ANNALES DE L'I. H. P., SECTION A

M. RAFIQUE
K. TAHIR SHAH

Invariant operators of inhomogeneous unitary group

Annales de l'I. H. P., section A, tome 21, nº 4 (1974), p. 341-345
<http://www.numdam.org/item?id=AIHPA_1974__21_4_341_0>

Ann. Inst. Henri Poincaré, Section A :
Vol. XXI, n° 4, 1974, p. 341-345. *Physique théorique.*

Invariant operators of inhomogeneous unitary group

by

M. RAFIQUE and K. TAHIR SHAH
International Centre for Theoretical Physics, Trieste, Italy

ABSTRACT. — Let $t_n \oplus u(n)$ be the semi-direct sum of an abelian Lie algebra with the Lie algebra of unitary group U(n). It is shown that an invariant of $u(n + 1)$ is also an invariant of $t_n \oplus u(n)$ if $\varphi : t_n \oplus u \to u(n + 1)$ is an expansion. Sixth-order invariants of semi-direct sum algebra are calculated by this method.

I. INTRODUCTION

In this note we discuss the problem of constructing invariants of inhomogeneous unitary group using a short-cut method which is specifically useful for the calculation of higher-order invariants and Casimir operators which are needed for the representation theory.

There are various methods of constructing invariants of the inhomogeneous unitary group. We use a method first given by Rosen [*1*] to compute some of the invariants of the inhomogeneous orthogonal group and which was later generalized by Nagel and Shah [*2*] to include all cases of inhomogeneous symplectic and orthogonal groups. We start from the Lie algebra of the group $T_n \otimes U(n_1, n_2)$ which is the semidirect product of n-dimensional translation with $U(n_1, n_2)$ group. This is expanded to $u(n_1, n_2 + 1)$ such that the generators of $u(n_1, n_2 + 1)$ are functions of the generators of $t_n \oplus u(n_1, n_2)$. Here t_n and $u(n_1, n_2)$ are the Lie algebras corresponding to the groups T_n and $U(n_1, n_2)$. From the Casimir operator of $u(n_1, n_2 + 1)$, which is easy to calculate, we extract invariants of $t_n \oplus u(n_1, n_2)$ by using a lemma given below.

In Sec. II, we describe the construction of the generators of the algebra $u(n_1, n_2 + 1)$ and give the commutation relations. Sec. III describes the method of calculation. In Sec. IV we apply this method to calculate the sixth-order invariants of $t_n \oplus u(n)$.

II. EXPANSION OF $t_n \oplus u(n_1, n_2)$ TO $u(n_1, n_2 + 1)$

In order to use the expansion process in the computation of invariants, we first have to define it. Let g be a Lie algebra. Consider the universal enveloping algebra $\varepsilon(\mathrm{I}g)$ of $\mathrm{I}g$ in its polynomial form. Here, $\mathrm{I}g$ represents an inhomogeneous algebra, being a semidirect sum of an abelian part I and a semisimple algebra g. Now extend the polynomial algebra $\varepsilon(\mathrm{I}g)$ to an algebra $\varepsilon_{\mathrm{E}}(\mathrm{I}g)$ such that not only complex numbers appear as coefficients but also functions $f(\mathrm{C}_1, \mathrm{C}_2, \ldots, \mathrm{C}_\mathrm{N})$ with $\mathrm{C}_1, \mathrm{C}_2, \ldots, \mathrm{C}_\mathrm{N}$ being elements in $\varepsilon(\mathrm{I}g)$ generating its centre. Then an injective homomorphism φ of a Lie algebra $\bar{g}$ into $\varepsilon_{\mathrm{E}}(\mathrm{I}g)$, i. e.,

$$\varphi : \bar{g} \rightarrow \varepsilon_{\mathrm{E}}(\mathrm{I}g)$$

is called an $\mathrm{I}g$ realization [3] in $\varepsilon_{\mathrm{E}}(\mathrm{I}g)$ or an expansion. One should note that, applying the later given mappings φ to the Lie algebras of the classical groups, one obtains $\bar{g}$ of the same type as g except their dimensions. The method is equally applicable to compact as well as noncompact groups. We consider $t_n \oplus u(n_1, n_2)$, where $n = n_1 + n_2$, is the real Lie algebra of complex unitary transformations of an n-dimensional vector space V_n. The generators of $t_n \oplus u(n_1, n_2)$ are

$$\begin{aligned} \mathrm{L}_{\mu\nu} &= -\,\mathrm{L}_{\nu\mu} \qquad (\mu, \nu = 1, 2, \ldots, n), \\ \mathrm{Q}_{\mu\nu} &= +\,\mathrm{Q}_{\nu\mu}, \end{aligned}$$

R_μ and S_μ, with the metric $g^{\mu\nu}$ given by

$$g^{\mu\nu} = \begin{bmatrix} \overbrace{\begin{matrix} +1 & & & \\ & +1 & & \\ & & \ddots & \\ & & & +1 \end{matrix}}^{n_1} & \\ & \overbrace{\begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix}}^{n_2} \end{bmatrix}$$

whereas the commutation relations are

$$\begin{aligned} [\mathrm{L}_{\mu\nu}, \mathrm{L}_{\lambda\sigma}] &= g_{\nu\lambda}\mathrm{L}_{\mu\sigma} - g_{\lambda\mu}\mathrm{L}_{\nu\sigma} + g_{\nu\sigma}\mathrm{L}_{\lambda\mu} - g_{\sigma\mu}\mathrm{L}_{\lambda\nu} \\ [\mathrm{L}_{\mu\nu}, \mathrm{Q}_{\lambda\sigma}] &= g_{\nu\lambda}\mathrm{Q}_{\mu\sigma} - g_{\lambda\mu}\mathrm{Q}_{\nu\sigma} + g_{\nu\sigma}\mathrm{Q}_{\lambda\mu} - g_{\sigma\mu}\mathrm{Q}_{\lambda\nu} \\ [\mathrm{Q}_{\mu\nu}, \mathrm{Q}_{\lambda\sigma}] &= -\,g_{\nu\lambda}\mathrm{L}_{\mu\sigma} - g_{\lambda\mu}\mathrm{L}_{\nu\sigma} + g_{\nu\sigma}\mathrm{L}_{\lambda\mu} + g_{\sigma\mu}\mathrm{Q}_{\lambda\nu} \\ [\mathrm{L}_{\mu\nu}, \mathrm{S}_\lambda] &= g_{\nu\lambda}\mathrm{S}_\mu - g_{\lambda\mu}\mathrm{S}_\nu \\ [\mathrm{Q}_{\mu\nu}, \mathrm{R}_\lambda] &= -\,g_{\nu\lambda}\mathrm{S}_\mu - g_{\lambda\mu}\mathrm{S}_\nu \\ [\mathrm{Q}_{\mu\nu}, \mathrm{S}_\lambda] &= g_{\nu\lambda}\mathrm{R}_\mu + g_{\lambda\mu}\mathrm{R}_\nu \\ [\mathrm{L}_{\mu\nu}, \mathrm{R}_\lambda] &= g_{\nu\lambda}\mathrm{R}_\mu - g_{\lambda\mu}\mathrm{R}_\nu \end{aligned}$$

Following [4], we construct the generators of the Lie algebra $u(n_1, n_2 + 1)$ as an $Iu(n)$ realization via the following mapping φ (the generators of $\varphi u(n_1, n_2 + 1)$ are denoted as $L'_{\mu\nu}$, etc.):

$$\text{Ⓐ} \begin{cases} L'_{\mu\nu} = L_{\mu\nu}\,; \qquad Q'_{\mu\nu} = Q_{\mu\nu} \\ L'_{n+1,\nu} = R_\nu + \dfrac{\lambda}{\sqrt{-T^2}}\bar{R}_\nu \\ Q'_{n+1,\nu} = S_\nu + \dfrac{\lambda}{\sqrt{-T^2}}\bar{S}_\nu \\ Q'_{n+1,n+1} = \dfrac{-V}{T^2} \end{cases}$$

where

$$T^2 = (g_{n+1,n+1})^{-1}(R_\mu R^\mu + S_\mu S^\mu)$$
$$\bar{R}_\nu = \{ S^\mu L_{\mu\nu} + R^\mu Q_{\mu\nu} \}$$
$$\bar{S}_\nu = \{ -R^\mu L_{\mu\nu} + S^\mu Q_{\mu\nu} \}$$

and

$$V = \{ -2R^\mu S^\nu L_{\mu\nu} + R^\mu R^\nu Q_{\mu\nu} + S^\mu S^\nu Q_{\mu\nu} \}.$$

The braces { , } mean that each term is to be symmetrized in R_μ and S_μ with respect to $L_{\mu\nu}$, $Q_{\mu\nu}$, $\bar{R}_\nu$ and $\bar{S}_\nu$ but not relative to one another, and divided by the number of terms needed for symmetrization. An invariant and a Casimir operator are defined as follows:

Invariant: let $I(L_{\mu\nu}, Q_{\mu\nu}, S_\mu, R_\mu)$ be a homogeneous polynomial in $L_{\mu\nu}$, $Q_{\mu\nu}$, S_μ and R_μ. Then I is said to be an invariant if it commutes with all $L_{\mu\nu}$, $Q_{\mu\nu}$, S_μ and R_μ. The order of an invariant is the order of the polynomial.

Casimir operator: a Casimir operator is an invariant which cannot be expressed as a linear combination of lower-order invariants. In general, an m^{th}-order Casimir operator $C_m(G)$ for a group G is

$$C_m(G) = X_{\mu_1\mu_2}X_{\mu_2\mu_3} \dots X_{\mu_m\mu_1}$$

where $X_{\mu_i\mu_j}$ are the generators of the Lie algebra of G.

III. COMPUTATION OF INVARIANTS OF $t_n \oplus u(n_1, n_2)$

Let $g_i \in t_n \oplus u(n_1, n_2)$ which is expanded to $u(n_1, n_2 + 1)$ and let $h_\alpha \in u(n_1, n_2 + 1)$ such that $h_\alpha = h_\alpha(g_1, g_2, \dots, g_N)$ where $N = n(n + 1)/2$.

The Casimir operators of $u(n_1, n_2 + 1)$ are functions of $g_1, \dots, g_N$ since

$$C_m(h_1, \dots, h_M) = C_m(g_1, \dots, g_N)$$

where C_m is the m^{th}-order Casimir operator of $u(n_1, n_2 + 1)$ and M is the number of generators of $u(n_1, n_2 + 1)$. Let X_1 be an invariant of $t_n \oplus u(n_1, n_2)$, then

$$[X_1, L_{\mu\nu}] = 0 = [X_1, Q_{\mu\nu}]; \qquad [X_1, R_\mu] = 0 = [X_1, S_\mu].$$

Let X_2 be an invariant of $u(n_1, n_2 + 1)$, then we have

$$[X_2, L'_{\mu\nu}] = 0; \qquad [X_2, Q'_{\mu\nu}] = 0; \qquad [X_2, L'_{\mu,n+1}] = 0;$$
$$[X_2, Q'_{\mu,n+1}] = 0; \qquad [X_2, Q'_{n+1,n+1}] = 0.$$

LEMMA. — *If $u(n_1, n_2 + 1)$ is realized in $\varepsilon_E(t_n \oplus u(n_1, n_2))$ through the expansion φ, then any operator X commuting with $h_\alpha \in u(n_1, n_2 + 1)$ also commutes with $g_i \in t_n \oplus u(n_1, n_2)$.*

Proof. — The generators $L'_{\mu\nu}$, $Q'_{\mu\nu}$, $L'_{\mu,n+1}$ and $Q'_{\mu,n+1}$ are, by construction, functions of $L_{\mu\nu}$, $Q_{\mu\nu}$, S_μ and R_μ. One can verify that any operator X which satisfies the relation satisfied by X_2 also satisfies the relation satisfied by X_1. We show that $[R_\mu, X]$ and $[S_\mu, X]$ are zero as follows. Consider the commutator $[Q'_{n+1,n+1}, X]$ and substitute for $Q'_{n+1,n+1}$ to get

$$[Q'_{n+1,n+1}, X] = \frac{1}{T^2}[\{ -2R^\mu S^\nu L_{\mu\nu} + R^\mu R^\nu Q_{\mu\nu} + S^\mu S^\nu Q_{\mu\nu} \}, X]$$

which reduces to an expression involving commutators of the type $[R^\mu S^\nu, X]$, $[R^\mu R^\nu, X]$ and $[S^\mu S^\nu, X]$ because $[L_{\mu\nu}, X]$ and $[Q_{\mu\nu}, X]$ vanish due to Ⓐ. Now $[L_{n+1,\nu}, X] = 0$ implies that

$$[\lambda^{-1}R_\nu + \{ S^\mu L_{\mu\nu} + R^\mu Q_{\mu\nu} \}, X] = 0$$

or

$$\lambda^{-1}[R_\nu, X] + [\{ S^\mu L_{\mu\nu} + R^\mu Q_{\mu\nu} \}, X] = 0.$$

Notice that we have divided by λ the last three equations of Ⓐ. Now we may remark that if $X = X(\alpha) = \sum_i \alpha^i X_i$; $i = 0, 1 \ldots$ with parameter α, and if I is another operator independent of α, then I commutes with X_i if and only if it commutes with $X(\alpha)$. This means that

$$[R_\nu, X] = 0 \qquad \text{and} \qquad [\{ S^\mu L_{\mu\nu} + R^\mu Q_{\mu\nu} \}, X] = 0.$$

Using further the commutation relations of $L_{\mu\nu}$ and $Q_{\mu\nu}$ with X, we have also $[S^\mu, X] = 0$. So, if X is an invariant of $u(n_1, n_2 + 1)$ then it is also an invariant of $t_n \oplus u(n_1, n_2)$. Q. E. D.

IV. SIXTH-ORDER INVARIANTS OF $t_n \oplus u(n_1, n_2)$

Using the method developed in Sec. III, starting from the second-order Casimir operator of $u(n_1, n_2 + 1)$ we determine invariants of second, fourth and sixth order for the Lie algebra $t_n \oplus u(n_1, n_2)$. The second-order Casimir operator of $u(n_1, n_2 + 1)$ is

$$C_2(u(n_1, n_2 + 1)) = C_2(u(n_1, n_2)) + 2Q'_{n+1,\nu}Q'_{\nu,n+1} - 2L'_{n+1,\nu}L'_{\nu,n+1} + (Q'_{n+1,n+1})^2 .$$

In polynomial form one can have

$$C_2(u(n_1, n_2 + 1)) = \sum_i \mu^i A_i$$

where μ is some parameter related to λ. Substituting from equations Ⓐ and using commutation relations we have the following invariants (all the indices are written as subscripts because for simplicity we take $g_{\mu\nu} = \delta_{\mu\nu}$):

$$A_1 = 2[S_\mu S_\mu - R_\mu R_\mu]$$

$$A_2 = 2[S_\mu S_\nu Q_{\mu\nu} - R_\mu R_\nu Q_{\mu\nu} + (2n + 1)R_\mu S_\mu - R_\mu S_\nu L_{\mu\nu}]$$

$$\begin{aligned} A_3 = {} & 2[8S_\mu R_\lambda Q_{\mu\nu} L_{\lambda\nu} - S_\mu S_\lambda L_{\mu\nu} L_{\lambda\nu} - R_\mu R_\lambda Q_{\mu\nu} Q_{\lambda\nu} - 2R_\mu S_\lambda Q_{\mu\nu} L_{\lambda\nu} \\ & + R_\mu S_\mu Q_{\lambda\lambda} + (n + 2)R_\mu S_\nu Q_{\mu\nu} + (7n + 9)(n - 1)S_\mu S_\mu + n(n + 1)R_\mu R_\mu] \\ & + 4R_\mu R_\lambda S_\nu S_\sigma L_{\mu\nu} L_{\lambda\sigma} - 4R_\mu R_\lambda R_\sigma S_\nu L_{\mu\nu} Q_{\lambda\sigma} - 4R_\mu S_\nu S_\lambda S_\sigma L_{\mu\nu} Q_{\lambda\sigma} \\ & + R_\mu R_\nu R_\lambda R_\sigma Q_{\mu\nu} Q_{\lambda\sigma} + 2R_\mu R_\nu S_\lambda S_\sigma Q_{\mu\nu} Q_{\lambda\sigma} + S_\mu S_\nu S_\lambda S_\sigma Q_{\mu\nu} Q_{\lambda\sigma} \\ & + 4R_\mu R_\nu R_\sigma S_\nu Q_{\mu\sigma} - 4R_\mu R_\sigma R_\mu S_\nu Q_{\nu\sigma} + 4R_\mu S_\nu S_\sigma S_\nu Q_{\mu\sigma} - 4R_\mu S_\mu S_\nu S_\sigma Q_{\nu\sigma} . \end{aligned}$$

ACKNOWLEDGMENTS

The authors wish to thank Prof. H. D. Doebner for correcting many errors, for discussion and critical reading of the manuscript. Thanks are due to Prof. Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

REFERENCES

[1] J. Rosen, *J. Math. Phys.*, t. **9**, 1968, p. 1305; *Nuovo Cimento*, t. **46B**, 1966, p. 1.

[2] J. G. Nagel and K. T. Shah, *J. Math. Phys.*, t. **11**, 1970, p. 1483.

[3] H. D. Doebner and T. D. Palev, ICTP, Trieste, preprint IC/71/104.

[4] J. G. Nagel, *Ann. Inst. Henri Poincaré*, t. **13**, 1970, p. 1.

(Manuscrit reçu le 17 décembre 1973)