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Homogeneous Fibered and Quantizable Dynamical Systems

by

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ABSTRACT. — Fibered dynamical systems whose total spaces are homogeneous integral cohomology spheres and homogeneous homotopy spheres are classified. In the latter case and in the case where the total space is an odd dimensional homogeneous simply connected Finslerian B_x-manifold, the total space is shown to be a standard sphere and the fibered dynamical system is then a quantizable dynamical system.

RÉSUMÉ. — On classe les systèmes dynamiques fibrés dont l'espace total est une sphère cohomologique, homogène et intégrale et ceux dont l'espace total est une sphère de homotopie homogène. Dans ce dernier cas et dans celui où l'espace total est une B_x-variété finslérienne homogène et simplement connexe de dimension impaire, on démontre que l'espace total est une sphère usuelle, et que le système dynamique fibré est donc un système dynamique quantifiable.

Recently in [6] dynamical systems topologically similar to harmonic oscillators have been examined. These can be described by pairs (E, G), where $G = S^1$ is the circle group acting differentiably and freely on a differentiable manifold E (*cf.* [6]). Then $\xi : G \to E \to M = \frac{E}{G}$ is a principal circle bundle and ξ or (E, G) is called a *fibered dynamical system* (FDS).

In [6], Prop. 2.2, 2.3 and 2.4 stated that if E is homotopically equivalent to the total space S^{2n+1} of the harmonic oscillator, then the

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FDS ξ is homotopically equivalent to the harmonic oscillator, that is the quantizable dynamical system (QDS), $G \to S^{2n+1} \to \mathbf{C}P(n)$; *cf.* [7]. Here S^{2n+1} is the $2n+1$ dimensional sphere and $\mathbf{C}P(n)$ is the n dimensional complex projective space. And conversely, if M is homotopically equivalent to the phase space $\mathbf{C}P(n)$ of the harmonic oscillator, then there is a QDS $\xi : G \to \Sigma^{2n+1} \to M$ over M and Σ is a homotopy sphere. In this note we examine the case when the total space of the FDS is a homogeneous manifold. That is, E is a coset space $\frac{K}{H}$ where K is a compact connected Lie group and H is a closed (not necessarily connected) subgroup. Following the work [6], we treat only two related cases; namely (*a*) E is an odd dimensional homogeneous integral cohomology sphere; and (*b*) E is an odd dimensional homogeneous homotopy sphere. Of course case (*b*) is equivalent to saying E is a simply connected homogeneous integral cohomology (or homology) sphere, so a special case of (*a*).

First we state a partial generalization of [6], Prop. 2.3 and 2.5 in the following result due to J. C. Su [11]. Let $\dim_{\mathbf{Z}}(E)$ denote the cohomology dimension of manifold E over the integers $\mathbf{Z}$ (*cf.* [4]). Let M be an integral cohomology projective space. Then there is a natural map $f : M \to \mathbf{C}P(n)$ (*see* [11], § 6) such that, as in the proof of [6], Prop. 2.5, there is a FDS over M, namely the principal S^1 bundle ξ induced by f. And by [6], Prop. 2.3, ξ is a QDS. Furthermore if $c \in H^2(\mathbf{C}P(n); \mathbf{Z})$ is the generator and $\alpha \in H^2(M; \mathbf{Z})$ is the generator, then $f^*(c) = \alpha$ and $p^* : H^2(M, \mathbf{Z}) \to H^2(E; \mathbf{Z})$ maps α into zero — i. e. p^* is trivial. Using the results of Su [11], § 5, we state.

Proposition 1.1. — *If* (E, G) *is a FDS and* E *is an integral* $2n+1$ *dimensional cohomology sphere with* $\dim_{\mathbf{Z}}(E) < \infty$, *then* $M = \frac{E}{G}$ *is an integral cohomology projective space. And conversely, if* M *is an integral cohomology projective space with* $\dim_{\mathbf{Z}}(M) < \infty$, [*or if* $\xi : G \to E \to M$ *is a FDS with* M *as stated and* $p^* : H^2(M; \mathbf{Z}) \to H^2(E; \mathbf{Z})$ *trivial*], *then there is as QDS over* M, $G \to E \to M$, *and* E *is an odd dimensional integral cohomology sphere with* $\dim_{\mathbf{Z}}(E) < \infty$ (*resp.* E *in the FDS* ξ *is as stated*).

Borel in [3], § 4.6 has completely classified all homogeneous homotopy spheres E. (Confer also the work of Montgomery-Samelson [9], Borel [1], [2], Wang [2], Matsushima [8], and Poncet [10].) Namely $E^n = \frac{SO(n+1)}{SO(n)} = S^n$ and also $E^6 = \frac{G_2}{SU(3)} = S^6$, $E^7 = \frac{\mathrm{Spin}(7)}{G^2} = S^7$ and $E^{15} = \frac{\mathrm{Spin}(9)}{\mathrm{Spin}(7)} = S^{15}$. (For notations, *cf.* [13].) Thus we have

PROPOSITION 1.2. — *If* (E, G) *is a FDS where* E *is an odd dimensional homogeneous homotopy sphere, then* E *must be a standard sphere* S^{2n+1}; *and so* (E, G) *is a QDS.*

Bredon in [5] has completely classified the homogeneous integral cohomology spheres E. Namely, E is either simply connected (so covered in Borel's work), or a circle S^1, or of the form $\frac{SO(3)}{I} \approx \frac{Sp(1)}{I'}$ where I and I′ are the icosahedral subgroups of SO (3) and Sp (1) respectively (*cf.* [13]). Note that the integral cohomology sphere $\frac{SO(3)}{I}$ is not simply connected, so it is not a homotopy sphere nor a sphere. Thus from Borel's results, E is either a standard sphere or $\frac{SO(3)}{I}$. Summarizing, we have

PROPOSITION 1.3. — *If* (E, G) *is a FDS where* E *is a homogeneous integral cohomology sphere, then* E *must be a standard sphere* S^n *or the nonsimply connected space* $\frac{SO(3)}{I}$.

In [6], § 3 we examined FDS's whose total spaces have another topological property of the total space S^{2n+1} of the harmonic oscillator; namely the property of being a Finslerian B_x-manifold. We showed that the only possible candidates for QDS's whose total spaces are odd dimensional simply connected Finslerian B_x-manifolds are the odd dimensional homotopy spheres. Thus from Proposition 1.2 we have immediately.

PROPOSITION 1.4. — *If* (E, G) *is a FDS whose total space is an odd dimensional simply connected homogeneous Finslerian* B_x*-manifold, then* E *must be a standard sphere* S^{2n+1}; *and so* (E, G) *is a QDS, namely the harmonic oscillator.*

In connection with [6], we note that Yang [14] has shown that the orbit space M of a FDS (E, G) possesses a natural triangulation.

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