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On momentum states in quantum mechanics

by

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ABSTRACT. — In the algebraic formulation of quantum mechanics a class of states, called the momentum states are defined. They correspond to plane wave and wave packet states of ordinary quantum mechanics. A uniqueness theorem is proved for this class of states. Furthermore a purely algebraic proof if given of the fact that in any irreducible representation of the CCR-algebra the von Neumann algebra generated by all translations is maximal abelian.

0. INTRODUCTION

The algebraic formulation of quantum mechanics for a system with n degrees of freedom consists in defining a dynamical system, given by the following triplet:

i) the abstract set of observables as the elements of the CCR-C*-algebra Δ [*1*] build on the sympletic space $H = R^{2n}$ and sympletic form σ:

$$\sigma((pq), (p'q')) = \frac{\hbar}{2} \sum_{i=1}^{n} (p_i q'_i - q_i p'_i); \qquad p, q, p', q' \in \mathbb{R}^n$$

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ii) a set of states on the C*-algebra Δ,

iii) an evolution for the state (Schrödinger or Liouville equation), we do not specify more on this point in this work.

We do not elaborate either the notion of such dynamical systems, but define a class of states, called momentum states. These momentum states correspond to the plane wave and wave packet states of ordinary quantum mechanics. The plane wave states are pure state invariant for space translations, and the wave packet states are integrals over plane wave states.

It is proved that all momentum states with pure point spectrum for the linear momentum operator are quasi-equivalent with a plane wave state. This property yields a uniqueness theorem for momentum states comparable with von Neumann's uniqueness theorem for Weyl states.

Furthermore it is proved that in any irreducible representation of the C*-algebra, the von Neumann algebra of all translations is a maximal abelian von Neumann subalgebra (Theorem I.6). This property is well known in Schrödinger quantum mechanics and there the proof is based on the fact that the spectrum of the momentum operator is the real line. In our case the spectrum of the momentum operator is not necessarily absolutely continuous, as can be seen in theorems I.4 and I.5. Finally we stress the fact that the proof of Theorem I.6 is purely algebraic.

I. MOMENTUM STATES

For a system with n degrees of freedom, consider the real vector space $H = R^{2n}$, the elements of H are denoted by $\psi = (p, q)$, $p, q \in R^n$. The space H has to be looked upon as the classical phase space of the canonical variables p and q.

Define on H the following sympletic form σ:

$$\sigma((pq), (p'q')) = \frac{\hbar}{2} \sum_{i=1}^{n} (p_i q'_i - q_i p'_i)$$

$\hbar$ is Planck's constant and from now on $\hbar = 1$.

Denote by Δ the CCR-C*-algebra [*1*] build on (H, σ): it is the smallest C*-algebra generated by the functions $\delta_\psi : H \to R$, $\psi \in H$, defined by

$$\delta_\psi(\Phi) = 1 \quad \text{if} \quad \psi = \Phi$$
$$\delta_\psi(\Phi) = 0 \quad \text{if} \quad \psi \neq \Phi$$

Addition is that of pointwise addition of functions, the product rule is defined by

$$\delta_\psi \delta_\varphi = \exp(-i\sigma(\psi, \varphi))\delta_{\psi+\varphi}$$

and the involution, indicated by a .*, is given by

$$(\delta_\psi)^* = \delta_{-\psi}$$

For more details see [*1*].

DEFINITION I.1. — A *quantum mechanical state* or shortly a *state* is any positive, linear, normalized form on the C*-algebra.

DEFINITION I.2. — A *momentum state* is any state ω such that the map $q \in \mathbb{R}^n \to \omega(\delta_{pq})$ is continuous for each fixed $p \in \mathbb{R}^n$.

Each state ω on Δ determines uniquely a representation Π_ω on a Hilbert space $\mathcal{H}_\omega$, containing a cyclic vector Ω_ω. For reasons of notational convenience we write up the results for 1-dimensional systems. The generalisation to n-dimensions is trivial.

LEMMA I.3. — Let ω be a momentum state, then there exists a self-adjoint operator P_ω on $\mathcal{H}_\omega$ such that $\Pi_\omega(\delta_{0q}) = \exp iqP_\omega$; P_ω is called the momentum operator.

Proof. — For any $\xi \in \mathcal{H}_\omega$ there exists an element

$$\mu = \sum_{i=1}^{n} c_i \Pi_\omega(\delta_{p_i q_i})\Omega_\omega \quad \text{such that} \quad \| \xi - \mu \| < \frac{\varepsilon}{4}$$

Hence

$$\begin{aligned} \| \Pi_\omega(\delta_{0q})\xi - \xi \| &= \| \Pi_\omega(\delta_{0q})(\xi - \mu) + \Pi_\omega(\delta_{0q})\mu - \mu + \mu - \xi \| \\ &\leqslant 2 \| \xi - \mu \| + \| \Pi_\omega(\delta_{0q})\mu - \mu \| \\ &\leqslant \frac{\varepsilon}{2} + \sqrt{2}[(\mu, \mu) - R_e(\Pi_\omega(\delta_{0q})\mu, \mu)]^{\frac{1}{2}} < \varepsilon \end{aligned}$$

for q small enough, because ω is a momentum state.

Hence the map $q \to \Pi_\omega(\delta_{0q})$ is strongly continuous, and by Stone's theorem we get the Lemma.

Q. E. D.

In the following theorems we characterize the momentum states, and prove properties of momentum states analogous in spirit as in the uniqueness theorem of von Neumann [2] for the Weyl states on the CCR.

THEOREM I.4. — Let ω_i be pure momentum states and P_i the corresponding momentum operators on $\mathcal{H}_{\omega_i}$ $(i = 1, 2)$. If $\Sigma_p(P_i)$ is not empty for $i = 1, 2$ then the states ω_1 and ω_2 induce unitary equivalent representations Π_{ω_1} and Π_{ω_2}.

Proof. — Consider any pure momentum state ω on Δ; let $(\Pi, \mathcal{H}, \Omega)$ be its GNS-triplet; P the momentum operator; let $\lambda \in \Sigma_p(P)$ and $\varphi_\lambda \in \mathcal{H}$ the corresponding normalized momentum eigenvector: $P\varphi_\lambda = \lambda\varphi_\lambda$. Then

$$\Pi(\delta_{0,q})\Pi(\delta_{\mu-\lambda,0})\varphi_\lambda = e^{i\mu q}\Pi(\delta_{\mu-\lambda,0})\varphi_\lambda$$

Denote $\varphi_\mu = \Pi(\delta_{\mu-\lambda,0})\varphi_\lambda$ then $P\varphi_\mu = \mu\varphi_\mu$. Hence $\Sigma_p(P) = \mathbb{R}$. Since the state ω is pure, each vector is cyclic, therefore φ_λ is cyclic, also

$$\Pi(\delta_{0,q}) = e^{i\mu q}\varphi_\mu, \qquad \Pi(\delta_{p,0})\varphi_\mu = \varphi_{\mu+p}\,.$$

Hence $\mathscr{H}$ is generated by the eigenvectors $\{ \varphi_\mu \mid \mu \in \mathrm{R} \}$ of P and the state ω is unitarily equivalent to the vector state $\omega_{\varphi_0} : \omega_{\varphi_0}(x) = (\varphi_0, \Pi(x)\varphi_0)$, $x \in \Delta$; φ_0 is the zero momentum eigenvector. Furthermore an easy calculation shows

$$\omega_{\varphi_0}(\delta_{pq}) = \delta_{p,0} \qquad \text{(Kronecker symbol)}$$

Q. E. D.

Denote by ω_k the state on Δ defined by $\omega_k(\delta_{p,0}) = \delta_{p,0}e^{ikq}$. It is clear that ω_k is a momentum state for all $k \in \mathrm{R}$. It is easily checked that all these states are pure states [3] and in particular that with the notations of above (proof of Theorem I.4) $\omega_k(\delta_{pq}) = (\varphi_k, \Pi(\delta_{pq})\varphi_k)$. These states ω_k are called plane wave states [3].

As compared with von Neumann's uniqueness theorem, remark that in view of Theorem I.4 an arbitrary momentum state ω is not necessarily a direct sum of copies of the vector state ω_0 ; e. g. the state

$$\omega(\delta_{pq}) = \int dk\, f(k)\omega_k(\delta_{pq}) ; \qquad f \geqslant 0, \qquad f \in \mathscr{L}^1(\mathbb{R}, dx)$$

is a state on Δ which is a direct integral over the state ω_k which are all equivalent with ω_0. However what we have is the following result.

THEOREM I.5. — Let ω be a momentum state, such that $\Sigma(\mathrm{P}_\omega) = \Sigma_p(\mathrm{P}_\omega)$, then ω is quasi-equivalent with ω_0.

Proof. — We have to prove that no subrepresentation π' of Π_ω is disjoint from Π_{ω_0}.

Let π' be any irreducible subrepresentation of Π_ω, let E be the projection of $\Pi_\omega(\Delta)'$ such that $\pi'(.) = \mathrm{E}\Pi_\omega(.)\mathrm{E}$ then $\Sigma(\mathrm{EP}_\omega\mathrm{E}) = \Sigma_p(\mathrm{EP}_\omega\mathrm{E})$ and by theorem I.4 the representation π' is equivalent with Π_{ω_0}.

Q. E. D.

Theorem I.4 proves that all pure momentum states with point momentum spectrum induce representation in the Hilbert space $\mathscr{H}_0$, the representation space of a plane wave state of zero momentum.

For reasonable potentials (tending to zero fast enough at infinity) we expect that the asymptotic behaviour of the quantum mechanical wave function is like a plane wave, hence for such systems it may be argued that the state will induce a representation in Π_0.

We can write down a configuration representation of $\mathscr{H}_0$ by the following identification : $\varphi_k \in \mathscr{H}_0$:

$$\mathrm{P}\varphi_k = k\varphi_k ; \; \varphi_k = e^{ikx}, \qquad x \in \mathbb{R} ;$$

so the elements φ_k are quasi-periodic functions, the scalar product is determined by the translation invariant mean:

$$(\varphi_k, \varphi_l) = \mathscr{M}(\overline{\varphi}_k\varphi_l) = \lim_{a\to\infty} \frac{1}{2a} \int_{-a}^{+a} dx\, e^{-ikx}e^{ilx}$$

This is the way in which the usual plane waves of quantum mechanics are imbedded in $\mathcal{H}_0$.

In the Hilbert space $\mathcal{H} = \mathcal{L}^2(\mathrm{R}^n, dx^n)$ it is well-known that the von Neumann algebra generated by the group of unitary translation operators forms a maximal abelian von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$. This implies e. g. all translation invariant operators in $\mathcal{B}(\mathcal{H})$ belong to this von Neumann sub-algebra.

As quantum mechanics on $\mathcal{H} = \mathcal{L}^2(\mathrm{R}^n, dx^n)$ corresponds to considering on the algebra Δ, only Weyl states (i. e.; continuous with respect to q and p; compare with the definition of momentum states (Definition I.2) where only continuity with respect to q is required), we generalise this theorem to all pure states on the C*-algebra Δ.

THEOREM I.6. — Let ω be a pure state on the C*-algebra Δ; Δ_p the C*-subalgebra generated by the elements $\delta_{p,0} p \in \mathrm{R}^n$ and Δ_q the C*-subalgebra generated by the elements $\delta_{0,q} q \in \mathrm{R}^n$.

Then $\Pi_\omega(\Delta_p)''$ and $\Pi_\omega(\Delta_q)''$ are maximal abelian von Neumann sub-algebra's of $\Pi_\omega(\Delta)'' = \mathcal{B}(\mathcal{H}_\omega)$.

Proof. — We prove that $\Pi_\omega(\Delta_p)''$ is maximal abelian in $\mathcal{B}(\mathcal{H}_\omega)$; therefore we must prove that $\Pi_\omega(\Delta_p)' = \Pi_\omega(\Delta_p)''$.

The proof for $\Pi_\omega(\Delta_q)''$ is analogous.

As $\Pi_\omega(\Delta_p)$ is abelian $\Pi_\omega(\Delta_p) \subset \Pi_\omega(\Delta_p)'$ hence

$$\Pi_\omega(\Delta_p)'' \subset \Pi_\omega(\Delta_p)'$$

Remains to prove that the sets are equal. Suppose however that $\Pi_\omega(\Delta_p)'$ is strictly greater than $\Pi_\omega(\Delta_p)''$, then there exists a projection x in $\Pi_\omega(\Delta_p)'$, not belonging to $\Pi_\omega(\Delta_p)''$. Let $\{ x \}''$ be the von Neumann algebra generated by x i. e. $\{ x \}''$ is the set

$$\{ \alpha x + \beta \mathbb{1} \},$$

with arbitrary $\alpha, \beta \in \mathbb{C}$, then

$$\{ x \}'' \subset \Pi_\omega(\Delta_p)' \quad (a)$$

$$\{ x' \} \supset \Pi_\omega(\Delta_p)'' \quad (b)$$

$$\{ x \}'' \cap \Pi_\omega(\Delta_p)'' = \mathbb{C}\mathbb{1} \quad (c)$$

From (b) and the abelianness of $\Pi_\omega(\Delta_q)$

$$\Pi_\omega(\Delta_p)'' \cup \Pi_\omega(\Delta_q)'' \subset \Pi_\omega(\Delta_p)'' \cup \Pi_\omega(\Delta_q)' \subset \{ x \}' \cup \Pi_\omega(\Delta_q)'$$

From the purity of the state or equivalently the irreducibility of the representation Π_ω:

$$\mathcal{B}(\mathcal{H}_\omega) = (\Pi_\omega(\Delta_p)'' \cup \Pi_\omega(\Delta_q)'')'' \subset \{ \{ x \}' \cup \Pi_\omega(\Delta_q)' \}'' \subset \mathcal{B}(\mathcal{H}_\omega)$$

or :

$$\mathcal{B}(\mathcal{H}) = (\{ x \}' \cup \Pi_\omega(\Delta_q)')''$$

hence

$$\mathbb{C}\mathbb{1} = (\{ x \}' \cup \Pi_\omega(\Delta_q)')' = \{ x \}'' \cap \Pi_\omega(\Delta_q)'' \qquad (d)$$

From (c) and (d):

$$\{ x \}'' \cap (\Pi_\omega(\Delta_q)'' \cup \Pi_\omega(\Delta_p)'') = \mathbb{C}\mathbb{1}$$

hence

$$(\{ x \}' \cup (\Pi_\omega(\Delta_q)'' \cup \Pi_\omega(\Delta_p)'')')'' = \mathcal{B}(\mathcal{H}) \qquad (e)$$

Again as Π_ω is irreducible

$$(\Pi_\omega(\Delta_q)'' \cup \Pi_\omega(\Delta_p)'')' = \mathbb{C}\mathbb{1}$$

and (e) becomes

$$(\{ x \}' \cup \mathbb{C}\mathbb{1})'' = \{ x \}' = \mathcal{B}(\mathcal{H})$$

or $\{ x \}'' = \mathbb{C}\mathbb{1}$ contradicting the above assumption.

Hence

$$\Pi_\omega(\Delta_p)' = \Pi_\omega(\Delta_p)'' .$$

Q. E. D.

Remark. — The proof of Theorem I.6 constitutes also a purely algebraic proof of the above mentionned property of the group of translation acting on $\mathcal{L}^2(\mathbb{R}^n, dx^n)$.

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