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## **A Group theoretical Model for Mass-splitting**

by

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**ABSTRACT.** — We consider the extensions of the orthochronous inhomogeneous Lorentzgroup with determinant  $+ 1$  by some internal symmetry group  $G_1$  and introduce a new quantal observable, called the « symmetry breaking operator »  $C(f)$  which measures the mass splitting of particles within some multiplet. We give a model for the mechanism of the experimentally observed symmetry breaking. I. e. this model exhibits the existence of nontrivial couplings of space-time symmetry and some internal symmetry.

**SOMMAIRE.** — Nous considérons les extensions du groupe de Lorentz orthochrone inhomogène de déterminant  $+ 1$  par un groupe de symétrie interne  $G_1$  et nous introduisons une nouvelle observable quantique  $C(f)$ , appelée « opérateur de violation de symétrie ». Cet opérateur mesure le scindement des masses des particules d'un même multiplet. Nous proposons un modèle qui explique le mécanisme de la violation de symétrie observée expérimentalement. En d'autres termes, ce modèle met en évidence l'existence de couplages non triviaux de la symétrie espace-temps et d'une symétrie interne.

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### **I. INTRODUCTION**

Mass splitting which is found to occur experimentally within the multiplets of particles can be explained as symmetry-breaking phenomena, i. e. the interaction-Hamiltonian is only approximately invariant under

the internal symmetry under consideration. The noninvariant part of this Hamiltonian then accounts for the mass-splitting.

Another approach for the investigation of the mechanism of symmetry breaking by pure group theoretical means has been advocated by McGlenn [1]. The idea is to explain mass splitting within the context of some higher symmetry group  $G$  which contains both the inhomogeneous Lorentzgroup  $\{(a, \Lambda)\}$  and some internal symmetry group  $G_1$  as analytic subgroups [2]. It turns out, however, that no mass splitting with discrete masses is possible using finite-dimensional Liegroups, since these yield a direct product-coupling  $\{(a, \Lambda)\} \times G_1 = G$ . Also in the case of a large class of infinite parameter Liegroups the aforementioned no-go theorems concerning mass splitting can be shown to hold [3]. Moreover, in the use of such a class of groups one is confronted with a trivial S-matrix  $S = I$  [4]. Thus the following problem arises: are there nontrivial couplings of space-time symmetry  $\{(a, \Lambda)\}$  and internal symmetry  $G_1$ ? And what is then the relationship between these groups?

Our goal is to show that by considering the second cohomology group [5] [6] of  $P_+^\dagger$  ( $P_+^\dagger$  stands for the orthochronous inhomogeneous Lorentzgroup with determinant + 1) over some internal symmetry  $G_1$  and some appropriate selfadjoint symmetry breaking operator which constitutes a new quantum mechanical observable one can give a model which describes a nontrivial coupling of space-time and internal symmetry and which contains, as a special case the direct product coupling of the afore-named authors.

## II. THE BASIC SETUP

As starting point we remark that the direct product  $E = G_1 \times P_+^\dagger$  is a special case of a group extension. Such special extensions are denoted by the following short exact sequence which is said to split:

$$(1) \quad 1 \longrightarrow G_1 \xrightarrow{j} E \xrightarrow[\substack{\Phi \\ u}]{\substack{\Phi \\ u}} P_+^\dagger \longrightarrow 1$$

where  $1 := \{e\}$  is the trivial group which consists of the neutral element alone

$\Phi \circ u = 1_{P_+^\dagger}$  : Identity automorphism of  $P_+^\dagger$

$u$  : Injective mapping (which is a homomorphism)

$G_1$  : Normal subgroup of  $E$  (i. e. one identifies  $j(G_1)$  with  $G_1$  and  $u(P_+^\dagger)$  with  $P_+^\dagger$ ).

An example of such a short exact sequence is

$$(2) \quad 1 \longrightarrow \mathbb{Z}_2 \longrightarrow \tilde{P}_+^\dagger \longrightarrow P_+^\dagger \longrightarrow 1$$

where  $\tilde{P}_+^\dagger$  is the semi-direct product  $\tilde{P}_+^\dagger = \{(a, I)\} \times SL(2, \mathbb{C})$  (where  $\{(a, I)\}$  denotes the translation group and  $SL(2, \mathbb{C})$  the simply connected universal covering group of  $L_+^\dagger$ ).

To begin with, we assume the internal symmetry group  $G_1$  to be abelian (It turns out, as seen below, that this is readily generalized for non-abelian groups). An example of an internal abelian symmetry is the one-dimensional unitary group of phases

$$(3) \quad U_1 = \{ e^{i\varphi} : -\pi < \varphi < +\pi \}$$

With the assumption of abelian internal symmetries, the group  $\text{Ext}(P_+^\dagger, G_1)$  of extensions of  $P_+^\dagger$  by  $G_1$  is isomorphic with the second cohomology group [5]

$$(4) \quad H^2(P_+^\dagger, G_1) \cong Z^2(P_+^\dagger, G_1)/B^2(P_+^\dagger, G_1)$$

If  $u$  is a homomorphism, the extension  $(E, \Phi)$  splits. In general  $u$  will not be a homomorphism and the deviation of  $u$  from a homomorphism is a map

$$(5) \quad f : P_+^\dagger \times P_+^\dagger \longrightarrow G_1$$

called factor set or deviation, with the property:

$$(6) \quad u(L_r)u(L_s) = f(L_r, L_s)u(L_r L_s) \quad L = (a, \Lambda) \in P_+^\dagger$$

From the law of associativity of  $E$ ,  $f$  is identified as a cocycle, i. e.

$$f \in Z^2(P_+^\dagger, G_1).$$

The cohomology class  $\{f\}$  which corresponds to the extension  $(E, \Phi)$  may be regarded as the obstruction against splitting  $(E, \Phi)$ ; the extension splits if and only if the two-dimensional cohomology class  $\{f\}$  equals  $\{0\}$ .

In the case where the internal symmetry  $G_1$  is not abelian, but possesses the properties stated in [6], which are:

$$(7) \quad P_+^\dagger \text{ operates on } \mathcal{C}(G_1) = G_2 \text{ (where } \mathcal{C}(G_1) \text{ denotes the center of } G_1) \text{ and}$$

$$(8) \quad (G_1, \theta) \text{ is extendible (to } G_1 \text{ corresponds the zero-element of the third cohomology group } \{f^3\} = \{0\}). \theta \text{ denotes the homomorphism}$$

$$\theta : P_+^\dagger \rightarrow \text{Aut } G_1 / \text{Int } G_1$$

$\text{Aut } G_1$  : Automorphisms of  $G_1$

$\text{Int } G_1$  : Inner Automorphism of  $G_1$

the set of inequivalent extensions of  $P_+^\uparrow$  by  $G_1$  may then be put into one-one correspondence with  $H^2(P_+^\uparrow, \mathcal{C}(G_1))$ .

Otherwise stated:  $P_+^\uparrow$  is not extendible by any internal symmetry group but only by those groups  $G_1$  which have the properties (7) and (8). Then, by virtue of the one-one correspondence of the sets  $\text{Ext}(P_+^\uparrow, G_1)$  and  $H^2(P_+^\uparrow, \mathcal{C}(G_1))$  the problem of finding all extensions of  $P_+^\uparrow$  by  $G_1$  is analogous to that of abelian groups.

### III. STATEMENT OF THE MASS-SPLITTING THEOREM

In order to treat interacting particles with spin we introduce the concept of vector particles [7]. These are adequately described by vector-valued wave-distributions which are, by definition, elements of the space

$$\mathfrak{H}^{\otimes 1} \subset \mathcal{D}'(\mathbb{R}^4, \mathfrak{H}_{G_1}) = \mathcal{L}(\mathcal{D}(\mathbb{R}^4), \mathfrak{H}_{G_1})$$

(= continuous injection;  $\mathfrak{H}^{\otimes 1}$  is equipped with a finer topology than the one induced by  $\mathcal{D}'$ ).

Here  $\mathfrak{H}^{\otimes 1}$ : denotes the one-particle space which consists of the positive energy solutions of the underlying differential equation.

$\mathfrak{H}_{G_1} = \mathbb{C}^n$ : Representation space of the internal symmetry (for electrons for instance, clearly  $\mathfrak{H}_{G_1} = \mathbb{C}^2$ )

$\mathcal{D}'(\mathbb{R}^4, \mathfrak{H}_{G_1})$ : Space of  $\mathfrak{H}_{G_1}$ -valued distributions on  $\mathbb{R}^4$  (or equivalently:  
 $\mathcal{L}(\mathcal{D}(\mathbb{R}^4), \mathfrak{H}(G_1))$ : space of the continuous linear maps  
 $\mathcal{D}(\mathbb{R}^4) \xrightarrow{\text{into}} \mathfrak{H}_{G_1}$ .)

#### MASS-SPLITTING-THEOREM

Let  $P_+^\uparrow$  be the orthochronous inhomogeneous Lorentzgroup with determinant 1 and  $G_1$  some internal symmetry group which satisfies the properties (7) and (8). We denote by  $(E, \Phi)$  an element of the set of extensions  $\text{Ext}(P_+^\uparrow, G_1)$ . Let  $H^2(P_+^\uparrow, \mathcal{C}(G_1))$  be the second cohomology group of  $P_+^\uparrow$  over  $\mathcal{C}(G_1)$  and let

$$C(f) \in \mathcal{L}\left(\bigoplus_{n=0}^{\infty} \mathfrak{H}^{\otimes 3}, \bigoplus_{n=0}^{\infty} \mathfrak{H}^{\otimes 3}\right)$$

be a selfadjoint symmetry breaking operator which is a function of the cohomology classes  $\{f\} \in H^2(P_+^\uparrow, \mathcal{C}(G_1))$ . The spectrum of the energy-

momentum vector  $P^\mu = \int p^\mu dE(p)$  ( $E(p)$  is a spectral measure) and of the mass-breaking operator  $C(f)$  is supposed to be contained in the forward cone  $\{p = 0\} \cup \bar{V}^+$ . Then, by virtue of the operator  $C(f)$  there exists a well-defined symmetry breaking mechanism which may be characterized as follows: each cohomology class  $\{f\} \neq \{0\}$  accounts for a non-trivial coupling of space-time and internal symmetry, i. e. each element of  $H^2(P_+^\dagger, \mathcal{C}(G_1))$  and therefore each similarity class of extensions

$$(E, \Phi) \in \text{Ext}(P_+^\dagger, G_1)$$

is uniquely related to a mass difference  $\Delta m_{ik} \neq 0$  ( $i, k = 1, \dots, n$ ) of the particles of an E-multiplet. In particular the trivial direct product-coupling  $E = G_1 \times P_+^\dagger$  attached to the class  $\{0\}$  is related to the mass differences  $\Delta m_{ik} = 0$  ( $i, k: 1, 2, \dots, \dim(E\text{-multiplet})$ ).

*Remarks.*

1. — Under our assumptions the internal symmetry-group may be nonabelian or abelian, but for the sake of simplicity of the proof, we shall consider the group  $G_1$  to be abelian:

$$G_1 = \mathcal{C}(G_1)$$

2. — The symmetry- (or mass-) breaking operator  $C(f)$  is to be understood in the following sense:

$$(9) \quad C: H^2(P_+^\dagger, G_1) \xrightarrow{\text{into}} \mathcal{L}\left(\bigoplus_{n=0}^{\infty} \mathfrak{H}^{\otimes n}, \bigoplus_{n=0}^{\infty} \mathfrak{H}^{\otimes n}\right)$$

is an injective linear operator-valued mapping which is constant on each cohomology class. That is, we require:

$$(9') \quad f_1 \not\sim f_2 \text{ mod } B^2(P_+^\dagger, G_1) \Leftrightarrow C(f_1) \neq C(f_2)$$

$\mathcal{L}\left(\bigoplus_{n=0}^{\infty} \mathfrak{H}^{\otimes n}, \bigoplus_{n=0}^{\infty} \mathfrak{H}^{\otimes n}\right)$  stands for the algebra of linear operators on the Fockspace  $\mathcal{F} = \bigoplus \mathfrak{H}^{\otimes n}$  (see remark 12).

3. — As  $C(f)$  is by assumption selfadjoint, this operator is to be considered as a quantummechanical observable whose real eigenvalues shall be some coefficients  $\pi_{ik}$  which are to be identified with the mass differences  $\Delta m_{ik}$ . That is,  $C(f)$  admits the following spectral resolution:

$$(10) \quad C(f) = \sum_{i,k} \pi_{ik}^f P_{ik} = \pi_{12}^f P_{12} + \pi_{13}^f P_{13} + \pi_{23}^f P_{23} + \dots$$

( $P_{ik}$ : projection operators,

$$f \in \{ f \} \in H^2(P_+^1, G_1))$$

and the still unspecified spectrum of  $C(f)$  is given by

$$(11) \quad \text{sp } C(f) = \{ \pi_{12}^f, \pi_{13}^f, \pi_{23}^f, \dots \}$$

4. —  $C(f)$  is considered to have been assigned to the commutators of the Liealgebras  $\mathcal{G}(P_+^1)$  and  $\mathcal{G}(G_1)$  respectively in the following way:

$$(12) \quad \begin{aligned} a) \quad [P^\mu, Q_k] = 0 &\rightarrow C(f) = 0 \leftrightarrow \Delta m_{ik} = 0 \quad \forall i, k \\ b) \quad [P^\mu, Q_k] \neq 0 &\rightarrow C(f) \neq 0 \leftrightarrow \Delta m_{ik} \neq 0 \end{aligned}$$

i. e. the lack of commutation of the selfadjoint infinitesimal symmetry-interaction generators  $Q_k$ ,  $k = 1, \dots, \dim G_1$  with the Lorentz translation-generators is related to a nonvanishing mass-breaking-operator.

5. — The proof of the mass-splitting-Theorem uses the Theorem of Kernels (Nuclear Theorem) of Schwartz which states:

$$(13) \quad \mathcal{L}(\mathcal{D}_x, \mathcal{D}'_y) = \mathcal{D}'_{xy}$$

algebraically and topologically. That is: the topological vector space  $\mathcal{L}(\mathcal{D}_x, \mathcal{D}'_y)$  of continuous linear maps from  $\mathcal{D}(V) = \mathcal{D}_x$  into  $\mathcal{D}'(W) = \mathcal{D}'_y$ , with the topology of bounded convergence is canonically isomorphic to the topological vector space  $\mathcal{D}'(V \times W) = \mathcal{D}'_{xy}$ . Here  $V$  and  $W$  are two  $C^\infty$ -manifolds. Then the following proposition is related to the Nuclear Theorem:

PROPOSITION [7]. — There is a one-to-one correspondence between the Hilbertspaces  $\mathfrak{H} \subset \mathcal{D}$  and the positive Kernel-distributions:

$$(14) \quad \mathfrak{H} \leftrightarrow K_{xy} \in \mathcal{D}'_{xy}$$

#### PROOF OF THE MASS-SPLITTING-THEOREM

Let  $\mathfrak{H}^{\otimes 1}$  be the single particle Hilbertspace (see remark 12). In the case of a nontrivial multiplet-structure (i. e. where particles within a multiplet have different masses)  $\mathfrak{H}^{\otimes 1}$  has necessarily to be written as the direct sum of the subspaces  $\mathfrak{H}_1^{\otimes 1}, \mathfrak{H}_2^{\otimes 1}, \dots, \mathfrak{H}_n^{\otimes 1}$ :

$$(15) \quad \mathfrak{H}^{\otimes 1} = \bigoplus_{j=1}^n \mathfrak{H}_j^{\otimes 1}$$

where

$$(16) \quad \mathfrak{H}_i^{\otimes 1} := \mathfrak{H}_{m_i}^{\otimes 1} := \mathfrak{H}(m_i, s)$$

( $s$ : spin of the  $n$  particles of mass  $m_i$ ).

The spaces  $(\mathfrak{H}(m_i, s))_{1 \leq i \leq n}$  carry inequivalent irreducible representations  $\{U_i(a, \Lambda)\}_i$  of the group  $P_+^\uparrow$  (see remark 13) such that

$$(17) \quad U(a, \Lambda) = \bigoplus_{j=1}^n \mathcal{U}_j(a, \Lambda) \quad \text{in } \mathfrak{H}^{\otimes 1}$$

Then, by means of the Nuclear Theorem and the following remark 9 we assign to each  $\mathfrak{H}(m_i, s)$  a positive Kernel-distribution

$$(18) \quad K_i \longleftarrow \mathfrak{H}(m_i, s) \quad i = 1, \dots, n$$

The associated kernels are, by the assumptions made in the Theorem, clearly elementary and so are these spaces (see definition in remark 10). This means that their Fourier-transformed positive measures on

$$\mathbb{R}^{4'} = \left\{ p/(p, x) = \sum_{\mu=0}^3 p_\mu x^\mu; \quad x \in \mathbb{R}^4 \right\}$$

(= topological locally compact dual space of  $\mathbb{R}^4$ ) are extremal, i. e. have their support located in one  $\{(a, \Lambda)\}$ -orbit and are proportional to  $\delta$ , that is, that they are of the form

$$(19) \quad \mu = c\delta(p^2 - m^2)$$

Thus by (18) and (19) clearly a well defined mass  $m_i$  is assigned to each elementary Hilbertspace  $\mathfrak{H}(m_i, s)$  and one is therefore led to analyze the following cases:

$$(20a) \quad m_1 = m_2 = \dots = m_n, \quad \text{i. e. } \Delta m_{ij} = 0 = m_i - m_j$$

$$(20b) \quad m_1 \neq m_2 \neq \dots \neq m_n, \quad \text{i. e. } \Delta m_{ij} = m_i - m_j \neq 0$$

$$\Rightarrow \quad (K_i - K_j)\varphi = \psi^{ij} \in \mathcal{D}'(\mathbb{R}^4, \mathfrak{H}_{G_1})$$

From (20a) and (20b) we have the following correspondence:

$$(21) \quad K_{ij} \longleftarrow \Delta m_{ij} \quad K_{ij} := K_i - K_j$$

We show now that the classification of the mass differences according to (20a) and (20b) is uniquely related to the cohomology classes  $\in H^2(P_+^\uparrow, G_1)$ .



We first relate the elements  $f \in Z^2(P_+^1, G_1)$  to the wave-distributions  $\psi^{ij}$  by the following definition:

$$(22) \quad \mathcal{U}(f(L_r, L_s))\Phi_0 = \psi_{(L_r, L_s)}^{ij}(\varphi_0) = \int_{\mathbb{R}^4} \psi^{ij}(L_r, L_s, x)\varphi_0(x)dx = \Phi \in \mathfrak{H}_{G_1}$$

where  $\Phi_0 \in \mathfrak{H}_{G_1}$ : fixed

$\varphi_0 \in \mathcal{D}(\mathbb{R}^4)$ : fixed

$x \in \mathbb{R}^4$

$\psi_{(L_r, L_s)}^{ij}(\varphi_0)$ : vectorvalued distribution  $\in \mathfrak{H}_{G_1}$  i. e.  $\psi_{(L_r, L_s)}^{ij} \in \mathcal{D}'(\mathbb{R}^4, \mathfrak{H}_{G_1})$ .

Then the following Lemma holds:

LEMMA 1. — Consider the continuous unitary representation

$$(23) \quad g \longmapsto U(g) = U(f(L_r, L_s)); \quad f \in Z^2(P_+^1, G_1)$$

characterized by definition (22). Let the restriction  $U' := U/G_1$  of  $U$  to the subgroup  $G'_1$  of  $G_1$ ,

$$(24) \quad G'_1 = \{ b(L_r, L_s) : b \in B^2(P_+^1, G_1); \quad (L_r, L_s) \in P_+^1 \times P_+^1 \}$$

be such that the sub-representations of  $U'$  are defined by a family  $(\mathfrak{H}_{G'_1}^{ij})$  of pairwise orthogonal cyclic subspaces of  $\mathfrak{H}_{G'_1}$ :

$$(25) \quad \overline{\mathfrak{H}_{G'_1}^{ij}} = \{ \psi_{(L_r, L_s)}^{ij}(\varphi_0) = \mathcal{U}(b(L_r, L_s))\psi_{0(L_r, L_s)}^{ij}(\varphi_0) : \psi^{ij} \in K_{ij}\mathcal{D} \subset \mathcal{D}'(\mathbb{R}^4, \mathfrak{H}_{G_1}); \varphi_0 \in \mathcal{D}, \varphi_0 : \text{fixed} \}$$

where  $\psi_{0(L_r, L_s)}^{ij}(\varphi_0)$  denotes the cyclic vector.

Then, to any given mass-difference  $\Delta m \neq 0$  corresponds one and only one cohomology class  $\{ f \}$ .

Remark 6. — As  $G_1$  is a group by assumption, and according to (5) we have:

$$f(L_r, L_s) \in G_1 \quad \forall f \in Z^2(P_+^1, G_1), \quad (L_r, L_s) \in P_+^1 \times P_+^1$$

one can define a binary operation in  $G_1$  as follows:

$$(26) \quad f_i(L_r, L_s) + f_j(L'_r, L'_s) = (f_i + f_j)(L_r L'_r, L_s L'_s) \\ = f_k(L''_r, L''_s), \quad L_r, L_s \text{ fixed}$$

where  $f_k \in Z^2(P_+^1, G_1)$ , since  $Z^2(P_+^1, G_1)$  is a group. Endowed with the binary operation (26),  $G'_1$  becomes a group, as  $B^2(P_+^1, G_1) \subset Z^2(P_+^1, G_1)$  is a group.

*Proof of Lemma 1.* — The cyclicity of the representation  $U/G'_1$  with respect to  $\psi_0^{ij}(\varphi_0) \in \mathfrak{H}_{G'_1}^{ij}$  entails the following property

$$(27) \quad \begin{matrix} \forall z \in Z^2(P^+_+, G_1) \\ z \notin B^2(P^+_+, G_1) \end{matrix} \quad \text{if } \psi'_{(L_r, L_s)}(\varphi_0) = \mathcal{U}(z(L_r, L_s))\psi_{0(L_r, L_s)}^{ij}(\varphi_0) \quad \text{holds}$$

then

$$\psi' \notin K_{ij}\mathcal{D} \quad (Z^2(P^+_+, G_1): 2\text{-dimensional cocycles})$$

Obviously the statement of cyclicity implies:

$$\mathfrak{H}_{G'_1} = \bigoplus_{ij} \mathfrak{H}_{G'_1}^{ij}$$

Therefore, one infers from the definition (25):

$$(28) \quad K_{ij}\mathcal{D} \cap K_{im}\mathcal{D} = \{0\}$$

which means the disjointness of the Kernel-distributions associated with different mass differences  $\Delta m_{ij}$  and  $\Delta m_{im}$ .

Let  $K_{ij}$  satisfy (21) and let  $\psi_e^{ij}$  and  $\psi_m^{ij}$  be two elements of  $K_{ij}\mathcal{D}$  which are related by (22) to  $f$  and  $f' \in Z^2(P^+_+, G_1)$  respectively. We have to show that

$$f \sim f' \text{ mod } B^2(P^+_+, G_1)$$

Indeed, the assumption of cyclicity yields:

$$\begin{aligned} \psi_e^{ij}_{(L_r, L_s)}(\varphi_0) &= \mathcal{U}(b_e(L_r, L_s))\psi_{0(L_r, L_s)}^{ij}(\varphi_0) \\ \psi_m^{ij}_{(L_r, L_s)}(\varphi_0) &= \mathcal{U}(b_m(L_r, L_s))\psi_{0(L_r, L_s)}^{ij}(\varphi_0) \end{aligned}$$

and definition (22) implies

$$\begin{aligned} \psi_e^{ij}_{(L_r, L_s)}(\varphi_0) &= \mathcal{U}(f(L_r, L_s))\Phi_0 \quad ; \quad \psi_m^{ij}_{(L_r, L_s)}(\varphi_0) = \mathcal{U}(f'(L_r, L_s))\Phi_0 \\ \psi_{0(L_r, L_s)}^{ij}(\varphi_0) &= \mathcal{U}(f_0(L_r, L_s))\Phi_0 \\ \Rightarrow \psi_e^{ij}_{(L_r, L_s)}(\varphi_0) &= \mathcal{U}(b_e(L_r, L_s))\mathcal{U}(f_0(L_r, L_s))\Phi_0 = \mathcal{U}[(b_e + f_0)(L_r, L_s)]\Phi_0 \\ \psi_m^{ij}_{(L_r, L_s)}(\varphi_0) &= \mathcal{U}(b_m(L_r, L_s))\mathcal{U}(f_0(L_r, L_s))\Phi_0 = \mathcal{U}[(b_m + f_0)(L_r, L_s)]\Phi_0 \end{aligned}$$

that is:

$$\begin{aligned} f &= b_e + f_0 \\ f' &= b_m + f_0 \end{aligned} \Rightarrow f \sim f' \in \{f_0\}$$

Conversely, if  $f \not\sim f' \text{ mod } B^2(P^+_+, G_1)$ , then  $\psi$  and  $\psi'$  are elements of  $K_{ij}\mathcal{D}$  and  $K_{im}\mathcal{D}$  respectively, i. e. they are related to different mass-differences. In fact:  $f \not\sim f' \text{ mod } B^2(P^+_+, G_1)$  means:

$$\exists z \in Z^2(P^+_+, G_1), \quad z \notin B^2(P^+_+, G_1): \quad f' = z + f.$$

Let  $\psi = \psi_e^{ij} \in K_{ij}\mathcal{D} : \psi_{\alpha(L_r, L_s)}^{ij}(\varphi_0) = \mathcal{U}(f(L_r, L_s))\Phi_0$   
 $= \mathcal{U}(b_e(L_r, L_s))\psi_{0(L_r, L_s)}^{ij}(\varphi_0)$

and

$$\begin{aligned} \psi'_{(L_r, L_s)}(\varphi_0) &= \mathcal{U}(f'(L_r, L_s))\Phi_0 = \mathcal{U}[(z + f)(L_r, L_s)]\Phi_0 \\ &= \mathcal{U}(z(L_r, L_s))\mathcal{U}(f(L_r, L_s))\Phi_0 = \mathcal{U}(z(L_r, L_s))\psi_{(L_r, L_s)}^{ij}(\varphi_0) \\ &= \mathcal{U}(z(L_r, L_s))\mathcal{U}(b_e(L_r, L_s))\psi_{0(L_r, L_s)}^{ij}(\varphi_0) \\ &= \mathcal{U}[(z + b_e)(L_r, L_s)]\psi_{0(L_r, L_s)}^{ij}(\varphi_0) \end{aligned}$$

but

$$z + b_e \notin B^2(\mathbf{P}_+^1, G_1) \Rightarrow \psi' \notin K_{ij}\mathcal{D}$$

Moreover one easily shows Lemma 2) to hold:

LEMMA 2. — If  $f \in Z^2(\mathbf{P}_+^1, G_1)$  corresponds to  $\psi^{ij} \in K_{ij}\mathcal{D}$  and  $f'$  corresponds to  $\psi^{im} \in K_{im}\mathcal{D}$  with  $K_{ij}\mathcal{D} \cap K_{im}\mathcal{D} = \{0\}$ , then  $f$  and  $f'$  are inequivalent mod  $B^2(\mathbf{P}_+^1, G_1)$  i. e.

$$f \not\sim f' \text{ mod } B^2(\mathbf{P}_+^1, G_1)$$

*Proof.* — By definition (22) and the assumption of cyclicity one infers:

$$\begin{aligned} \psi_{(L_r, L_s)}^{im}(\varphi_0) &= \mathcal{U}(z(L_r, L_s))\mathcal{U}(f_0(L_r, L_s))\Phi_0 \\ &= \mathcal{U}[z(L_r, L_s) + f_0(L_r, L_s)]\Phi_0 \end{aligned}$$

and

$$\psi_{(L_r, L_s)}^{im}(\varphi_0) = \mathcal{U}(f'(L_r, L_s))\Phi_0$$

$\Rightarrow$

$$(29) \quad f' = z + f_0, \quad z \notin B^2(\mathbf{P}_+^1, G_1) \Rightarrow f' \not\sim f_0 \text{ mod } B^2(\mathbf{P}_+^1, G_1)$$

Moreover:

$$\begin{aligned} \psi_{(L_r, L_s)}^{ij}(\varphi_0) &= \mathcal{U}(b(L_r, L_s))\mathcal{U}(f_0(L_r, L_s))\Phi_0 \\ &= \mathcal{U}[b(L_r, L_s) + f_0(L_r, L_s)]\Phi_0 \end{aligned}$$

and

$$\psi_{(L_r, L_s)}^{ij}(\varphi_0) = \mathcal{U}(f(L_r, L_s))\Phi_0$$

$\Rightarrow$

$$(30) \quad f = b + f_0 \Rightarrow f \sim f_0 \text{ mod } B^2(\mathbf{P}_+^1, G_1)$$

By (29) and (30) it follows:  $f \not\sim f'$  mod  $B^2(\mathbf{P}_+^1, G_1)$ . This achieves the proof of Lemma 2.

According to remark 3) and the preceding lemmas, we have:

$$\{f_\alpha\} \leftrightarrow \Delta^\alpha m_{er} \quad (e, r: \text{fixed})$$

and

$$\text{sp } C(f_\alpha) = \{\pi_{ij}^{f_\alpha}\} \quad \text{with} \quad \begin{cases} \pi_{er}^{f_\alpha} = \Delta^\alpha m_{er} \\ \pi_{ij}^{f_\alpha} = 0 \quad i \neq e, j \neq r \end{cases}$$

that is, the mass-breaking operator  $C(f)$  is defined in such a way that its expectation value becomes:

$$(31) \quad (\xi, C(f_\alpha)\xi) = \Delta^\alpha m$$

$\xi$ : eigenstate of  $C(f_\alpha)$ .

The spectrum of  $C(f)$  then turns out to be:

$$(32) \quad \begin{aligned} \text{sp } C(f_\alpha) &= \{ 0, 0 \dots 0, \Delta m_{lr}^\alpha, 0 \dots 0 \} \\ \text{sp } C(f_\beta) &= \{ 0, \Delta m_{ij}^\beta, 0 \dots 0, 0 \dots \} \end{aligned}$$

where

$$f_\alpha \not\sim f_\beta \text{ mod } \mathbf{B}^2(\mathbf{P}_+^\dagger, G_1)$$

Hence, we obtain the following

**COROLLARY.** — The mass differences  $\Delta^0 m_{ik} = 0$  are associated with the cohomology class

$$\{ f_0 \} = \{ 0 \}$$

*Proof.* — As  $C(f)$  is constant on each class  $\{ f \} \in \mathbf{H}^2(\mathbf{P}_+^\dagger, G_1)$  this entails

$$\begin{aligned} C(f) &= C(f') = C(f + b) & f &\sim f' \text{ mod } \mathbf{B}^2(\mathbf{P}_+^\dagger, G_1) \\ C(f + b) &= C(f) + C(b) & \text{as } C &\text{ is a linear map} \end{aligned}$$

$\Rightarrow$

$$(33) \quad C(b) = 0 \quad \forall b \in \mathbf{B}^2(\mathbf{P}_+^\dagger, G_1) = \{ 0 \}$$

On the other hand we have (remark 4)

$$[\mathbf{P}^\mu, \mathbf{Q}_k] = 0 \Leftrightarrow \Delta m_{ik} = 0 \quad \forall i, k \leftrightarrow C(f_0) = 0$$

From (9') and (33) it then follows:  $f_0 \in \mathbf{B}^2(\mathbf{P}_+^\dagger, G_1)$

$$f_0 \sim 0 \text{ mod } \mathbf{B}^2(\mathbf{P}_+^\dagger, G_1)$$

That is:  $\Delta m_{ik} = 0, \forall i, k$  belongs to the cohomology class

$$\{ 0 \} = \{ 0 + \mathbf{B}^2(\mathbf{P}_+^\dagger, G_1) \}$$

*Remark 8.* — The corollary and remark 2) yield:

$$\Delta m_{ij} \neq 0 \Leftrightarrow \{ f \} \neq \{ 0 \}$$

that is, to non-vanishing mass-differences can only correspond non-zero cohomology classes.

## DISCUSSION

Consider the unitary symmetries  $SU(n)$ . These simple-compact groups do not fit in our model, since following Michel [10] the inequivalent central extensions of the inhomogeneous Lorentz group  $P$  by finite or compact simple Lie groups are of the form

$$(40) \quad E_\alpha = G \times \tilde{P}/\mathbb{Z}_2(\alpha)$$

where  $\tilde{P}$  denotes the covering group of  $P$  and  $\mathbb{Z}_2(\alpha)$  a two element group. For  $\alpha = 1$ ,  $E_1 = E$  is the direct product  $G \times P$ . But all these inequivalent extensions (40) are isomorphic to the direct product and therefore not of any interest in connection with our proposed mass-splitting model. Galindo has shown [12] that the existence of essential noncentral extensions of  $P$  by some internal analytic group implies that this be necessarily nonsemisimple and noncompact. Only in this case one finds, within our mass-splitting model, physically interesting mixings of the Lorentz- and internal symmetry groups.

*Final remarks:*

9. — A Kernel-distribution  $K_{xy}$  on  $\mathbb{R}^4 \times \mathbb{R}^4$  is said to be positive ( $K_{xy} > 0$ ) if it satisfies  $\forall \varphi \in \mathcal{D}$  the condition

$$(42) \quad K_{xy}(\varphi(x)\bar{\varphi}(y)) \geq 0$$

Clearly only if property (42) holds one can assign a Hilbertspace to a Kernel-distribution, as  $\forall \Phi = K\varphi \in \mathfrak{H}$

$$\begin{aligned} (\Phi, \Phi) &= (K\varphi, K\varphi) \\ &= \bar{K}_{xy}(\varphi(x)\bar{\varphi}(y)) \geq 0 \end{aligned}$$

10. — A necessary and sufficient condition for a particle to be elementary is that its associated Kernel-distribution be elementary. By definition  $K_{xy}$  is called  $G$ -elementary (where  $G$  is the structure group under consideration) if for any other  $G$ -invariant positive Kernel  $K_1$  the following condition holds:

$$(43) \quad K_1 \leq K, \text{ i. e. } K_1 = \lambda K \text{ for some } 0 \leq \lambda \leq 1$$

This property (43) of elementarity means that the Fourier-transformed measure  $\mu$  will be extremal, i. e.

$$a) \quad \mu = c\delta(p^2 - m^2)$$

and

b) its support lies in one masshyperboloid i. e.

$$\text{supp } \mu \subset \bar{V}_m^+ = \{ p \in \mathbb{R}^{4'} : p^2 = m^2, p^0 \geq 0 \}$$

Clearly a measure of this type refers to stable elementary particles, i. e. discrete eigenvalues of the mass operator  $M^2$ .

11. — The Kernel-distributions  $K_{ij}$  are obviously not elementary.

12. — The local coupled quantized fields which are associated with our symmetry-breaking model are characterized by the following algorithm [8] [9], to which belong

a) The separable one-particle Hilbertspace  $\mathfrak{H}^{\otimes 1} = \mathfrak{H}$ .

b) The associated « Green-function »  $K_{xy} \in \mathcal{D}'_{xy}$  (according to the multiplet structure of  $\mathfrak{H}^{\otimes 1}$  given by formula (15) this Green-function is given by the following Kernel-distribution:

$$(44) \quad K = K_1 + K_2 + \dots + K_n \longleftrightarrow \bigoplus_{k=1}^n \mathfrak{H}_k^{\otimes 1} = \mathfrak{H}^{\otimes 1}$$

c) The Fockspace  $\bigoplus_{n=0}^{\infty} \mathfrak{H}^{\otimes n}$  over  $\mathfrak{H}^{\otimes 1}$

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathfrak{H}^{\otimes n} = \left\{ \Phi = (\varphi_0, \varphi_1, \dots, \varphi_n, \dots) \mid \varphi_k \in \mathfrak{H}^{\otimes k} : \sum_{k=0}^{\infty} \|\varphi_k\|^2 < \infty \right\}$$

where:  $\mathfrak{H}^{\otimes 0} = \langle \Omega \rangle = \{ c\Omega \mid c \in \mathbb{C} \}$      $\Omega$ : vacuum  
 $\mathfrak{H}^{\otimes n} = \underbrace{\mathfrak{H} \otimes \dots \otimes \mathfrak{H}}_n$

Then the quantized fields which operate on  $\mathcal{F}$  as linear operators are defined by

$$(45) \quad A(f, \alpha) = A^+(f, \alpha) + A^-(f, \alpha) = a^+(Kf, \alpha) + a^-(Kf, \alpha)$$

Where the Kernel-distribution  $K$  is given by property (44) and  $\alpha$  characterizes the transformation properties of the particles with respect to the internal symmetry group  $G_1$ .

Regarding the operator-valued fields (45) it must be stressed that there is not necessarily any connection between the number of fields in a theory and the number of particles described by this theory. That is, with the exception of very special cases, there exists no simple correlation between the fields and the number of particles.

13. — In the limit of exact symmetry, that is by switching off a part of the interaction, the following situation occurs:

$$\mathfrak{H}^{\otimes 1} = L^2(\mathbb{R}^{4'}, \mu) \otimes \mathfrak{H}_{G_1}$$

Then  $m_i = m_k, \forall i, k$  and the representations are equivalent, i. e.:

$$\{ U_i(a, \Lambda) \} \sim \{ U_k(a, \Lambda) \}$$

## REFERENCES

- [1] W. D. MCGLINN, *Phys. Rev. Letters*, t. **12**, 1964, p. 16.
- [2] G. C. HEGERFELDT and J. HENNIG, Preprint University Marburg and references cited in there.
- [3] C. v. WESTENHOLZ, *Act. Phys. Austr.*, t. **31**, 1970, p. 54-60.
- [4] C. v. WESTENHOLZ, *Act. Phys. Austr.*, t. **30**, 1969, p. 260-270.
- [5] S. EILENBERG, *A. M. S. Bull.*, t. **55**, 1949, p. 3-37.
- [6] S. EILENBERG and S. MACLANE, *Ann. Math.*, t. **48**, 1947, p. 326.
- [7] L. SCHWARTZ, *Tech. Rep.*, Nr. 7, *Contr.*, Nr. 222 (60). Office of Naval Research.
- [8] D. KASTLER, *Introduction à l'électrodynamique quantique*. Dunod, Paris, 1961.
- [9] C. v. WESTENHOLZ, BMWF-FB K 69-30. Zentralstelle für Atomkernenergie-Dokumentation (ZAED).
- [10] L. MICHEL, *Nucl. Phys.*, t. **57**, 1964, n° 1, p. 356-386.
- [11] G. W. MACKEY, *The theory of group representations*. Dept. of Mathematics, University of Chicago, 1955.
- [12] A. GALINDO, *J. Math. Phys.*, t. **8**, n° 4, 1967, p. 768.

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