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# A note on quantising Kolmogorov systems

by

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ABSTRACT. — It is shown that there exists a quantisation of classical systems of a certain mathematical structure. The method gives the usual commutation relations for the electro-magnetic field.

RÉSUMÉ. — Il existe une quantification des systèmes classiques d'une certaine structure mathématique. La méthode donne les relations de commutations ordinaires pour le champ électro-magnétique.

One method of quantising a classical system is to construct a Weyl system on it. A symplectic space (S, $\sigma$) is obtained from the classical system and a Weyl map W is defined from S to unitary operators on a Hilbert space $\mathscr{H}$, such that

$$W(s_1)W(s_2) = W(s_1 + s_2) \exp\left(-\frac{1}{2}\sigma(s_1, s_2)\right). \tag{1}$$

and R($s$), where $W(s) = e^{iR(s)}$, exists. The dynamics of the classical system is defined by a 1-parameter symplectic group U($t$) on S. If (S, $\sigma$, U($t$)) is Hilbertisable-i. e., S can be represented as a complex Hilbert space, $\sigma$ the imaginary part of the inner product and U($t$), a unitary group with positive generator, then Weinless [6] showed there exists a quantisation of the system with a unique vacuum state. This is the case if the classical system is a real Kolmogorov system.

A real Kolmogorov system is a triple $\{ H, U(t), D \}$ of a real Hilbert space H, $U(t) = e^{itC}$, a 1-parameter group of orthogonal operators on H, and D a subspace of H, with $U(t)D \subset D$, $t \geq 0$, $\bigcap_t U(t)D = \{ 0 \}$,

$\bigcup_t U(t)D = H$. Sinai [5] proved a representation theorem for such systems.

THEOREM. — For any K-system $\{ H, U(t), D \}$ there exists an isometric isomorphism $B : H \to L_2(-\infty, \infty ; N)$, unique up to isomorphism of the real Hilbert space N, so that

(*i*) B maps $U(t)$ onto translation by $t$ to the right.
(*ii*) $BD = L_2(0, \infty ; N)$.

It is a simple corollary of this theorem that such systems are Hilbertizable symplectic spaces-namely H is isomorphic to the complex Hilbert space $L_2(0, \infty ; N + iN)$ and $U(t)$ has a generator with positive spectrum. The equivalence is given by the map $T : L \to \sqrt{2}P_+\mathcal{F}Bh$, where $\mathcal{F}$ is the Fourier transform and $P_+$ is the projection operator from $L_2(-\infty, \infty ; N + iN)$ onto $L_2(0, \infty ; N + iN)$. $\sigma$ is chosen to be the bilinear form that becomes the imaginary part of the inner product under T. If W is the Weyl map over $L_2(0, \infty ; N + iN)$, the classical element $h \in H$ is identified with the quantum observable $R(C^{\frac{1}{2}}Th)$. Since $C^{\frac{1}{2}}$ is an unbounded operator it is necessary to restrict W to the subspace of H for which

$$\int_0^\infty \langle (Th)(x), (Th)(x) \rangle x dx < \infty \tag{2}$$

—the closure of the domain of $C^{\frac{1}{2}}$.

Such K-systems occur in classical scattering theory [*2*] and stationary stochastic processes [*1*, *3*]. An example of such a K-system is the free electro-magnetic field, as was outlined in [*2*] and [*4*]. The classical field is a solution of

$$\frac{\partial \underline{\Omega}}{\partial t}(\underline{x}, t) = A(\underline{x})\underline{\Omega}(\underline{x}, t) = \begin{pmatrix} 0 & \text{curl} \\ -\text{curl} & 0 \end{pmatrix}\underline{\Omega}(\underline{x}, t) \tag{3}$$

where $\underline{\Omega}^T(\underline{x}, t) = (\underline{E}(\underline{x}, t), \underline{H}(\underline{x}, t))$ and $\text{div}\,\underline{E} = \text{div}\,\underline{H} = 0$. Take H to be the space of initial data of (3) with finite energy. This is a Hilbert space with $\|\Omega\|^2 = \int |\underline{\Omega}(\underline{x}, 0)|^2 d\underline{x}$, and D is the space of data which are zero at $x = 0$ for all positive time. The K-system representation of this space as $L_2(-\infty, \infty ; L(S_2) \oplus L(S_2))$ where $L(S_2)$ is the space of bounded operators on the sphere, is given in [*2*] and [*4*]. If $f_i^t$ is an element of the initial data space so that at time $t$, $\underline{\Omega}_j(\underline{x}, t) = \delta_{ij}f(x)$, then the quantum field operators are given by $\underline{E}_i(f, t) = R(C^{\frac{1}{2}}Tf_i^t)$, and $\underline{H}_j(g, t) = R(C^{\frac{1}{2}}Tg_{j+3}^t)$. The commutation relations of the quantum system are given by the Weyl representation

$$\begin{aligned} [E_i(f, t), H_j(g, t')] &= [R(C^{\frac{1}{2}}Tf_i^t), R(C^{\frac{1}{2}}Tg_{j+3}^{t'})] \\ &= i\text{Im}\langle C^{\frac{1}{2}}Tf_i^t, C^{\frac{1}{2}}Tg_{j+3}^{t'} \rangle . \end{aligned} \tag{4}$$

These equations give the commutation relations normally associated with the electro-magnetic field. This method works for all K-systems, and in particular one can calculate the quantisation of the e. m. f. outside a perfect conductor.

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