

# ANNALES DE L'I. H. P., SECTION A

J. MICKELSSON

J. NIEDERLE

**On integrability of discrete representations  
of Lie algebra  $u(p, q)$**

*Annales de l'I. H. P., section A*, tome 19, n° 2 (1973), p. 171-177

[http://www.numdam.org/item?id=AIHPA\\_1973\\_\\_19\\_2\\_171\\_0](http://www.numdam.org/item?id=AIHPA_1973__19_2_171_0)

© Gauthier-Villars, 1973, tous droits réservés.

L'accès aux archives de la revue « Annales de l'I. H. P., section A » implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

## On integrability of discrete representations of Lie algebra $u(p, q)$

by

**J. MICKELSSON (\*) and J. NIEDERLE**

Institute of Physics, Czechoslovak Academy of Sciences,  
Na Slovance 2, Prague 8

---

ABSTRACT. — It is proved that every representation of the discrete series of hermitian representations of Lie algebra  $u(p, q)$  constructed by the Gel'fand-Graev method is differential of a unitary one-valued representation of Lie group  $U(p, q)$ .

### 1. INTRODUCTION

In 1965 Gel'fand and Graev [1] described a method for constructing discrete series of hermitian irreducible representations of Lie algebra  $u(p, q)$ , i. e. series of irreducible hermitian representations of  $u(p, q)$  characterized by a finite number of integers. The question of integrability of these representations to the corresponding connected simply-connected (universal covering) Lie group of  $u(p, q)$  was not discussed. Recently theorems concerning integrability criteria of representations of finite dimensional real Lie algebra appear ([2], [3]) which complete the study of Nelson [4] and give us powerful tools for proving integrability of discrete representations of  $u(p, q)$ .

In section 2 a brief description of the discrete series of (skew-symmetric) irreducible representations of Lie algebra  $u(p, q)$  is given. Section 3 contains the proof that the discrete representations of  $u(p, q)$  are integrable.

---

(\*) Permanent address : University of Jyväskylä, Finland.

## 2. DISCRETE SERIES OF REPRESENTATIONS OF $u(p, q)$

According to Gel'fand and Graev [1] a basis for the (real) Lie algebra  $u(p, q)$ ,  $p + q = n$ ,  $p \geq q$ , is given by

$$(1) \quad \left\{ \begin{array}{l} M_{kk} = i A_{kk} \quad (k = 1, 2, \dots, n). \\ M_{jk} = i (A_{jk} + A_{kj}), \quad \tilde{M}_{jk} = (A_{jk} - A_{kj}) \\ \quad (j < k \leq p \text{ or } p < j < k), \\ N_{jk} = i (A_{jk} - A_{kj}), \quad \tilde{N}_{jk} = (A_{jk} + A_{kj}) \quad (j \leq p < k) \end{array} \right.$$

the commutation relations of which follow from the commutation relations of  $A_{jk}$  :

$$(2) \quad [A_{ij}, A_{km}] = \delta_{jk} A_{im} - \delta_{mi} A_{kj}.$$

Irreducible representations of  $u(p, q)$  by skew-symmetric operators are described by all inequivalent systems of operators satisfying (2) and the condition of skew-symmetry

$$(3) \quad \left\{ \begin{array}{l} A_{jk}^+ = A_{jk} \quad \text{for } j \leq p, k < p \text{ and } j > p, k > p; \\ A_{jk}^+ = -A_{kj} \quad \text{for } j \leq p, k > p \text{ and } j > p, k \leq p \quad (1). \end{array} \right.$$

The discrete irreducible representation of  $u(p, q)$ ,  $p \geq q$ , by skew symmetric operators in a Hilbert space  $\mathcal{H}$  is characterized by  $n = p + q$  integers  $m_n = (m_{1n}, m_{2n}, \dots, m_{nn})$ ,  $m_{1n} \geq m_{2n} \geq \dots \geq m_{nn}$  and by the decomposition  $p = \alpha + \beta$ ,  $\alpha, \beta$  being non-negative integers.

Any state in  $\mathcal{H}$  may be written as a linear combination of basis states  $|m\rangle$  which are mutually orthonormal and labeled by integers  $m_{j,k}$ ,  $j \leq k$ , satisfying the following inequalities [1] :

$$(4) \quad \left\{ \begin{array}{l} \text{(i)} \quad m_{j,k+1} \geq m_{jk} \geq m_{j+1,k+1} \\ \quad \quad (j = 1, 2, \dots, k; \\ \quad \quad k = 1, 2, \dots, p - 1 \text{ or } j = \alpha + 1, \alpha + 2, \dots, k - \beta; \\ \quad \quad k = p + 1, p + 2, \dots, n - 1), \\ \text{(ii)} \quad m_{1k} \geq m_{1,k+1} + 1 \geq m_{2k} \geq m_{2,k+1} + 1 \geq \dots \\ \quad \quad \quad \quad \quad \quad \quad \quad \geq m_{\alpha,k} \geq m_{\alpha,k+1} + 1 \\ \quad \quad (k = p, p + 1, \dots, n - 1), \\ \text{(iii)} \quad m_{k-\beta+2,k+1} - 1 \geq m_{k-\beta+1,k} \geq m_{k-\beta+3,k+1} - 1 \geq \dots \\ \quad \quad \quad \quad \quad \quad \quad \quad \geq m_{k+1,k+1} - 1 \geq m_{kk} \\ \quad \quad (k = p, p + 1, \dots, n - 1). \end{array} \right.$$

---

(1) Generators and their representations will be denoted by the same letters.

The basis states  $|m\rangle$  may be expressed as Gel'fand-Zetlin patterns which are a geometrical transcription of the above inequalities (for more detail see [1]).

The action of generators of  $u(p, q)$  in  $\mathcal{H}$  can easily be calculated by specifying the action of  $A_{jk}$  on the basis  $|m\rangle$  in  $\mathcal{H}$ . In fact, it is sufficient to specify the action of  $A_{kk}$ ,  $A_{k-1,k}$  and  $A_{k,k-1}$  ( $k = 1, \dots, n$ ), since the action of the other  $A_{jk}$  can be calculated by using commutation relations (2).

The action of  $A_{jk}$  on the basis in  $\mathcal{H}$  is given by [1] :

$$(5) \quad \left\{ \begin{aligned} A_{kk} |m\rangle &= \left[ \sum_{i=1}^k m_{ik} - \sum_{i=1}^{k-1} m_{ik-1} \right] |m\rangle, \\ A_{k,k-1} |m\rangle &= \sum_{j=1}^{k-1} a_{k-1}^j(m) |m_{k-1}^j - 1\rangle, \\ A_{k-1,k} |m\rangle &= \sum_{j=1}^{k-1} b_{k-1}^j(m) |m_{k-1}^j + 1\rangle, \end{aligned} \right.$$

where  $k = 1, 2, \dots, n$  and

$$(6) \quad \left\{ \begin{aligned} a_{k-1}^j(m) &= \frac{\left\{ \prod_{i=1}^k (m_{ik} - m_{j,k-1} - i + j + 1) \right\} \times \left\{ \prod_{i=1}^{k-2} (m_{i,k-2} - m_{j,k-1} - i + j) \right\}}{\left\{ \prod_{\substack{i=1 \\ i \neq j}}^{k-1} (m_{i,k-1} - m_{j,k-1} - i + j + 1) \right\} \times (m_{i,k-1} - m_{j,k-1} - i + j)} \right\}^{1/2}, \\ b_{k-1}^j(m) &= \frac{\left\{ \prod_{i=1}^k (m_{ik} - m_{j,k-1} - i + j) \right\} \times \left\{ \prod_{i=1}^{k-2} (m_{i,k-2} - m_{j,k-1} - i + j - 1) \right\}}{\left\{ \prod_{\substack{i=1 \\ i \neq j}}^{k-1} (m_{i,k-1} - m_{j,k-1} - i + j) \right\} \times (m_{i,k-1} - m_{j,k-1} - i + j - 1)} \right\}^{1/2}. \end{aligned} \right.$$

$|m'_{k-1} - 1\rangle$  and  $|m'_{k-1} + 1\rangle$  are Gel'fand-Zetlin patterns which are obtained from  $|m\rangle$  by changing there  $m_{j,k-1}$  into  $m_{j,k-1} - 1$  and  $m_{j,k-1} + 1$  respectively.

Moreover, in order to define the action of  $A_{jk}$  uniquely we take

$$\arg a'_{k-1} = \arg b'_{k-1} = \begin{cases} 0 & (k \neq p + 1), \\ \frac{\pi}{2} & (k = p + 1). \end{cases}$$

### 3. INTEGRABILITY OF DISCRETE REPRESENTATIONS OF $u(p, q)$

First we state a result (Corollary 2) proved by Simon [3] : Let  $T$  be a representation of a real finite dimensional Lie algebra  $g$  defined on a dense domain  $D$  in a Hilbert space  $H$ , invariant under  $T(g)$ , by skew symmetric operators. Suppose that there exists a set of generators  $\{x_1, \dots, x_s\}$  of  $g$  <sup>(2)</sup> such that  $D$  is a domain of analytic vectors for the operators  $X_i = T(x_i)$  ( $1 \leq i \leq s$ ) then  $T$  is the differential (on  $D$ ) of a unitary representation of the connected simply connected real Lie group  $G$  (the Lie algebra of which is  $g$ ) on Hilbert space  $H$ .

Since the action of skew symmetric generators of  $u(p, q)$  on an arbitrary basis vector  $|m\rangle$  of  $\mathcal{H}$  can be calculated by using (5) the results of Simon may be applied provided that  $D$  is considered as all finite linear combination of  $|m\rangle$  and for each generator  $x_i$  ( $i = 1, \dots, s$ ) from the set of generators of  $u(p, q)$  any vector  $|m\rangle$  is an analytic vector, i. e. for each vector  $|m\rangle$  there exists  $t < 0$  such that

$$\sum_{n=0}^{\infty} \frac{1}{n!} t^n \| (X_i)^n |m\rangle \| < +\infty \quad (i = 1, 2, \dots, s).$$

This is equivalent to show that for each  $x_i$  and for each  $|m\rangle$  there exists a constant  $C > 0$  such that

$$(7) \quad \| (X_i)^n |m\rangle \| \leq n! C^n.$$

First let remark that the set of generators  $x_i$  of  $u(p, q)$  is formed by generators  $M_{11}, M_{k-1,k}$  ( $k = 2, 3, \dots, p; k = p + 2, p + 3, \dots, p + q$ ) and  $N_{p,p+1}$  defined in (1) <sup>(3)</sup>.

<sup>(2)</sup> A set of generators of  $g$  is a set of vectors  $\{x_1, \dots, x_s\}$  in  $g$  such that  $g$  is generated by linear combinations of the vectors  $x_1, x_2, \dots, x_s, [x_i, x_j], [x_i, [x_i, x_j]], \dots$  when  $1 \leq i, j, \dots \leq s$ .

<sup>(3)</sup> Really, taking commutator  $[M_{12}, M_{11}]$  we get  $\tilde{M}_{12}$  and taking  $[M_{12}, \tilde{M}_{12}]$  we obtain  $M_{22}$ . Then  $[M_{23}, M_{22}]$  leads to  $\tilde{M}_{23}$  and from  $[M_{23}, \tilde{M}_{23}]$  we derive  $M_{33}$ , and so on. The generators  $N_{jk}$  are derived from  $N_{p,p+1}$  by using commutators with  $\tilde{M}_{p+1,p+2}; \tilde{M}_{p+2,p+3}, \dots, \tilde{M}_{p+q-1,p+q}, \tilde{M}_{p-1,p}, \tilde{M}_{p-2,p-1}, \dots, \tilde{M}_{1,2}$ .

Thus we may distinguish three cases :

(i)  $M_{11}$  : The constant C in (7) trivially exists since

$$\| (M_{11})^n | m \rangle \| = (m_{11})^n.$$

(ii)  $M_{k-1, k}$  ( $k = 2, 3, \dots, p$  and  $k = p + 2, p + 3, \dots, p + q$ ) : In this case the subspace of  $\mathcal{A}$  spanned by vectors  $\{ (M_{k-1, k})^n | m \rangle \}_{n=1}^\infty$ ,  $k$  and  $| m \rangle$  fixed but arbitrary, are finite dimensional (generators  $M_{k-1, k}$  change  $k - 1$  row in  $| m \rangle$  that for  $k = 2, 3, \dots, p$  and  $k = p + 2, p + 3, \dots, p + q$  contains  $m_{ik-1}$  ( $i = 1, \dots, k - 1$ ) which are bounded [see (1), (4), (5)] and thus C obviously exists).

(iii)  $N_{\rho, \rho+1}$  : In this case

$$N_{\rho, \rho+1} | m \rangle = i \sum_{j=1}^{\rho} [b_{\rho}^j(m) | m_{\rho}^j + 1 \rangle - a_{\rho}^j(m) | m_{\rho}^j - 1 \rangle].$$

Let us first consider the numbers  $b_{\rho}^j(m)$ . If  $j \leq \alpha$  :

$$\begin{aligned} (8) \quad b_{\rho}^j(m) &= \prod_{i=1}^{j-1} \left( \frac{m_{i\rho-1} - m_{j\rho} - i + j - 1}{m_{i\rho} - m_{j\rho} - i + j - 1} \right)^{1/2} \\ &\times \prod_{i=j}^{\rho-1} \left( \frac{m_{i\rho-1} - m_{j\rho} - i + j - 1}{m_{i+1\rho} - m_{j\rho} - (i + 1) + j} \right)^{1/2} \\ &\times \prod_{i=1}^{j-1} \left( \frac{m_{i\rho+1} - m_{j\rho} - i + j}{m_{i\rho} - m_{j\rho} - i + j} \right)^{1/2} \\ &\times \prod_{i=j}^{\alpha-1} \left( \frac{m_{i\rho+1} - m_{j\rho} - i + j}{m_{i+1\rho} - m_{j\rho} - (i + 1) + j - 1} \right)^{1/2} \\ &\times \prod_{i=\alpha+2}^{\rho+1} \left( \frac{m_{i\rho+1} - m_{j\rho} - i + j}{m_{i-1, \rho+1} m_{j\rho} - (i - 1) + j - 1} \right)^{1/2} \times \text{phase factor} \\ &\times (- (m_{\alpha, \rho+1} - m_{j\rho} - \alpha + j) \\ &\times (m_{\alpha+1, \rho+1} - m_{j\rho} - (\alpha + 1) + j))^{1/2}. \end{aligned}$$

Using the inequalities (4) one can easily show that the absolute values of all of the factors, except of the last one, are smaller or equal to 1. Therefore,

$$\begin{aligned} (9) \quad | b_{\rho}^j(m) | &\leq | (m_{\alpha, \rho+1} - m_{j\rho} - \alpha + j) \\ &\quad \times (m_{\alpha+1, \rho+1} - m_{j\rho} - (\alpha + 1) + j) |^{1/2} \\ &\leq (m_{1\rho} - m_{\rho\rho} + \rho) \quad (j \leq \alpha). \end{aligned}$$

If  $j > \alpha$  instead of (8) one writes

$$\begin{aligned}
 (8') \quad b_p^j(m) = & \prod_{i=1}^{j-1} \left( \frac{m_{i,p-1} - m_{j,p} - i + j - 1}{m_{i,p} - m_{j,p} - i + j - 1} \right)^{1/2} \\
 & \times \prod_{i=j}^{p-1} \left( \frac{m_{i,p-1} - m_{j,p} - i + j - 1}{m_{i+1,p} - m_{j,p} - (i+1) + j - 1} \right)^{1/2} \\
 & \times \prod_{i=1}^{\alpha} \left( \frac{m_{i,p+1} - m_{j,p} - i + j}{m_{i,p} - m_{j,p} - i + j} \right)^{1/2} \\
 & \times \prod_{i=\alpha+3}^{j+1} \left( \frac{m_{i,p+1} - m_{j,p} - i + j}{m_{i-2,p} - m_{j,p} - (i-2) + j} \right)^{1/2} \\
 & \times \prod_{i=j+2}^{p-1} \left( \frac{m_{i,p+1} - m_{j,p} - i + j}{m_{i-1,p} - m_{j,p} - (i-1) + j} \right)^{1/2} \times \text{phase factor} \\
 & \times [ - (m_{\alpha+1,p+1} - m_{j,p} - (\alpha+1) + j) \\
 & \quad \times (m_{\alpha+2,p+1} - m_{j,p} - (\alpha+2) + j) ]^{1/2}.
 \end{aligned}$$

As before we get

$$(9') \quad |b_p^j(m)| \geq m_{1,p} - m_{p,p} + p \quad (j > \alpha).$$

In a similar way we can show that

$$(10) \quad a_p^j(m) \leq m_{1,p} - m_{p,p} + p.$$

Consequently

$$\begin{aligned}
 (11) \quad \| (N_{p,p+1})^n |m\rangle \| = & \left\| \sum a_p^{j_1}(m^{(n-1)}) a_p^{j_2}(m^{(n-2)}) b_p^{j_3}(m^{(n-3)}) \dots \right. \\
 & \left. a_p^{j_{n-1}}(m^{(1)}) b_p^{j_n}(m^{(0)}) |m^{(n)}\rangle \right\| \\
 & \leq \sum |a_p^{j_1}(m^{(n-1)}) \dots b_p^{j_n}(m^{(0)})| \\
 & \leq (2p)^n \cdot \Delta(\Delta+1) \dots (\Delta+n) \\
 & \leq \Delta \cdot n! (2p(\Delta+1))^n
 \end{aligned}$$

where  $\Delta = m_{1,p} - m_{p,p} + p$  and the sum is over all possible combinations of three things the  $a_p$  and  $b_p$  factors and  $m^{(k)}$  ( $k = 1, 2, \dots, n-1$ ),  $m^{(0)} = m$ .

Numbers  $m^{(k)}$  are obtained from numbers  $m^{(k-1)}$  by adding  $\pm 1$  to one of the numbers  $m_{j,p}^{(k-1)}$  ( $j = 1, 2, \dots, p$ ), i. e.,  $|m^{(k)}\rangle$  represents any vector in  $\mathcal{H}$  which can be reached from  $|m^{(k-1)}\rangle$  by acting once by operator  $N_{p,p+1}$ .

Thus we have proved that every basis vector  $|m\rangle$  in  $\mathfrak{X}$  is analytic for the given set of generators of  $u(p, q)$  and consequently, that every discrete skew symmetric representation of  $u(p, q)$  is the differential (on  $D$ ) of a unitary representation (on  $\mathfrak{X}$ ) of a connected and simply connected Lie group  $\widehat{U(p, q)}$ . Since, in this unitary representation, all elements of the discrete center of  $\widehat{U(p, q)}$  are represented by the unit operator in  $\mathfrak{X}$  ( $m_i$  are integers), the unitary representation of  $\widehat{U(p, q)}$  is a one-valued unitary representation of group  $U(p, q)$ .

#### ACKNOWLEDGEMENT

One of the authors (J. M.) would like to thank to Dr. J. Sedláč for hospitality at the Institute of Physics of the Czechoslovak Academy of Sciences in Prague.

#### REFERENCES

- [1] I. M. GEL'FAND and M. I. GRAEV, *Irreducible representations of the Lie algebra of groups  $U(p, q)$*  (*Proceedings of the International Spring School for Theoretical Physics of the Joint Institute for Nuclear Research, Jalta, 1966*, p. 216).
- [2] M. FLATO, J. SIMON, H. SNELLMAN and D. STERNHEIMER, *Simple Facts About Analytic Vectors and Integrability* (*Ann. scient. Éc. Norm. Sup.*, t. 5, 1972, p. 423).
- [3] J. SIMON, *On the integrability of Representation of Finite Dimensional Real Lie Algebras* (*Commun. Math. Phys.*, t. 28, 1972, p. 39).
- [4] E. NELSON, *Analytic vectors* (*Ann. Math.*, t. 81, 1959, p. 547).

(Manuscrit reçu le 1<sup>er</sup> mars 1973.)