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Section A
Physique théorique.

On the approach to the critical point (1)

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ABSTRACT. — The mass perturbation yields a new construction for $\mathscr{P}(\varphi)_2$ = Even + Linear quantum field models. For φ^4 models, the Schwinger functions are differentiable in the parameter $\sigma = \delta m^2$, in the single phase region, and are continuous as $\sigma \searrow \sigma_c$. For the Ising_d model, $d \geq 2$ classical bounds on the critical exponents, *e. g.* $1 \leq \gamma \leq 2\nu$, are a consequence of an Ornstein-Zernike type upper bound on the two point function.

1. INTRODUCTION

For certain critical values of the parameters, the Schwinger functions of quantum field theory or correlation functions of statistical mechanics are expected to exhibit nonexponential decay,

$$S^{(2)}(x) \equiv \int \varphi(x)\varphi(0)dq \sim |x|^{-d+2-\eta}$$

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where $0 \leq \eta \leq 2$ and with the result that the corresponding integrated quantities, for example the susceptibility,

$$\chi \equiv \int S^{(2)}(x)dx$$

become infinite. Let $S^{(n)}(\sigma; x)$ denote the dependence of the n-point Schwinger function in an even $\mathscr{P}(\varphi)_2$ model on the mass perturbation

$$\frac{1}{2}\sigma \int :\varphi(x)^2: dx.$$

An elementary application of Griffiths' inequalities gives three ranges of parameter values as follows: $\sigma < \sigma_c^-$ defines a multiple phase theory, $\sigma_c^- < \sigma < \sigma_c^+$ defines a zero mass (critical) theory, and $\sigma_c^+ < \sigma$ defines a positive mass, single phase theory. It is expected that $\sigma_c^- = \sigma_c = \sigma_c^+$; however in some cases, σ_c may not define a zero mass theory, see chapter 2.

As $\sigma \to +\infty$, the limit $S^{(n)}(\sigma, x) \to 0$, $n \geq 1$, follows from the cluster expansion [*15*] [*16*]. Callan and Symanzik, see [*2*] [*29*], propose the mass perturbation, as in the formula

$$S^{(n)}(\sigma_c, x) = -\int_{\sigma_c}^{\infty} \partial_\sigma S^{(n)}(\sigma, x)d\sigma, \tag{1.1}$$

for the study of critical phenomena. In fact with g denoting the physical charge, hypotheses *e. g.* on

$$\beta(g) \equiv \frac{m}{2}\left(\frac{dg}{d\sigma}\frac{d\sigma}{dm}\right)_{\mathscr{P}=\text{const.}}$$

near $\sigma = \sigma_c$ lead to the scaling law and calculation of critical exponents in terms of *e. g.* $\beta'(g_c)$, see [*3*] [*24*].

In this paper, we show that the derivatives in (1.1) exist for φ^4 models in the single phase region, as a consequence of the Lebowitz inequalities [*12*]. The Lebowitz inequalities dominate $\partial S^{(n)}/\partial\sigma$ by a sum of products of two point functions,

$$0 \leq -\partial S^{(n)}/\partial\sigma \leq \sum \prod_{i,j\neq k,l} S^{(2)}(x_i - x_j) \times \int S^{(2)}(x_k - y)S^{(2)}(x_l - y)dy, \tag{1.2}$$

see chapter 3.

2. CONSTRUCTION OF THE $\mathscr{P}$ = EVEN − $\mu\varphi$ MODELS

We give a new construction of the $\mathscr{P}$ = Even + Linear model in two dimensions. We start from a polynomial $\mathscr{P}$ with a sufficiently large bare mass, so that it yields a Wightman theory as a consequence of a convergent cluster expansion [15] [16]. We study the associated Schwinger functions under a two parameter family of perturbations, of the form $\frac{1}{2}\sigma\varphi^2 - \mu\varphi$. Using a Griffiths inequality, we show that for $\sigma \leq 0$ and $\mu \geq 0$, the Schwinger functions converge monotonically to Schwinger functions for an Osterwalder-Schrader theory [23], with the possible exception of clustering. In the case of mixed phases, the decomposition into pure phases satisfying all the Osterwalder-Schrader axioms is given in [1] [5]. Each pure phase then gives rise to a $\mathcal{P}(\varphi)_2$ Wightman theory.

We consider the perturbation for the action,

$$\delta V = \frac{1}{2}\sigma\int_{\Lambda_1} :\varphi^2: dx - \mu\int_{\Lambda_2}\varphi dx, \tag{2.1}$$

where $\mu, -\sigma \geq 0$, and where Λ is a bounded space-time region with volume $|\Lambda|$. We suppose that the linear term in $\mathcal{P}$ has a nonpositive coefficient.

PROPOSITION 2.1. — Under the perturbation δV, the $\mathcal{P}(\varphi)_2$ Schwinger functions increase monotonically in $-\sigma, \mu, |\Lambda_i|$.

Proof [18]. — By the first Griffiths inequality, $S^{(n)}(X) \geq 0$, and by the second Griffiths inequality

$$-\frac{d}{d\sigma}S^{(n)}(X) = \int_{\Lambda_1}[S^{(n+2)}(Xzz) - S^{(n)}(X)S^{(2)}(zz)]dz \geq 0. \tag{2.2}$$

For the lattice approximation to field theory, the Griffiths inequalities follow from [6], for $\mathcal{P}$ = even $-\mu\varphi$, $\mu \geq 0$. These inequalities follow in field theory from convergence of the lattice approximation ([18] for $\mathcal{P}(\varphi)_2$, [25] for φ_3^4). Here we note the Wick ordering constant cancels between the two terms on the right of (2.2). This proves that $S^{(n)}$ increases monotonically in $-\sigma$, $|\Lambda_1|$. The proof of monotonicity in μ, $|\Lambda_2|$ is similar.

We remark that the Schwinger functions are uniformly bounded in Λ_i, σ, and μ, so long as σ and μ remain bounded (This is a consequence of Hamiltonian φ-bounds [8]. See Fröhlich [5] for the translation of these bounds to the Schwinger functions). Hence the Schwinger functions converge as $|\Lambda_i| \to \infty$, as $\sigma \searrow \bar{\sigma} > -\infty$, and as $\mu \nearrow \bar{\mu} < \infty$. Clearly the limiting Schwinger functions are Euclidean covariant, and the remaining Osterwalder-Schrader axioms (except clustering) follow from the properties of the approximate Schwinger functions.

THEOREM 2.2. — For each polynomial $\mathcal{P}(\varphi)$ with a convergent cluster expansion, we obtain a two parameter family of (possibly mixed phase) Osterwalder-Schrader theories, with Schwinger functions $S^{(n)}(\sigma, \mu; X)$.

Each theory $\{ S^{(n)}(\sigma, \mu; X) \}$ can be decomposed into a direct integral of pure phases, with each component an Osterwalder-Schrader theory.

REMARK 2.3. — We note that the theories $S^{(n)}(\sigma, \mu; X)$ are uniquely defined by the bare parameters, together with the bare mass. Thus with $\sigma_i \leq 0$, $\mu_i \geq 0$, we can add successively the perturbations

$$\delta V_i = \frac{1}{2}\sigma_i \int_{\Lambda_i} : \varphi^2 : dx - \mu_i \int_{\Lambda_i} \varphi dx,$$

taking the limit $\Lambda_i \nearrow R^2$ in any order, and obtain $S^{(n)}(\sigma, \mu; X)$, where $\sigma = \Sigma\sigma_i$, and $\mu = \Sigma_i\mu_i$. The fact that the limit $S^{(n)}(\sigma, \mu; X)$ is independent of the order in which the perturbations are added has the following interpretation. If a phase transition occurs at the parameter values σ, μ, we can regard $S^{(n)}(\sigma, \mu; X)$ as defined by a δV_i perturbation, starting from the boundary conditions $S^{(n)}(\sigma^{(i)}, \mu^{(i)}; X)$, where $\sigma^{(i)} = \Sigma_{j\neq i}\sigma_j$ and $\mu^{(i)} = \Sigma_{j\neq i}\mu_j$. Since $S^{(n)}(\sigma, \mu; X)$ is independent of the order in which the δV_i perturbations are added, the various boundary conditions $\sigma^{(i)}$, $\mu^{(i)}$ define the same theory. Allowing for an infinite series σ_j, μ_j, we see that

$$S^{(n)}(\sigma, \mu ; X) = \lim_{\sigma \searrow \sigma', \mu' \nearrow \mu} S^{(n)}(\sigma', \mu' ; X)$$

Because of the possibility of phase transitions, the iterated limits with some or all $\mu_i \leq 0$ may not coincide with the $S^{(n)}(\sigma, \mu; X)$. In fact the iterated limits with each $\sigma_i\mu_i \leq 0$ are always defined (in any order) by monotonicity, and should provide a general set of boundary conditions. In particular, for φ^4, both pure phases as well as the even mixture could be constructed by this method. From this point of view, the construction of Theorem 2.2 with a single δV perturbation could be called *weak coupling boundary conditions.*

REMARK 2.4. — For φ^6 and higher order $\mathscr{P}(\varphi)$ theories, it is possible that curves of phase transitions can occur in the σ, μ plane for $\mu \neq 0$. Because the above iterated limits coincide, any such curve in the $\mu > 0$ half plane must have a positive slope in the σ, μ plane, at least in the case of a first order phase transition with respect to order parameter $\langle \varphi \rangle$. Standard mean field approximations suggest that there should be a phase transition only at $\mu = 0$ if the φ^4, φ^6, ... coefficients in an even $\mathscr{P}$ are fixed and $\sigma \ll 0$. This picture also suggests that the curves for $\mu > 0$ or $\mu < 0$ move in toward and meet the $\mu = 0$ line as σ decreases. It is consistent with this picture that $\sigma_c^+ = \sigma_c^-$ (as defined in chapter 3) could give the value of σ for which two such curves meet the $\mu = 0$ line. In this case it is reasonable to suppose that $m(\sigma_c) > 0$. See figure 1. The variation of the φ^4 and higher coefficients is expected to lead to tricritical points, see [*26*]. There heuristic calculations indicate classical tricritical exponents for $d = 3$, in contrast to an ordinary critical point.

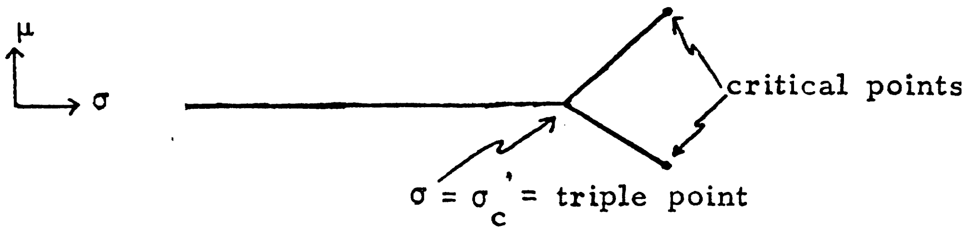

FIG. 1. — Postulated phase transition curves for $\varphi^6 + \lambda\varphi^4$ models, $\lambda \ll 0$.

REMARK 2.5. — The above construction (weak coupling boundary conditions) coincides with the theory defined by Dirichlet boundary data. In fact if the weakly coupled $\mathcal{P}(\varphi)_2$ theory defined above by a cluster expansion has Dirichlet data on the boundary of a region Λ_0, then $S(\sigma)$ is monotone increasing in Λ_0. The limits $\Lambda_0 \nearrow R^2$, $\Lambda_1 \nearrow R^2$, $\Lambda_2 \nearrow R^2$ may be taken in any order. With $\Lambda_0 \nearrow R^2$ first, we have weak $\mathcal{P}(\varphi)_2$ boundary conditions, since the cluster expansion defines a theory which is independent of boundary conditions. With $\Lambda_0 = \Lambda_1 = \Lambda_2 \nearrow R^2$, we have the conventional Dirichlet theory.

The weak coupling boundary conditions also coincide with free boundary data, provided only that $m_0 \gg 1$ and that the space cutoff in the mass term $\frac{1}{2}\sigma\int :\varphi^2: dx$ is removed after the space cutoff in the other terms of $H_I = \int \mathcal{P}(\varphi)dx$. It follows that the estimates of [*8*], established for the C*-algebra construction of $\mathcal{P}(\varphi)_2$ field theory [*7*], also have consequences for the theory constructed here; see section 5. An extensive analysis of even boundary conditions is given in [*19*]. It is an open question whether the periodic mass converges in the infinite volume limit to the mass of the infinite volume theory. A positive answer to this question would show that for φ^4, $m(\sigma) \searrow = 0$ as $\sigma \searrow \sigma_c$.

3. THE CRITICAL POINT IN EVEN $\mathcal{P}(\varphi)_2$ MODELS

The variation of the mass parameter σ in an even $\mathcal{P}(\varphi)_2$ model (see chapters 1, 2) yields three characteristic intervals:

a) a single phase interval (σ_c^+, ∞),
b) a critical interval (σ_c^-, σ_c^+),
c) a multiple phase interval $(-\infty, \sigma_c^-)$ with symmetry breaking.

The endpoints $\sigma_c^\pm$ may also be included in the intervals *a*) *c*).

We let $\mu = 0$, and let

$$M(\sigma)^2 = \lim_{|x|\to\infty} S^{(2)}(\sigma\,;\,x).$$

We define $m(\sigma)$ as the exponential decay rate of $S^{(2)}(\sigma; x) - M(\sigma)^2$. Consider the two conditions,

$$m(\sigma) > 0 \tag{3.1}$$

$$M(\sigma) = 0. \tag{3.2}$$

DEFINITION 3.1. — Let σ_c^+ be the infimum of the values of σ satisfying both (3.1) and (3.2). Let σ_c^- be the supremum of the values of σ which violate (3.2).

REMARK 3.2. — With the above definitions:

(*i*) $\sigma_c^- \leq \sigma_c^+ < \infty$.
(*ii*) On $(-\infty, \sigma_c^-)$, (3.2) is violated.
(*iii*) On (σ_c^-, σ_c^+), (3.2) holds, but $m = 0$.
(*iv*) On (σ_c^+, ∞), both (3.1) and (3.2) hold.

By scaling, $\sigma \to +\infty$ is the weak coupling limit. For weak coupling, the cluster expansion [*15*] [*16*] yields (3.1) and (3.2), so $\sigma_c^+ < \infty$. The remaining statements follow by the monotone decrease of $S^{(2)}(x; \sigma)$ in σ. The bound $|\sigma_c^-| < \infty$ is announced in [*4*]; see also [*5*].

THEOREM 3.3. — Let $\mu = 0$. Then $m(\sigma)$ is continuous on (σ_c^+, ∞) at all points of continuity of $S^{(2)}(\sigma; x)$.

THEOREM 3.4. — Let $\mu = 0$, and suppose (3.2) holds for $\sigma = \sigma_c^+$. Then $m(\sigma) \searrow m(\sigma_c^+)$ as $\sigma \searrow \sigma_c^+$.

THEOREM 3.5. — Let $\mu = 0$ and suppose $m(\sigma) \not\to 0$ as $\sigma \searrow \sigma_c^+$. Then (3.2) holds for $\sigma = \sigma_c^+$, and so $m(\sigma) \searrow m(\sigma_c^+)$ as $\sigma \searrow \sigma_c^+$.

REMARK 3.6. — Assuming φ_3^4 to be a Wightman theory, $M(\sigma_c) = 0$. This follows from the fact that $S^2(\sigma; x) \leq |x|^{-1}$. Thus $m(\sigma) \searrow m(\sigma_c^+)$.

REMARK 3.7. — See Figure 1. In that case, the limit $\sigma \searrow \sigma_c$ should yield the pure phase associated with $M(\sigma_c) = 0$, but with $m(\sigma_c) > 0$.

REMARK 3.8. — $\mathcal{P}(\varphi)_2$ critical theories, *i. e.* $m = 0 = M$, are not scale invariant. The continuum of such theories is parameterized by a scale length, see ([*16*], p. 155).

Proof of Theorems 3.3-3.5. — Since $S^{(2)}$ is monotone decreasing for $\sigma \in (-\infty, \infty)$, and continuous from above, it follows that $m(\sigma)$ is monotone increasing on (σ_c^-, ∞). If σ_0 satisfies (3.2), then $m(\sigma) \searrow m(\sigma_0)$ as $\sigma \searrow \sigma_0$, for the exponential decay rate of $S^{(2)}$ defines $m(\sigma_0)$. This proves Theorem 3.4. The same argument applied to convergence from below completes the proof of Theorem 3.3. Theorem 3.5 follows from the observation that if $m \not\to 0$, then $S^{(2)}$ has a uniform exponential decay as $\sigma \searrow \sigma_c$.

THEOREM 3.9. — For a φ_2^4 theory with $\mu = 0$, the derivatives $\partial S^{(n)}(\sigma; x)/\partial\sigma$ exist on (σ_c^+, ∞), and are bounded by (1.2). Furthermore $\partial Z(\sigma, f)/\partial\sigma$ exists, where $Z(\sigma, f) = \langle e^{\Phi(f)} \rangle_\sigma$, and $| f |_{L_p} \ll 1$.

Proof. — We use the Lebowitz inequalities

$$S^{(n+2)}(12U) - S^{(n)}(U)S^{(2)}(12) \leq \sum_{X\,\text{odd}} S(1X)S(2 \sim X), \tag{3.3}$$

where X is a subset of U, see [*12*, eq. 5]. By induction, the right side of (3.3) can be bounded by a sum of products of two point functions of the stated form. The sum in (1.2) ranges over the $\leq (n+1)!$ terms which arise from this bound. The inductive hypothesis we use is the following: $S^{(2n)}$ is bounded by a sum of at most $(2n-1)!$ products of n two point functions.

We first differentiate with respect to a change in σ in a finite volume Λ. By Remark 2.3, with $\delta\sigma \leq 0$,

$$\begin{aligned} &S^{(n)}(\sigma) - S^{(n)}(\sigma + \delta\sigma) \\ &\quad = \lim_{\Lambda\to\infty} \int_{\sigma+\delta\sigma}^{0} \int_{\Lambda} [S^{(n+2)}(s, \Lambda, z, z) - S^{(n)}(s, \Lambda, U)S^{(2)}(s, \Lambda, z, z)]dzds \end{aligned}$$

where $S(s, \Lambda)$ has a mass perturbation σ in all of R^2, and a mass perturbation $s - \sigma$, $\delta\sigma \leq s - \sigma \leq 0$, in Λ. We substitute (3.3) and (1.2) as an upper bound for the integrand. By the Lebesgue monotone convergence theorem, we may take the limit $\Lambda \to \infty$ under the integral sign. Convergence of the z integration follows from the positive mass and exponential decay of $S^{(2)}$.

The derivative of $Z(\sigma, f)$ exists by summation of the exponential series.

REMARK 3.10. — The existence of the derivatives of the connected Euclidean Green's functions $S_T^{(n)}(x) \equiv G^{(n)}(x)$ follows from this result. The derivative $dG^{(n)}/d\sigma$ is bounded in the norms of [*9*].

THEOREM 3.11. — For a φ_2^4 theory with $\mu = 0$, the Euclidean vertex functions $\Gamma^{(n)}(x)$ have bounded derivatives $d\Gamma^{(n)}(x)/d\sigma$, on the interval (σ_c^+, ∞).

Proof. — By the definition of the $\Gamma^{(n)}(x)$ in terms of the $G^{(n)}(x)$ and $\Gamma^{(2)} = -G^{(2)-1}$, and Remark 3.10, it is sufficient to prove the boundedness of $d\Gamma^{(2)}/d\sigma$ as a map from H_1 to H_{-1}. By definition,

$$\frac{d\Gamma^{(2)}}{d\sigma} = \Gamma \frac{dG^{(2)}}{d\sigma} \Gamma.$$

Since Γ is a bounded transformation from H_1 to H_{-1}, we need only show that $dG^{(2)}/d\sigma$ is bounded from H_{-1} to H_1. However, as in (1.2) or [11],

$$0 \leq -\frac{dG^{(2)}(x)}{d\sigma} \leq \int G^{(2)}(x-z)G^{(2)}(z)dz. \tag{3.4}$$

For $\sigma \in (\sigma_c^+, \infty)$, we have $m(\sigma) > 0$, so the Fourier transform $G^{(2)\sim}(p)$ is bounded by $m(\sigma)^{-2}$. Since $G^{(2)}(x)$ is bounded from H_{-1} to H_1, it follows that $G^{(2)}{}_*G^{(2)}$ is also. Hence so is $dG^{(2)}/d\sigma$, by (3.4).

4. THE ISING MODEL CRITICAL POINT

In this section we obtain bounds on critical exponents in Ising models. The analysis of Section 3 may also be applied to the Ising model, with $\beta \nearrow \beta_c$ replacing $\sigma \searrow \sigma_c$. The bound on $dS^{(n)}/d\beta$ by a product of two point functions is known [20]. In our 'analysis of the Ising model we assume an Ornstein-Zernike upper bound on the two point function,

$$S^{(2)}(\beta\,;x) \leq \text{const.} \int \frac{e^{-ipx}}{p^2+m^2}dp = S_0^{(2)}(m^2\,;x) \tag{4.1}$$

where the constant is independent of β and where $m = \xi^{-1}$ is the inverse correlation length. In field theory, (4.1) follows from the Lehmann spectral formula. From this hypothesis, we have the bound $\eta \geq 0$ on the anomalous dimension. We now use (4.1) to derive other bounds on critical exponents.

THEOREM 4.1. — Assume the bound (4.1). Then:

(*a*) For the ferromagnetic Ising model with $\beta < \beta_c$, $d \geq 2$,

$$0 \leq \frac{d\chi}{d\beta} \leq 0(1)\chi^2. \tag{4.2}$$

(*b*) Assuming the existence of critical exponents, they satisfy

$$1 \leq \gamma \leq 2\nu \tag{4.3}$$

$$\Delta_4 \leq \frac{1}{2}(d\nu + \gamma) \tag{4.4}$$

$$\Delta_4 + \Delta_6 \leq d\nu + \gamma \tag{4.5}$$

REMARK 4.2. — The inequality

$$0 \leq -\frac{dm}{d\beta} \leq 0(m^{-1}) \tag{4.6}$$

also follows by slight modification of [11], using a transfer matrix formalism [22]. The bounds (4.1-6) hold for a φ^4 field theory.

Proof. — The inequality (4.2) follows as in [*11*], and yields $1 \leq \gamma$. Combined with (4.1) this yields (4.3). The inequalities (4.4) and (4.5) are a consequence of (4.1) and the bounds of [*13*].

5. PRESERVATION OF ESTIMATES

In this section, we sketch how to transfer uniform estimates, *e. g.* estimates of the form

$$e^{-(\mathrm{H}\pm\mathrm{A})} \leq \text{const.}$$

established for $\mathrm{V} < \infty$, to the infinite volume limit. This justifies the statement in Remark 2.5 that the Hamiltonian estimates [*8*] can be used in the infinite volume limit with weak coupling boundary conditions.

We are concerned with perturbations of the form

$$\mathrm{A}_j = \int : \varphi^j : (x) h_j x dx, \tag{5.1}$$

with $j \leq \deg \mathscr{P} \equiv d$, with suppt. $h_j \subset [-a, a]$, and with $h_j \in \mathrm{L}_{d/d-j}$, see [*5*]. In case $j = d$, we also require $\| h_j \|_\infty$ small. We also introduce the mass perturbation

$$\mathrm{B}_2 = \frac{1}{2}\sigma \int_{-b}^{b} : \varphi^2 : (x) dx. \tag{5.2}$$

The estimates will be formulated in semigroup language, because the properties [*5*] of

$$z \to e^{-t(\mathrm{H}+z_2\mathrm{B}_2+z_j\mathrm{A}_j)} \tag{5.3}$$

permit control of the semigroups without considering questions of operator domains. We use the path space notation implicit in the fact that the theories constructed here are Osterwalder-Schrader reconstructions [*23*].

THEOREM 5.1. — The Feynman-Kac density

$$e^{-\int_0^{\mathrm{T}}[\mathrm{B}_2(q(t))+\mathrm{A}_j(q(t))]dt}$$

defines a semigroup on the physical Hilbert space $\mathscr{H}$.

Proof. — This statement follows by limits from the approximate theories, given the convergence of the weak coupling Schwinger functions with φ^j vertices, and the uniform bounds on the analytic functions (5.3).

In particular, with b and a fixed, the bounds of [*8*] are transfered into the weak coupling infinite volume limit by the argument of this section. See Remark 2.9.

The semigroup of Theorem 5.1 will be called $\exp[-t\mathrm{H}(\sigma, j)]$. We identify

$$\mathrm{H}(\sigma, j) = \mathrm{H} + \mathrm{B}_2 + \mathrm{A}_j \tag{5.4}$$

on the domain

$$\mathscr{D} = \{ e^{-sH}\mathscr{R}\Omega : s > 0 \}$$

where $\mathscr{R}$ is a polynomial in fields and Ω is the ground state of H. By the reconstruction theorem [23]. $\mathscr{D} \subset D(H)$ and by the cluster expansion $\mathscr{D} \subset D(B_2) \cap D(A_j)$. The limits of the difference quotients from finite volume then establish (5.4) on $\mathscr{D} \times \mathscr{D}$, and hence by the above domain inclusions (5.4) holds on $\mathscr{D}$. In the case of the mass perturbation, $H + B_2$, (5.4) extends to $D(H) \times D(H)$ by Spencer's local N_τ estimate [28]. The fact that $N_{\tau,\text{loc}} \geq 0$ allows the transfer of the first order estimate on $N_{\tau,\text{loc}}$ to the infinite volume limit [1]. Let

$$\delta E(\sigma, b, h_j) = - \ln \| \exp [- H(\sigma, j)] \|, \tag{5.5}$$

$$\delta E(\sigma, b) = \delta E(\sigma, b, \phi) = \inf \text{spect.} (H + B_2). \tag{5.6}$$

We normalize the Wick ordering so that

$$\langle : \varphi^2 : \rangle = 0, \tag{5.7}$$

with expectations taken in the weak coupling physical Hilbert space. Then $\langle : \varphi^j : \rangle = 0$, with the standard definition of Wick order. The absence of domain difficulties follows from the cluster expansion, see [27].

Let $\delta E^*(\sigma, b, h_j)$ be the lim sup of the finite volume (weak coupling) approximates to $\delta E(\sigma, b, h_j)$. Then

$$- \delta E \leq - \delta E^*, \tag{5.8}$$

as in the proof of Theorem 5.1. Our main result is

THEOREM 5.2. — With constants uniform in b,

$$- \delta E(\sigma, b, h_j) \leq - \delta E(\sigma, b) + 0\left(a + \| h_j \|_{d/d-j}^{d/d-j}\right). \tag{5.9}$$

LEMMA 5.3.

$$- \delta E(\sigma, b, h_j) = \lim_{T \to \infty} (2T)^{-1} \ln \int e^{-\int_{-T}^{T}[B_2 + A_j]dt} dq. \tag{5.10}$$

Proof. — Let ψ be a bounded function on path space, measurable with respect to the σ-algebra generated by $\phi(x, t)$ for $t \geq 0$, let Θ denote reflection about $t = 0$, and let $\psi \to \psi_T$ denote a time translation. Then

$$\begin{aligned} - \delta E(\sigma, b, h_j) &= \sup_{\psi} \lim_{T\to\infty} T^{-1} \ln \langle \psi, e^{-TH(\sigma,j)}\psi \rangle \\ &= \sup_{\psi} \lim_{T\to\infty} T^{-1} \ln \int \Theta\psi \exp \left[- \int_0^T (B_2 + A_j)dt \right] \psi_T dq \\ &\leq \sup_{\psi} \lim_{T\to\infty} \left\{ T^{-1} \ln \| \psi \|_\infty + T^{-1} \ln \int \exp \left[- \int_0^T (B_2 + A_j)dt \right] dq \right\}. \end{aligned}$$

This completes the proof, since the function $\Omega \equiv 1$ is an allowed choice of ψ.

LEMMA 5.4. — $-\delta E(\sigma, b, h_j)$ is non-negative and convex in σ, h_j.

Proof. — The convexity follows from a Schwarz inequality relative to the inner product defined by the Euclidean weak coupling measure dq, combined with Lemma 5.3 $-\delta E \geq 0$ by the normalization $\langle : \varphi^j : \rangle = 0$.

LEMMA 5.5. — $-\delta E(\sigma, b)$ is convex and monotone increasing, as a function of b.

Proof. — Since $-\delta E \geq 0$ and $-\delta E(\sigma, 0) = 0$, the derivative $d(-\delta E)/db$ is non-negative for $b = 0$. Assuming convexity, the derivative is increasing, and hence is always non-negative, so $-\delta E$ is monotonic increasing. To prove convexity, we use a Schwarz inequality relative to the Osterwalder-Schrader inner product defined by a line $x = \text{const}$. Thus

$$-\delta E(\sigma, b) \leq -\frac{1}{2}(\delta E(\sigma, b') + \delta E(\sigma, b''))$$

where $b = \frac{1}{2}(b' + b'')$. Here $x = \text{const.}$ determines the choice of b' and b'', and by varying the constant we obtain an arbitrary division.

Remark. — The lemma applies to any localized perturbation, *e. g.* $B_2 + A_j$, so long as each h_j is the characteristic function of Λ_2.

Proof of Theorem 5.2. — We follow [*17*], which simplifies the original method of [*8*]. It is no loss of generality to assume $b \geq a$. As above, we use a Schwarz inequality relative to the lines $x = \pm a$. For $x \in [-a, a]$ we use the norm of the semigroup (with x and t interchanged). Since dq is rotation invariant, it is invariant under rotation by the angle $\theta = \pi/2$. The vacuum energy for fixed x is bounded by $0(1)[1 + |h(x)|^{d/d-j}]T$, where the second factor contributes for $|h|$ large. The factor $0(1)$ contains the σ-dependence. The bound is linear in T by the linear bound on the vacuum energy in the volume. Integrating over x, gives $0(T)\big(a + \| h \|_{d/d-j}^{d/d-j}\big)$ and $-\delta E(\sigma, b, h_j) \leq -\delta E(\sigma, b) + 0(1)\big(a + \| h_j \|_{d/d-j}^{d/d-j}\big)$.

In order to apply Theorem 5.2, we let $T_2 \nearrow \infty$, $b \nearrow \infty$, where $[-T_2, T_2]$ is the time interval in which B_2 is inserted in the path space integral. The $T_2 \nearrow \infty$ limit is taken first. In this limit, the effect of the ϕ^j perturbation is estimated using Theorem 5.2. We require,

LEMMA 5.6. — Let $\Omega \equiv 1$ and let

$$\widehat{H} = H + B_2 - \delta E(\sigma, b).$$

Then

$$\lim_{T \to \infty} \frac{\| e^{-(T-a)\widehat{H}}\Omega \|}{\| e^{-T\widehat{H}}\Omega \|} = 1. \tag{5.11}$$

Proof. — We let $\widehat{\mathrm{H}} = \int \lambda d\mathrm{E}_\lambda$. By Lemma 5.4, $0 = \inf$ suppt. $d\langle \mathrm{E}_\lambda\Omega, \Omega \rangle$ $\equiv$ inf suppt. $d\mu(\lambda)$.
We consider

$$dv_t(\lambda) = \frac{e^{-t\lambda}d\mu(\lambda)}{\int_0^\infty e^{-t\lambda}d\mu(\lambda)}.$$

Thus

$$\frac{\langle \Omega, e^{-(t-a)\widehat{\mathrm{H}}}\Omega \rangle}{\langle \Omega, e^{-t\widehat{\mathrm{H}}}\Omega \rangle} = \int_0^\infty e^{a\lambda}dv_t(\lambda).$$

Splitting the integral, $\int_0^\infty = \int_0^\varepsilon + \int_\varepsilon^\infty$, we find the contribution $\int_0^\varepsilon \to 1$, and $\int_\varepsilon^\infty \to 0$. which completes the proof.

By Theorem 5.1,

$$\int e^{-\int_{-t_j}^{t_j} \mathrm{A}_j(t)dt} e^{-\int_{-\mathrm{T}_2}^{\mathrm{T}_2} \mathrm{B}_2(t)dt} dq \Big/ \int e^{-\int_{-\mathrm{T}_2}^{\mathrm{T}_2} \mathrm{B}_2(t)dt} dq = \langle \theta_{\mathrm{T}_2}, e^{-2t_j\mathrm{H}_1}\theta_{\mathrm{T}_2} \rangle$$

where

$$\mathrm{H}_1 = \mathrm{H} + \mathrm{B}_2 - \delta\mathrm{E}(\sigma, b) + \mathrm{A}_j = \widehat{\mathrm{H}} + \mathrm{A}_j$$

and

$$\theta_{\mathrm{T}_2} = \frac{e^{-(\mathrm{T}_2 - t_j)\widehat{\mathrm{H}}}\Omega}{\| e^{-\mathrm{T}_2\widehat{\mathrm{H}}}\Omega \|}.$$

By Theorem 5.2, and Lemma 5.6,

$$\lim_{\mathrm{T}_2\to\infty} \langle \theta_{\mathrm{T}_2}, e^{-2t_j\mathrm{H}_1}\theta_{\mathrm{T}_2} \rangle / \| \theta_{\mathrm{T}_2} \|^2 \leq \exp t_j 0(1)\big(a + \| h_j \|_{d/d-j}^{d/d-j}\big).$$

Thus by Lemma 5.3,

$$\ln \| e^{-\mathrm{H}_1} \| \leq 0(1)\big(a + \| h_j \|_{d/d-j}^{d/d-j}\big)$$

in the limit $b = \infty$. Thus we have proved

THEOREM 5.7. — Let $\mathrm{H}(\sigma)$ be the even + linear $\mathscr{P}(\phi)_2$ Hamiltonian with weak coupling boundary conditions given by the mass perturbation

$$\frac{1}{2}\sigma \int : \phi^2 : dx, \ \sigma \leq 0.$$

Then,

$$e^{-(\mathrm{H}(\sigma) \pm \mathrm{A}_j)t} \leq e^{\mathrm{c}t} \tag{5.12}$$

where

$$c = 0(1)\big(a + \| h_j \|_{d/d-j}^{d/d-j}\big). \tag{5.13}$$

REMARK 5.8. — The same argument may be used with the Hamiltonian $H(\sigma, \mu)$, for a $\mathscr{P}(\phi)_2$ = even + linear Hamiltonian with weak coupling boundary conditions given by the perturbation $\frac{1}{2}\sigma\int :\phi^2: dx - \mu\int \phi dx$, with $-\sigma, \mu \geq 0$. For a general $\mathscr{P}(\phi)_2$ theory ($\mathscr{P} \neq$ even + linear), Theorem 5.7 remains valid, but in this case the $b = \infty$ infinite volume limit is defined by convergent subsequences.

REMARK 5.9. — The same argument can be used to obtain semigroup estimates of the type (5.12-13) for infinite volume, weakly coupled $\mathscr{P}(\phi)_2$ models. In this case, the convergence of the Schwinger functions as $T \to \infty$ follows from the cluster expansion, rather than monotonicity.

REMARK 5.10. — We improve on (5.13), in order to allow perturbations A_j and test functions h_j which do not have compact support. Using the bounds (5.12)-(5.13), valid for $b \leq \infty$, we have

$$\begin{aligned} -\delta\widehat{E}(\sigma, b, h_j) &\equiv \ln \| \exp(-\widehat{H} + A_j) \| \\ &\leq 0(1)\left(a + \| h_j \|_{d/d-j}^{d/d-j}\right). \end{aligned}$$

Since $-\delta\widehat{E}$ is convex in h_j and vanishes for $h_j = 0$, the above inequality implies

$$-\delta\widehat{E} \leq 0(1)\left(a^{j/d}\| h_j \|_{d/d-j} + \| h_j \|_{d/d-j}^{d/d-j}\right).$$

We write $h_j = \sum_{n=-\infty}^{\infty} \chi_{[n,n+1]}h_j$ and use convexity again to obtain the bound

$$(5.14) \qquad -\delta\widehat{E} \leq 0(1)\left(\|(1 + |x|)^r h_j \|_{d/d-j} + \| h_j \|_{d/d-j}^{d/d-j}\right),$$

for some r. This bound holds uniformly for $b \leq \infty$ and for any h_j with $(1 + |x|)^r h_j \in L_{d/d-j}$.

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