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Section A :
Physique théorique.

An exponentiation theorem for unbounded derivations

by

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ABSTRACT — We give a sufficient (and necessary) condition to define the exponential of unbounded derivations in C*-algebras.

1. DEFINITIONS

Let $\mathscr{A}$ be a Banach algebra, a derivation is a linear function D from a dense sub-algebra $\mathscr{A}^{(1)}$ of $\mathscr{A}$, into $\mathscr{A}$, such that

$$\forall x \in \mathscr{A}^{(1)} \quad \forall y \in \mathscr{A}^{(1)} \qquad D(xy) = D(x)y + xD(y) \tag{1.1}$$

For a *-Banach algebra $\mathscr{A}$, the derivation D is said to be hermitian if:

$$\forall x \in \mathscr{A}^{(1)} \qquad x^* \in \mathscr{A}^{(1)} \quad \text{and} \quad D(x^*) = (D(x))^*.$$

The set of the elements x in $\mathscr{A}$ such that the function

$$\zeta \to \sum_{n=0}^{\infty} \frac{\zeta^n}{n!} D^n(x)$$

exists and is analytic in some neighbourhood of 0, is called « the set of the analytic elements » with respect to this derivation and is written $\mathscr{A}^{(a)}$.

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2. THEOREM

Let $\mathscr{A}$ be a C*-algebra, D an hermitian closed derivation of $\mathscr{A}$, such as $\mathscr{A}^{(a)}$ is dense in $\mathscr{A}$, then D induces a strongly continuous group $\{\alpha_t \mid t \in \mathbb{R}\}$ of automorphisms of $\mathscr{A}$.

Proof. — If $x \in \mathscr{A}^{(a)}$, $\exists t_x > 0$ such that $t \in \mathrm{R}$, $|t| \leqslant t_x$ we can define:

$$\alpha_t(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathrm{D}^n(x)$$

which is absolutely convergent in $\mathscr{A}$.

$\alpha_t(x) \in \mathscr{A}^{(a)}$, since for $|t'| < t_x - |t|$ we shall show that:

$$\alpha_{t'}(\alpha_t(x)) = \alpha_{t+t'}(x). \tag{2.1}$$

We write

$$y = \alpha_t(x); \quad y_j = \sum_{n=0}^{j} \frac{\mathrm{D}^n(x)}{n!} t^n; \quad y_j \in \mathscr{A}^{(1)}.$$

$$\mathrm{D}(y_j - y_k) = \sum_{n=k+1}^{j} \frac{\mathrm{D}^{n+1}(x)}{n!} t^n$$

now $\displaystyle\sum_{n=0}^{\infty} \frac{\mathrm{D}^n(x)}{n!} t^n$ is analytic on $]-t_x, t_x[$, therefore ([1], 9.3.5) $(\alpha_t(x))'_t$ is absolutely and uniformly converging on the same interval

$$(\alpha_t(x))'_t = \sum_{n=1}^{\infty} \frac{\mathrm{D}^n(x)}{(n-1)!} t^{n-1} = \sum_{n=0}^{\infty} \frac{\mathrm{D}^{n+1}(x)}{n!} t^n$$

We write $z_j = \displaystyle\sum_{n=0}^{j} \frac{\mathrm{D}^{n+1}(x)}{n!} t^n$, then $(z_j)_j$ is a Cauchy sequence for $\|\cdot\|$ and:

$$z_j - z_k = \sum_{n=k+1}^{j} \frac{\mathrm{D}^{n+1}(x)}{n!} t^n = \mathrm{D}(y_j - y_k).$$

So $\mathrm{D}(y_j - y_k)$ converges to 0 as j and k go to infinity. Let

$$z = \lim_{j,\infty} \mathrm{D}(y_j).$$

Now, $y = \lim_{j,\infty} y_j$. As D is closed, $z = \mathrm{D}(y)$

$$\mathrm{D}(y_j) = \sum_{n=1}^{j} \frac{\mathrm{D}^n(\mathrm{D}(x))}{n!} t^n, \qquad \lim_{j,\infty} \mathrm{D}(y_j) = \alpha_t(\mathrm{D}(x)).$$

hence

$$\mathrm{D}(\alpha_t(x)) = \alpha_t(\mathrm{D}(x)) \tag{2.2}$$

and consequently

$$\alpha_{t'}(\alpha_t(x)) = \lim_{l,\infty} \sum_{k=0}^{l} \frac{\mathrm{D}^k}{k!} \left(\sum_{n=0}^{\infty} \frac{\mathrm{D}^n(x)}{n!} t^n \right) t'^k$$

$$= \lim_{l,\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{l} \frac{\mathrm{D}^{k+n}(x)}{k!n!} t^n t'^k$$

which is absolutely converging as l goes to infinity, so we can rearrange the terms:

$$\sum_{m=0}^{\infty} \frac{\mathrm{D}^m(x)}{m!} (t + t')^m = \alpha_{t+t'}(x).$$

Through elementary calculations, taking advantage of the absolutely convergence of the series and of the continuity of * one gets:

$$\alpha_t(\lambda x + \lambda y) = \lambda \alpha_t(x) + \mu \alpha_t(y) \tag{2.3}$$

$$\alpha_t(xy) = \alpha_t(x)\alpha_t(y) \tag{2.4}$$

$$\alpha_t(x^*) = (\alpha_t(x))^* \tag{2.5}$$

for $t \in \mathrm{R}$ sufficiently small.

Moreover $\forall t \in \mathrm{R}$, $\exists m \in \mathrm{N} \,|$, $|t| < mt_x$; we write

$$\alpha_t(x) = \left[\frac{\alpha_t}{m} \right]^m (x)$$

α_t is now well defined for all $t \in \mathrm{R}$ on $\mathscr{A}^{(a)}$ and fulfils (2.1) and (2.2) for every x in $\alpha^{(a)}$,

α_t is a *-algebra isomorphism applying $\mathscr{A}^{(a)}$ into $\mathscr{A}^{(a)}$ and $\forall x \in \mathscr{A}^{(a)}$, $t \to \alpha_t(x)$ is an analytic function. We shall extend α_t to $\mathscr{A}$. We can assume that $\mathscr{A}$ has a unit element, for, if not, we can define D on $\tilde{\mathscr{A}} = \mathrm{C} \times \mathscr{A}$, the algebra obtained from $\mathscr{A}$ by adjunction of a unit element,

$$\mathrm{D}(\lambda, x) = (0, \mathrm{D}(x)).$$

Moreover, we can assume that $e \in \mathscr{A}^{(1)}$; because if not one settles: $\mathrm{D}(e) = 0$.

Note that $\alpha_t(e) = e$ because $D(e) = 0$. If $y = \alpha_0(y)$ is invertible, there exists a neighbourhood of 0 such that $\alpha_t(y)$ is invertible. Now if $t \to \alpha_t(y)$ is analytic, then $t \to (\alpha_t(y))^{-1}$ is also analytic. We can put $\alpha_t(y^{-1}) = (\alpha_t(y))^{-1}$ so $y \in \mathscr{A}^{(a)} \Rightarrow y^{-1} \in \mathscr{A}^{(a)}$ for $x \in \mathscr{A}^{(a)}$; $\lambda \in \mathbb{C}$.

$$x - \lambda e \text{ invertible} \Rightarrow \exists y \text{ and } (x - \lambda e)y = e$$
$$\Rightarrow \alpha_t \text{ is well defined on } y \text{ and } [\alpha_t(x) - \lambda e]\alpha_t(y) = e$$

therefore $(\alpha_t(x) - \lambda e)$ is invertible; hence $\text{Spec}' \alpha_t(x) \subset \text{Spec}' x$.

On the other hand, for an hermitian element y of $\mathscr{A}$:

$$\| y \| = \sup_{\zeta \in \text{Spec}' y} | \zeta |$$

([1], 15.4.14.1); hence:

$$\| \alpha_t(x) \|^2 = \| \alpha_t(x^*x) \| = \sup_{\zeta \in \text{Spec}' \alpha_t(x^*x)} | \zeta | \leqslant \sup_{\zeta \in \text{Spec}\, x^*x} | \zeta | = \| x^*x \| = \| x \|^2$$

and finally $\| \alpha_t(x^*x) \| = \| x \|$ on $\mathscr{A}^{(a)}$. We extend α_t to $\mathscr{A}$ (2.1) to (2.5) still hold $\forall x \in \mathscr{A}$, $\exists (y_n)n$, $y_n \in \mathscr{A}^{(a)}$ and $x = \lim_n y_n$. Therefore

$$\lim_n \| \alpha_t(x) - \alpha_t(y_n) \| = 0.$$

$t \to \alpha_t(x)$ is continuous as a uniform limit of continuous functions. So that the one-parameter unitary group $\{ \alpha_t \mid t \in \mathrm{R} \}$ is strongly continuous.

Comment. — We get an extension to C*-algebras of the work of E. Nelson on Hilbert spaces ([5]).

3. CONVERSE PROPOSITION

We give a new proof of the result of Kastler-Pool-Poulsen [4], which improves some one of I. Guelfand [3].

Let $\mathscr{E}$ be a Banach space, $\{ \alpha_t \}_{t \in \mathbb{R}}$ a strongly continuous one-parameter group of uniformly bounded linear operators, i. e.

$$\exists M > 0 \forall t \in \mathbb{R} \qquad \| \alpha_t \| \leqslant M$$

$\forall x \in \mathscr{E}$, $\forall \rho \in \mathscr{L}^1_{\mathbb{C}}(\mathbb{R})$; let $\alpha(\rho)x = \int_{-\infty}^{+\infty} \alpha_t(x)\rho(t)dt$, which exists in the Bochner's sense since $\| \alpha_t(x)\rho(t) \| \leqslant M \| x \| \, | \rho(t) |$ and one has that:

$$t \to \| \alpha_t(x)\rho(t) \| \in \mathscr{L}^1_{\mathbb{C}}(\mathbb{R}).$$

PROPOSITION. — $\mathscr{E}^{(e)}$ $(= \{ x \in \mathscr{E} \mid t \in \mathbb{R} \to \alpha_t(x) \text{ is entire} \})$ is dense in $\mathscr{E}$.

Proof. — Let ρ be a function in $C^{\mathbb{R}}$ so that $\hat{\rho} \in \mathscr{D}$. Then $\rho \in \mathscr{S}$, and $\rho \in \mathscr{L}^1_{\mathbb{C}}(\mathbb{R})$. Moreover, suppose that $\int_{-\infty}^{+\infty} \rho(t)dt = 1$. We notice that $\forall \varepsilon > 0, \exists \eta > 0$ so that

$$\int_{-\infty}^{-\eta} |\rho(t)|\, dt \leqslant \varepsilon \quad \text{and} \quad \int_{\eta}^{\infty} |\rho(t)|\, dt \leqslant \varepsilon.$$

Now if $\rho_n^{(t)} = n\rho(nt)$, $\int_{-\infty}^{-\frac{\eta}{n}} |\rho_n(t)|\, dt \leqslant \varepsilon$ and $\int_{\frac{\eta}{n}}^{\infty} |\rho_n(t)|\, dt \leqslant \varepsilon$

$$\|\alpha(\rho_n)x - x\| = \left\| \int_{-\infty}^{+\infty} \rho_n(t)\alpha_t(x)dt - \int_{-\infty}^{+\infty} \rho_n(t)x dt \right\|$$
$$\leqslant \int_{-\infty}^{-\frac{\eta}{n}} |\rho_n(t)|\; \|\alpha_t(x) - x\|\, dt + \int_{-\frac{\eta}{n}}^{+\frac{\eta}{n}} \ldots + \int_{\frac{\eta}{n}}^{\infty} \ldots$$

Now, $\forall \varepsilon > 0, \exists n_0$ such that $\forall n \geq n_0$ and $|t| \leqslant \dfrac{\eta}{n} \Rightarrow \|\alpha_t(x) - x\| \leqslant \varepsilon$.
On the other hand $\|\alpha_t(x)\| \leqslant M\|x\|$, hence

$$\|\alpha(\rho_n)x - x\| \leqslant [2(M + 1)\|x\| + 1]\varepsilon \quad \text{and} \quad x = \lim_n \alpha(\rho_n)x.$$

We prove that $\alpha(\rho_n)x \in \mathscr{E}^{(e)} \;\; \forall x \in \mathscr{E}$. Indeed:

$$\alpha_r(\alpha(\rho_n)x) = \alpha_r\left(\int_{-\infty}^{+\infty} \rho_n(t)\alpha_t(x)dt \right) = \int_{-\infty}^{+\infty} \rho_n(t)\alpha_{r+t}(x)dt$$
$$= \int_{-\infty}^{+\infty} \rho_n(t - r)\alpha_t(x)dt$$
$$= (\rho_n * h)(r)$$

where $h(r) = \alpha_r(x)$.

Now, h being continuous and bounded, $\rho_n * h \in \mathscr{S}$ and $\widehat{\rho_n * h} = \hat{\rho}_n \hat{h}$ is a distribution (cf. [6]) with compact support, hence due to the Paley-Wiener theorem $\rho_n * h$ is an entire function.

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