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Null Hypersurfaces in General Relativity Theory

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ABSTRACT. — In this article a procedure is discussed for obtaining solutions to Einstein's field equations in a region of space-time bounded by a null hypersurface on which hold the junction conditions proposed by O'Brien and Synge. Other formulations of null hypersurface conditions are shown to be inappropriate.

SOMMAIRE. — Dans cet article une méthode est proposée pour obtenir des solutions des équations d'Einstein dans une région limitée par une hypersurface nulle où sont obtenu les conditions de raccordement proposées par O'Brien et Synge. D'autres formulations des conditions sont montrées d'être inappropriées.

1. INTRODUCTION

The junction conditions which must hold at a hypersurface, S, of discontinuity in General Relativity have been formulated in several ways, but these formulations have been shown [9] to be equivalent in the case when S is not null. The purpose here is to discuss the situation when S is a null hypersurface.

Suppose that ds^2 is the metric of four-dimensional Riemannian space-time defined by (¹) :

(1) $$ds^2 = g_{ij}\, dx^i\, dx^j,$$

(¹) Latin indices $i, j, \ldots$ take values in the range 1-4 and Greek indices $\alpha, \beta, \ldots$ in the range 1-3. The convention of summation over repeated indices is used.

where g_{ij}, g^{ij} denote the covariant, contravariant components of the metric tensor, and that a three-dimensional null hypersurface, S, is defined by the equation

$$x^4 - a = 0, \tag{2}$$

where a is a constant. The condition that S is null may be expressed by

$$g^{44} = 0. \tag{3}$$

The components of the covariant normal to S and its covariant derivatives are denoted by N_i and $N_{i|j}$, respectively.

First consider what restrictions are placed on the metric tensor, g_{ij}, and its partial derivatives, $\frac{\partial g_{ij}}{\partial x^k}$, by the condition that the first and second fundamental forms [5], defined by

$$ds^2 = g_{ij}\, dx^i\, dx^j,$$
$$\Phi = N_{i|j}\, dx^i\, dx^j,$$

should be continuous at S for arbitrary dx^i consistent with the condition

$$N_i\, dx^i = 0. \tag{4}$$

When S is defined by equation (2) N_i has components (0, 0, 0, 1) so that condition (4) states $dx^4 = 0$, and the continuity of the forms ds^2 and Φ is equivalent to the continuity of the quantities $g_{\alpha\beta}$ and $N_{\alpha|\beta}$ ($\alpha, \beta = 1, 2, 3$). Now the quantities $N_{\alpha|\beta}$ are defined by the equations

$$N_{\alpha|\beta} = \frac{\partial N_\alpha}{\partial x^\beta} - \Gamma^i_{\alpha\beta}\, N_i \qquad (\alpha, \beta = 1, 2, 3),$$

where Γ^i_{jk} are the Christoffel symbols of the second kind, and consequently they are continuous at S if and only if the quantities $\Gamma^4_{\alpha\beta}$, defined by

$$\Gamma^4_{\alpha\beta} = g^{4i}\left(\frac{\partial g_{\alpha i}}{\partial x^\beta} + \frac{\partial g_{\beta i}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^i}\right),$$

are continuous at S. Since g^{44} is zero it follows that the continuity at S of $\Gamma^4_{\alpha\beta}$ (and hence of $N_{\alpha|\beta}$), is assured provided $g_{\alpha\beta}$ are continuous at S.

In other words, the first and second fundamental forms are continuous at S if and only if the components, $g_{\alpha\beta}$ ($\alpha, \beta = 1, 2, 3$), are continuous at S. These appear to be extremely weak conditions on the metric tensor and are not investigated further.

O'Brien and Synge [6] suggested that all the components, g_{ij}, of the metric tensor and the following four combinations of derivatives of the metric tensor

$$g^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial x^4}, \qquad g^{4\alpha}\frac{\partial g_{\alpha\beta}}{\partial x^4}, \tag{5}$$

($\alpha, \beta = 1, 2, 3$), should be continuous across a null hypersurface defined by equation (2). It is shown, in section 2, that these conditions impose restrictions on the energy-momentum tensor, and in section 3, that they give sufficient data on S to admit solutions to Einstein's field equations.

2. ENERGY-MOMENTUM TENSOR AT NULL HYPERSURFACES

It is shown in this section that if the metric tensor, g_{ij}, and the expressions (5) are continuous at S then the components, E_i^4 ($i = 1\text{-}4$), of the Einstein tensor are continuous at S, and consequently, through the field equations, the components, T_i^4 ($i = 1\text{-}4$), of the energy-momentum tensor are also continuous at S.

As a simplification, and without loss of generality, the further condition that the partial derivatives $\frac{\partial g_{4i}}{\partial x^4}$ ($i = 1\text{-}4$), are continuous at S may be assumed since a coordinate transformation, which does not alter any tensor quantities at S, may be introduced to impose this condition even when S is null [9].

It is worth pointing out that the continuity of the four combinations of partial derivatives (5) still allows some discontinuities in the six derivatives $\frac{\partial g_{\alpha\beta}}{\partial x^4}$ ($\alpha, \beta = 1, 2, 3$).

Before considering the possible discontinuities in the components of the Einstein tensor the following preliminary results are derived. Taking the partial derivatives of the expressions (5) with respect to x^γ, it is observed that the following expressions must be continuous at S,

$$\begin{cases} \dfrac{\partial g^{\alpha\beta}}{\partial x^\gamma}\dfrac{\partial g_{\alpha\beta}}{\partial x^4} + g^{\alpha\beta}\dfrac{\partial^2 g_{\alpha\beta}}{\partial x^\gamma\,\partial x^4}, \\ \dfrac{\partial g^{4\alpha}}{\partial x^\gamma}\dfrac{\partial g_{\alpha\beta}}{\partial x^4} + g^{4\alpha}\dfrac{\partial^2 g_{\alpha\beta}}{\partial x^\gamma\,\partial x^4}. \end{cases} \tag{6}$$

The contravariant components $g^{\alpha j}$ are defined by the following system of equations

$$g^{\alpha j}\, g_{jl} = \delta^\alpha_l, \tag{7}$$

where δ^i_j are the components of the Kronecker delta function.

Differentiating these equations with respect to x^γ derives the following equations

$$\frac{\partial g^{\alpha j}}{\partial x^\gamma} g_{jl} + g^{\alpha j} \frac{\partial g_{jl}}{\partial x^\gamma} = 0.$$

Using these and equations (7), it is easily seen that the expressions (6) may be written as follows

$$(6') \qquad \begin{cases} g^{\alpha\beta} \dfrac{\partial^2 g_{\alpha\beta}}{\partial x^\gamma \partial x^4} - g^{\alpha i} g^{\beta j} \dfrac{\partial g_{ij}}{\partial x^\gamma} \dfrac{\partial g_{\alpha\beta}}{\partial x^4}, \\ g^{4\alpha} \dfrac{\partial^2 g_{\alpha\beta}}{\partial x^\gamma \partial x^4} - g^{4l} g^{\alpha j} \dfrac{\partial g_{lj}}{\partial x^\gamma} \dfrac{\partial g_{\alpha\beta}}{\partial x^4}, \end{cases}$$

and these must be continuous at S.

Now consider the possible discontinuities at S in the components, R_{ijkl}, of the Riemann tensor when g^{44} is zero and the O'Brien-Synge conditions hold at S. From their definitions (*see* for example, [12]), it is easily seen that, at S, $R_{\alpha\beta\gamma\delta}$ may be expressed as follows

$$\begin{aligned} R_{\alpha\beta\gamma\delta} = {} & g^{4\varepsilon}([\alpha\delta, 4][\beta\gamma, \varepsilon] - [\alpha\gamma, 4][\beta\delta, \varepsilon]) \\ & + g^{4\varepsilon}([\alpha\delta, \varepsilon][\beta\gamma, 4] - [\alpha\gamma, \varepsilon][\beta\delta, 4]) + [C], \end{aligned}$$

where [C] denotes terms continuous at S, and $[ij, k]$ are the Christoffel symbols of the first kind defined by

$$[ij, k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right).$$

Using this definition the following expressions for $R_{\alpha\beta\gamma\delta}$ may be obtained

$$(7) \qquad \begin{aligned} R_{\alpha\beta\gamma\delta} = \frac{g^{4\varepsilon}}{2}\Big(& -\frac{\partial g_{\alpha\delta}}{\partial x^4}[\beta\gamma, \varepsilon] + \frac{\partial g_{\alpha\gamma}}{\partial x^4}[\beta\delta, \varepsilon] \\ & - \frac{\partial g_{\beta\gamma}}{\partial x^4}[\alpha\delta, \varepsilon] + \frac{\partial g_{\beta\delta}}{\partial x^4}[\alpha\gamma, \varepsilon]\Big) + [C]. \end{aligned}$$

Similarly, some of the other components of the Riemann tensor at S may be expressed in the following way

$$(8) \qquad \begin{aligned} R_{4\beta\gamma\delta} & = R_{\gamma\delta 4\beta} = -R_{\delta\gamma 4\beta} = -R_{\gamma\delta\beta 4} \\ & = \frac{1}{2}\left(\frac{\partial^2 g_{\beta\gamma}}{\partial x^4 \partial x^\delta} - \frac{\partial^2 g_{\beta\delta}}{\partial x^4 \partial x^\gamma}\right) + \frac{g^{\varepsilon\omega}}{2}\left(\frac{\partial g_{\delta\varepsilon}}{\partial x^4}[\beta\gamma, \omega] - \frac{\partial g_{\gamma\varepsilon}}{\partial x^4}[\beta\delta, \omega]\right) \\ & \quad + g^{4\omega}([4\delta, \omega][\beta\gamma, 4] - [4\gamma, \omega][\beta\delta, 4]) + [C] \end{aligned}$$

where it has been assumed here that $\dfrac{\partial g_{4i}}{\partial x^4}$ $(i = 1 - 4)$, are continuous at S.

Also when g^{44} is zero, the components $R_{\beta\gamma}$ of the Ricci tensor may be expressed as follows

(9) $R_{\beta\gamma} = g^{ij} R_{i\beta\gamma j}$, i. e. $R_{\beta\gamma} = g^{\alpha\delta} R_{\alpha\beta\gamma\delta} + g^{4\delta} R_{4\beta\gamma\delta} + g^{\alpha 4} R_{\alpha\beta\gamma 4}$.

Furthermore the components E^4_4 of the Einstein tensor may be written

$$E^4_4 = R^4_4 - \frac{1}{2} g^{ij} R_{ij}, \qquad \text{i. e.} \quad E^4_4 = -\frac{1}{2} g^{\beta\gamma} R_{\beta\gamma}.$$

Thus, substituting from equations (7) and (8) into equations (9), multiplying the resulting equations by $-\frac{1}{2} g^{\beta\gamma}$, summing over β and γ ($\beta, \gamma = 1, 2, 3$), and using the result that the expressions (5) are continuous at S, the following equation may be derived

$$\begin{aligned} E^4_4 = &- \frac{g^{\beta\gamma} g^{\alpha\delta} g^{4\varepsilon}}{2} \frac{\partial g_{\alpha\gamma}}{\partial x^4} \left(\frac{\partial g_{\beta\varepsilon}}{\partial x^\delta} - \frac{\partial g_{\beta\delta}}{\partial x^\varepsilon} \right) \\ &- \frac{g^{\beta\gamma} g^{4\delta}}{2} \left(\frac{\partial^2 g_{\beta\gamma}}{\partial x^4 \partial x^\delta} - \frac{\partial^2 g_{\beta\delta}}{\partial x^4 \partial x^\gamma} \right) + [C]. \end{aligned}$$

Using the result that the expressions (6′) are continuous at S, it may be seen from this that the component E^4_4 of the Einstein tensor is continuous at S.

Similarly, the components E^4_γ ($\gamma = 1, 2, 3$), of the Einstein tensor may be written

$$E^4_\gamma = g^{4\beta} R_{\beta\gamma},$$

Again, substituting from equations (7) and (8) into equations (9), multiplying the resulting equations by $g^{4\beta}$, summing over β ($\beta = 1, 2, 3$), and using the result that the expressions (5) are continuous at S, the following equations may be derived

$$\begin{aligned} E^4_\gamma = &\frac{g^{4\beta} g^{4\varepsilon} g^{\delta\omega}}{2} \frac{\partial g_{\delta\gamma}}{\partial x^4} \left(\frac{\partial g_{\beta\varepsilon}}{\partial x^\omega} - \frac{\partial g_{\beta\omega}}{\partial x^\varepsilon} \right) \\ &+ \frac{1}{2} g^{4\delta} g^{4\beta} \frac{\partial^2 g_{\beta\gamma}}{\partial x^4 \partial x^\delta} - \frac{1}{2} g^{4\delta} g^{4\beta} \frac{\partial^2 g_{\beta\delta}}{\partial x^4 \partial x^\gamma} + [C]. \end{aligned}$$

Using the condition that the second set of expressions in (6′) are continuous, it follows that these equations may be written

$$E^4_\gamma = \frac{1}{2} B^\alpha \frac{\partial g_{\alpha\gamma}}{\partial x^4} + [C],$$

where B^α ($\alpha = 1, 2, 3$), are defined by

$$B^\alpha = g^{4\beta} g^{4\varepsilon} g^{\alpha\delta} \frac{\partial g_{\beta\varepsilon}}{\partial x^\delta},$$

i. e.

$$B^\alpha = g^{\alpha\delta} g^{4\beta} \frac{\partial}{\partial x^\delta}(g^{4\varepsilon} g_{\beta\varepsilon}) - g^{\alpha\delta} g^{4\beta} g_{\beta\varepsilon} \frac{\partial g^{4\varepsilon}}{\partial x^\delta}.$$

However, since when g^{44} is zero,

$$g^{4\varepsilon} g_{\beta\varepsilon} = \delta^4_\beta = 0,$$

it follows that B^α ($\alpha = 1, 2, 3$), are zero, and consequently that the components E^4_γ ($\gamma = 1, 2, 3$), of the Einstein tensor are continuous at S.

3. NULL HYPERSURFACES AND EINSTEIN'S EQUATIONS

Suppose on a null hypersurface, S, defined by equation (2) and bounding a region, V, of space-time, the components, g_{ij}, of the metric tensor, the combinations (5) of the partial derivatives of the metric tensor and the components T^4_i of the energy-momentum tensor are given. The purpose of this section is to show how the ten independant Einstein field equations, defined by

$$E^i_j = -\varkappa T^i_j,$$

where $\varkappa$ is the gravitational constant, may be solved in V.

Following Synge [12], it is assumed that the six components $g_{\alpha\beta}$ ($\alpha, \beta = 1\text{-}3$), of the metric tensor and the four components, T^4_i ($i = 1\text{-}4$), of the energy tensor are to be determined in V from the field equations when g_{4i} and T^α_i are chosen in V and the above values are prescribed on S through the junction conditions. $\left(\text{The choice of } g_{4i} \text{ in V is subject to the restriction that they are continuous at S. The derivatives } \frac{\partial g_{4i}}{\partial x^4} \text{ may or may not be continuous at S}\right)$.

Notice that (5) constitute only four combinations of the six first derivatives $\frac{\partial g_{\alpha\beta}}{\partial x^4}$. Therefore, to obtain a solution, the field equations must determine two equations for these first derivatives at S as well as the values of the second derivatives of the metric tensor $\frac{\partial^2 g_{\alpha\beta}}{(\partial x^4)^2}$ and the first derivatives of the energy-momentum tensor $\frac{\partial T^4_i}{\partial x^4}$ at S.

Consider then the six independent equations included in the following (2) :

$$E^\alpha_\beta = -\varkappa T^\alpha_\beta,$$
$$E^\alpha_4 = -\varkappa T^\alpha_4.$$

(2) Recall that the four equations $E^4_i = -\varkappa T^4_i$ are satisfied identically at S.

Using t to denote terms which do not include second derivatives with respect to x^4 of the metric tensor, these equations may be expressed as follows :

$$-\frac{1}{2}g^{\alpha 4}g^{\gamma 4}\frac{\partial^2 g_{\gamma\beta}}{(\partial x^4)^2}+\frac{1}{2}\delta^\alpha_\beta g^{\gamma 4}g^{\delta 4}\frac{\partial^2 g_{\gamma\delta}}{(\partial x^4)^2}+t=-\varkappa T^\alpha_\beta,$$
$$\frac{1}{2}g^{\alpha 4}g^{\gamma\delta}\frac{\partial^2 g_{\gamma\delta}}{(\partial x^4)^2}-\frac{1}{2}g^{\gamma 4}g^{\alpha\beta}\frac{\partial^2 g_{\gamma\beta}}{(\partial x^4)^2}+t=-\varkappa T^\alpha_4,$$

since g^{44} is zero. These, in turn, may be conveniently expressed by

$$\text{(10)}\qquad \begin{cases} g^{\alpha 4}A_\beta-\delta^\alpha_\beta g^{\gamma 4}A_\gamma=l^\alpha_\beta,\\ g^{\alpha 4}A_4-g^{\alpha\beta}A_\beta=l^\alpha_4\end{cases}$$

where the derivatives $\frac{\partial g_{\alpha\beta}}{\partial x^4}$ and $\frac{\partial^2 g_{\alpha\beta}}{\partial x^\gamma\,\partial x^4}$ and the quantities T^α_i occur in the expressions l^α_i, and A_i ($i=1\text{-}4$), are defined as follows :

$$\text{(11)}\qquad \begin{cases} A_\beta=g^{4\alpha}\dfrac{\partial^2 g_{\alpha\beta}}{(\partial x^4)^2},\\ A_4\;=g^{\alpha\beta}\dfrac{\partial^2 g_{\alpha\beta}}{(\partial x^4)^2}.\end{cases}$$

[None of the second derivatives $\frac{\partial^2 g_{\alpha\beta}}{(\partial x^4)^2}$ occur in the l^α_i.]

It must be made clear that there are precisely six independent equations in the set (10), and so if explicit expressions are found for the four combinations, A_i, at S in terms of the g^{ij} and l^α_j from any four of them, it necessarily follows that there remain two other independent equations in the set into which these expressions may be substituted to give two equations between the values of g^{ij} and l^α_i at S. These serve as the two required relationships for the $\frac{\partial g_{\alpha\beta}}{\partial x^4}$ (α, $\beta = 1, 2, 3$).

It is now shown how explicit expressions for A_i ($i=1\text{-}4$), may be obtained from equations (10). Two cases arise.

First, suppose none of the $g^{4\alpha}$ ($\alpha=1, 2, 3$), are zero at S. Then the following equations from (10) give A_i ($i=1\text{-}4$), directly :

$$\text{(12)}\qquad \begin{cases} g^{14}A_2=l^1_2,\\ g^{24}A_1=l^2_1,\\ g^{14}A_3=l^1_3,\\ g^{14}A_4-g^{11}A_1-g^{12}A_2-g^{13}A_3=l^1_4.\end{cases}$$

Second, suppose at least one of $g^{4\alpha}$ ($\alpha=1, 2, 3$), is zero at S. Without loss of generality it is assumed g^{41} is zero. Then the following equations may be obtained from (10) :

$$(13)\quad \begin{cases} g^{11} A_1 + g^{12} A_2 + g^{13} A_3 = - l^1_4, \\ g^{21} A_1 + g^{22} A_2 + g^{23} A_3 - g^{24} A_4 = - l^2_4, \\ g^{31} A_1 + g^{32} A_2 + g^{33} A_3 - g^{34} A_4 = - l^3_4, \\ g^{42} A_2 + g^{43} A_3 = - l^4_1 \end{cases}$$

and these can always be solved for A_i ($i = 1\text{-}4$), provided the following condition holds

$$\det \begin{bmatrix} g^{11} & g^{12} & g^{13} & 0 \\ g^{21} & g^{22} & g^{23} & g^{24} \\ g^{31} & g^{32} & g^{33} & g^{34} \\ 0 & g^{42} & g^{43} & 0 \end{bmatrix} \neq 0.$$

However, when g^{14} is zero, the left-hand side of this is just the determinant $| g^{ij} |$, and so the inequality must always hold in this case.

Thus, it may be concluded that unique expressions for A_i ($i = 1\text{-}4$), can always be found from four of the six independent equations (10). These expressions may be substituted into the two remaining equations of the set (10) to obtain two partial differential equations, E, in $\frac{\partial g_{\alpha\beta}}{\partial x^4}$, $\frac{\partial^2 g_{\alpha\beta}}{\partial x^\gamma \partial x^4}$ and known functions at S. Because these equations, E, in general, involve the terms $\frac{\partial}{\partial x^\gamma}\left(\frac{\partial g_{\alpha\beta}}{\partial x^4}\right)$ at S it follows that their solutions for $\frac{\partial g_{\alpha\beta}}{\partial x^4}$ at S will, in general, involve arbitrary functions of x^γ.

Having obtained $\frac{\partial g_{\alpha\beta}}{\partial x^4}$ at S from these equations the other four equations in (10), i. e. equations (12) or (13), may be used to determine the values of A_i ($i = 1\text{-}4$), directly in terms of these arbitrary functions of x^γ. This done, the four equations

$$E^4_i = - \varkappa T^4_i$$

may be differentiated with respect to x^4 and the values of $\frac{\partial T^4_i}{\partial x^4}$ at S may be found-again in terms of the arbitrary functions of x^γ.

In this way the first derivatives with respect to x^4 of all the surface data may be found in terms of arbitrary functions of x^γ.

Comparing the four combinations A_i of second derivatives defined by equations (11), with the four combinations (5) of first derivatives, it is observed that the next stage, (and each subsequent stage), in the solution procedure, i. e. finding the second, (and higher), derivatives with respect to x^4 of the surface data, is precisely the same as the first stage outlined above. Arbitrary functions of x^γ are introduced as each higher derivative is obtained.

It is worth noting that if the Lichnerowicz junction conditions [4], stating that the components, g_{ij}, of the metric tensor and all its first derivatives, $\frac{\partial g_{ij}}{\partial x^k}$, should be continuous at S, were imposed, then the surface data would be over-prescribed, since the set of equations (10) would constitute a set of six equations in the four unknowns A_i ($i = 1\text{-}4$). These, in general, would not be consistent since the values T_i^α are chosen arbitrarily in V.

Example. — As illustration of the foregoing procedure consider the following metric

$$ds^2 = e^{2\varphi}(du^2 + 2\,du\,dx) - u^2(e^{2\beta}\,dy^2 + e^{-2\beta}\,dz^2), \tag{14}$$

where β, φ are functions of u only. Let S be a null hypersurface, defined by

$$u - a = 0,$$

where a is a constant, and separating space-time into two regions V_1 and V_2 defined by

$$\begin{aligned} &u - a \leqq 0, \\ &u - a \geqq 0, \quad \text{respectively.} \end{aligned}$$

Suppose V_1 is a flat space-time region so that in V_1 (14) is the Minkowski metric given by [1] :

$$\beta = \beta_0 + \ln u, \tag{15}$$

$$\varphi = \varphi_0 + \frac{1}{2}\ln u, \tag{16}$$

where φ_0, β_0 are constants.

According to the above procedure the functions g_{4i} and T_i^α may be chosen freely in V_2, and an interesting choice for T_i^α is that they are all zero except T_4^1 which takes the form

$$T_4^1 = u^{-2}\,e^{-2(\varphi+\beta)}\left(\frac{dA}{du}\right)^2$$

where A is a function of u. This form of energy-momentum tensor may be interpreted [13] as an electromagnetic plane wave progressing in the x direction (the function A being the only non-zero component of the electromagnetic potential). The functions g_{4i} may be chosen, for example, to be the same in V_2 as in V_1, i. e. φ may be chosen to have the form (16)in V_2 as well as in V_1.

For the metric (14) the condition that the combinations (5) of the first derivatives should be continuous at S imposes no restriction at all on the function β. The other junction conditions require the components T_i^4 ($i = 1\text{-}4$), to be zero at S and β to have the value $\beta_0 + \ln a$ at S.

The only non-trivial field equation in the set (10) is the following

$$E_4^1 = -\varkappa T_4^1$$

which may be expressed as

$$(17) \qquad 2\, e^{-2\varphi}\, u^{-1} \left[2\, \frac{d\varphi}{du} - u \left(\frac{d\beta}{du} \right)^2 \right] = -\varkappa\, e^{-2(\varphi+\beta)} \left(\frac{dA}{du} \right)^2.$$

This, being a first order differential equation for β, may be solved subject to the boundary conditions that β equals $\beta_0 + \ln a$ when $u = a$, whenever the function A is chosen. $\Bigg($Since it is assumed in this example that β is a function of u only no terms involving $\dfrac{\partial^2 \beta}{\partial x\, \partial u}$, $\dfrac{\partial^2 \beta}{\partial y\, \partial u}$ or $\dfrac{\partial^2 \beta}{\partial z\, \partial u}$ occur in this equation. Even so, the value of $\dfrac{d\beta}{du}$ at S is arbitrary up to a sign.$\Bigg)$

When φ has the form (16), a particular choice for the function A is given by

$$A = c\, e^{\beta},$$

where c is a constant. This leads to a solution of equation (17) (when $\varkappa$ is put equal to $8\,\pi$) given by

$$\beta = \beta_0 + \ln \left[\frac{a\,(u \pm \sqrt{4\,\pi\, c^2 + u^2})}{(a \pm \sqrt{4\,\pi\, c^2 + a^2})} \right].$$

Notice if $\dfrac{d\beta}{du}$ were chosen to be continuous at S the equation (17) would be inconsistent except when A were chosen to be constant.

Finally, the left-hand sides of the equations

$$E_i^4 = -\varkappa T_i^4$$

may be evaluated and the components T_i^4 in V_2 obtained. It turns out, in fact, that all the components T_i^4 are zero in V_2 no matter what choice is made for the function A. (This last remark is easily verified from the conservation laws

$$T^i_{j|i} = 0$$

remembering that T_i^4 are zero on S, and that the solutions are independent of x, y and z.)

4. DISCUSSION

If in the procedure outlined in section 3 the values of T_i^{λ} at S and of T_i^{α} in V are all chosen to be zero, then it follows from the conservation equations that all the components of the energy-momentum tensor in V must be zero, i. e. a vacuum solution is obtained thereby.

Many authors have studied the initial value problem for vacuum fields. Bondi *et al.* [2] and Sachs [10] when discussing the asymptotic behaviour of such fields implicitly assumed the existence of discontinuities in the first derivatives of the metric tensor at null hypersurfaces (the arbitrary functions being the « news » functions).

The possibility of first order discontinuities at null hypersurfaces (bearing no singular matter distribution) was shown by Papapetrou and Treder ([7], [8]) to be consistent with the vacuum equations. Propagation equations for these discontinuities (which are equivalent to the equations E discussed in section 3), were explicitly obtained.

The arbitrary functions introduced by equations E cannot be fixed without knowledge of the solution off the null hypersurface, S. In the vacuum case, Sachs [11] has determined unique solutions by giving the metric on a second null hypersurface and on its intersection with S. Choquet-Bruhat [3] has constructed unique solutions of the vacuum equations in harmonic coordinates when the metric and its first derivatives are given at the apex of the null cone formed by S. (The harmonic coordinate condition corresponds to choosing $g_{\lambda i}$ in V.)

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