}{2}\right\} + 1, \dots, L$, only once for $m > 3$ and each of the subspaces $\mathfrak{H}^l_{\pm}(H_{n+2}^n)$, $l = -\left\{\frac{n-3}{2}\right\}, -\left\{\frac{n-3}{2}\right\} + 1, \dots, L$, only once for $m = 3$. If $\mathfrak{H}^L(H_{m+n}^n)$ were equal to the sum

$$\sum_{l=-\left\{\frac{m+n-5}{2}\right\}}^L \oplus \mathfrak{H}^l(H_{m+n-1}^n),$$

then every irreducible representation of the subgroup $SO(m-1) \otimes SO(n)$ would appear at most $L + \left\{\frac{m+n-5}{2}\right\} + 1$ times, i. e. a finite number of times. This is in contradiction to the structure of the representations $D^L(H_{m+n}^n)$ describes in Section 2. Hence on the basis of Lemma 4.1 there is, besides a discrete sum, a direct integral

$$\int_0^\infty n_+(\lambda) \mathfrak{H}_{m-1,n}^{\lambda,+} d\lambda \oplus \int_0^\infty n_-(\lambda) \mathfrak{H}_{m-1,n}^{\lambda,-} d\lambda \tag{4.23}$$

in the reduction of the representation $D^L(H_{m+n}^n)$. Here, $n_{\pm}(\lambda)$ are multiplicities of the representations $D_{m-1,n}^{\lambda,\pm}$. In other words, we know that

there is an integer $l_{\left\{\frac{m}{2}\right\}}$ such that the function $C_{\alpha, m_H}^{L, l_{\left\{\frac{m}{2}\right\}}, l_H}(\theta, \lambda) \neq 0$ on

$\mathbb{R}_x \times [0, \infty)$. Because of the Lemma 4.3 the function $C_{\alpha, m_H}^{L, l_{\left\{\frac{m}{2}\right\}}, l_H}(\theta, \lambda)$ may be zero in λ only on a denumerable set of points from $[0, \infty)$ so that $n_{\pm}(\lambda)$ are constant functions n_{\pm} almost everywhere on the interval $[0, \infty)$. The multiplicity of the representations, i. e. the numbers n_{\pm} , can be determined in the same way as for the subspaces $\mathfrak{H}^l(H_{m+n-1}^n)$. That is, the subspaces $\mathfrak{H}_{m-1,n}^{\lambda}$ determined by the vectors $\widehat{\chi}^{L, \lambda}(f)$ for a fixed θ , where

$$\widehat{\chi}^{L, l}(f) = \left\{ \chi_{m_H}^{L, \lambda, l_{\left\{\frac{m}{2}\right\}}, l_H} (f) \middle| l_{\left\{\frac{m}{2}\right\}}, l_H, m_H \in \mathcal{N}_L(H_{m+n}^n), f \in \mathfrak{D}^L(H_{m+n}^n) \right\}$$

and for $f = Y_{m_G}^{L, l_G}$

$$\chi_{m_H}^{L, \lambda, l_{\left\{\frac{m}{2}\right\}}, l_H} (Y_{m_G}^{L, l_G}) = \int d\mu(\Omega) \overline{Y_{m_H}^{\lambda, l_H}(\Omega)} Y_{m_G}^{L, l_G}(\theta, \Omega),$$

are invariant with respect to the representation of the group H . Using the reflection operator P and keeping in mind that the eigenvectors Y_{mG}^{L, l_G} belong to a definite eigenspace of the reflection operator P , we easily find that $n_{\pm} = 1$ in accordance with the formulae (4.16)-(4.19) describing the reduction of the representations $D^L(H_{m+n}^n)$.

To reduce the representation $D^L(H_{m+n}^m)$ we remark that the expression for the operators $Q_{G, \alpha}$ is the same as in (4.5) and (4.6) so that a similar analysis can be made. Only in the case $n = 2$ is there no discrete spectrum of the selfadjoint extensions of the operators $\Delta(H_{m+1}^m)$ so that the decomposition is of the form (4.22). Q. E. D.

APPENDIX

Let $R_1 \in t \rightarrow \{g_{kl}(t)\} \subset SO_0(m, n)$ be the one parameter subgroup defined as that subgroup of $SO_0(m, n)$ elements g of which satisfy

$$g(\Gamma_{m,n}^{m,n} - (\Gamma_{m,n}^{k+1,l+1} - \Gamma_{m,n}^{k,l})) = \Gamma_{m,n}^{m,n} - (\Gamma_{m,n}^{k+1,l+1} - \Gamma_{m,n}^{k,l}).$$

To every one parameter subgroup $\{g_{kl}(t)\}$ we associated the generator X_{kl} . The generators of the compact and noncompact one parameter subgroups will be denoted by L_{kl} and B_{kl} respectively. The Lie algebra $\mathfrak{so}(m, n)$ of the group $SO_0(m, n)$ can be described as in (6.1) of [6]. The subalgebra $\mathfrak{so}(m-1, n-1)$ of the Lie algebra $\mathfrak{so}(m, n)$ which belongs to the subgroup $SO_0(m-1, n-1)$ is generated by elements L_{ij} , B_{ij} $i, j = 1, \dots, m-1, m+1, \dots, m+n-1$. Hence the elements L_{im} , $i = 1, 2, \dots, m-1$; $L_{j,m+n}$, $j = m+1, m+2, \dots, m+n-1$; B_{im} , $i = m+1, m+2, \dots, m+n$; $B_{j,m+n}$, $j = 1, 2, \dots, m$ are outside of $\mathfrak{so}(m-1, n-1)$. Let us construct the generators

$$Y_i = L_{im} + B_{im}, \quad i = 1, 2, \dots, m-1, \quad Y_j = L_{j,m+n} + B_{j,m+n}, \quad j = m+1, \dots, m+n-1.$$

It is easy to see that the smallest subalgebra generated by Y_j , $i = 1, 2, \dots, m-1, m+1, m+2, \dots, m+n-1$ is a commutative $(m+n-2)$ -dimensional subalgebra \mathfrak{a}^{m+n-2} of $\mathfrak{so}(m, n)$ and $[\mathfrak{so}(m-1, n-1), \mathfrak{a}^{m+n-2}] = \mathfrak{a}^{m+n-2}$. Hence T^{m+n-2} can be taken as that commutative subgroup of $SO_0(m, n)$, which has the Lie algebra \mathfrak{a}^{m+n-2} .

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[1] See for example

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- [8] N. LIMIĆ, J. NIEDERLE and R. RACZKA, *J. Math. Phys.*, t. **8**, 1967, p. 1091.
- [9] The group G is unimodular if $|\det Ad_G(g)| = 1$ for all $g \in G$, where $Ad_G(g)$ is the automorphism of the Lie algebra \mathfrak{g} of the group G defined by $Ad_G(g): \mathfrak{g} \ni X \rightarrow Ad_G(g)X = dI(g)_e X$ and $I(g)$ is the inner isomorphism of G onto itself. The group $SO_0(r, s)$ is a semi-simple (in fact simple) Lie group, hence unimodular. For the group $G = T^{m-n-2} \square SO_0(m-1, n-1)$ we also have $|\det Ad_G(g)| = 1$, as follows from the following argument. G is a connected group and therefore

every neighbourhood $U(e)$ of the identity element $e \in G$ generates the whole group G . As $G \ni g \rightarrow Ad(g) \in GL(g)$ is the homomorphism, it suffices to prove that $|\det Ad(g)| = 1$ for a $g \in U(e)$. We choose such $U(e)$ for which a neighbourhood $V(o) \ni g \rightarrow \exp X \in U(e)$ is the diffeomorphism. Then for every $g = \exp X \in U(e)$ we have $|\det Ad(\exp X)| = \exp \{ \text{Tr} adX \}$. In the basis of the Lie algebra of the considered group G , which is the union of the basis of Lie algebras of the groups T^{m+n-2} and $SO_0(m-1, n-1)$, one easily calculates that $\text{Tr} adX = 0$.

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[11] The Hilbert space $\mathfrak{S}(M)$ is a Hilbert space vector of which the equivalence classes of complex valued measurable functions $f(p)$ on M such that

$$\int_M |f(p)|^2 d\mu(p) < \infty$$

and the scalar product is defined by

$$(f, h) = \int_M \overline{f(p)} \times h(p) d\mu(p), f, h \in \mathfrak{S}(M).$$

Addition of vectors and multiplication of vectors by complex numbers is defined as the corresponding operations with the complex valued functions.

[12] The invariant C_2 is a Casimir operator $g^{\mu\nu} X_\mu X_\nu$ where $g^{\mu\nu}$ is the Cartan metric tensor of the Lie algebra $\mathfrak{so}(m, n)$ in a basis $X_1, X_2, \dots, X_{\left[\frac{m+n}{2}\right]}$.

[13] Here and elsewhere we use brackets for indices defined as follows:

$$\left[\begin{array}{l} a \\ 2 \end{array} \right] = \begin{cases} \frac{a}{2} & \text{for } a = 2r \\ \frac{a-1}{2} & \text{for } a = 2r+1, \end{cases} \quad \left\{ \begin{array}{l} a \\ 2 \end{array} \right\} = \begin{cases} \frac{a}{2} & \text{for } a = 2r \\ \frac{a+1}{2} & \text{for } a = 2r+1, \end{cases} \quad r = 1, 2, \dots$$

[14] For instance all UI representations of the group $SO_0(m, n)$ $m \geq n \geq 2$ related with three homogeneous spaces M which are classified by the same real number $\Lambda \in (0, \infty)$ and the same eigenvalue of the operator P are equivalent.

[15] E. C. TITCHMARSH, *Eigenfunction expansions*, Part I, Clarendon Press, Oxford, 1962.

[16] N. J. VILENKIN, *Special functions and theory of group representation*, NAUKA, Moscow, 1965 (In Russian).

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