ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 22, nº 2 (1975), p. 123-130

<http://www.numdam.org/item?id=AIHPA_1975__22_2_123_0>

Ann. Inst. Henri Poincaré,
Vol. XXII, n° 2, 1975, p. 123-130.

Section A :
Physique théorique.

On the duality of local observables

by

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ABSTRACT. — The necessary and sufficient conditions are shown for the duality, twisted duality and weak twisted duality of observable algebra and field algebra.

In the axiomatic quantum field theory, one associates to each region $\mathcal{O}$ in the space-time a von Neumann algebra $\mathfrak{A}(\mathcal{O})$ on a Hilbert space $\mathcal{H}$. $\mathfrak{A}(\mathcal{O})$ is called the *observable algebra, i. e.*, the algebra generated by all the observables which can be measured in $\mathcal{O}$. If $\mathcal{O}_1$ and $\mathcal{O}_2$ are two regions totally spacelike with respect to each other, then the measurement of observables in $\mathcal{O}_1$ should not disturb the measurement in $\mathcal{O}_2$. Hence, the observable algebras $\mathfrak{A}(\mathcal{O}_1)$ and $\mathfrak{A}(\mathcal{O}_2)$ are required to commute each other.

More specifically, if we denote by $\mathfrak{A}(\mathcal{O}')$ the C*-algebra generated by all $\mathfrak{A}(\mathcal{O}_i)$ with $\mathcal{O}_i$ totally space-like to $\mathcal{O}$, then the above arguments require that

$$\mathfrak{A}(\mathcal{O}')^- \subseteq \mathfrak{A}(\mathcal{O})',$$

called the *locality* for $\mathcal{O}$. Here S^- denotes the weak-closure of a set of bounded operators S on $\mathcal{H}$, and S' denotes the commutant of S. Then, the *duality* of observable algebra for $\mathcal{O}$ is

$$\mathfrak{A}(\mathcal{O}')^- = \mathfrak{A}(\mathcal{O})'.$$

It is well-known [*1*] that the duality holds for the free Boson fields. Recently, Eckman and Osterwald [*2*] have applied the Tomita-Takesaki theory [*3*] to the Fock space, computed explicitly the relevant operators in the theory, and shown the duality for the free Boson fields.

(*) Supported in part by NSF grant DID71-04010-A02

In their computations, the very special way of the modular involution in the Fock space plays a crucial role in the whole analysis. Indeed, the modular involution is an essential (necessary) condition for the duality to be true in the free Boson fields.

In the present note, we will show that the modular involution is not only a necessary condition but also a sufficient condition for the duality of any observable algebra (Proposition 1). Furthermore, we show that it characterizes essentially the twisted duality and weak twisted duality of field algebras (Propositions 4 and 5); and it is also a necessary condition for the weak duality for observable algebra (Proposition 6).

Given an involution J on a Hilbert space $\mathcal{H}$ (*i. e.* $J^2 = 1$), we define

$$j(A) = JAJ$$

for any bounded operator A on $\mathcal{H}$. We note that $j^2 = j \circ j = 1$.

Let M, N be von Neumann algebras on $\mathcal{H}$. If M has a cyclic and separating vector $\Omega \in \mathcal{H}$, then, from Tomita-Takesaki's theory [3], there is an involution J_Ω such that $J_\Omega M J_\Omega = M'$, or $j_\Omega(M) = M'$. N is said j-*implementable* if

$$j_\Omega(N) = N'.$$

In particular, if $N \subseteq M$, then $N = M$ whenever N is j-implementable.

In the quantum field theory, it is well-known that the vacuum Ω is a cyclic and separating vector for $\mathfrak{A}(\mathcal{O})$, hence $j_\Omega[\mathfrak{A}(\mathcal{O})] = \mathfrak{A}(\mathcal{O})'$. We shall deal with this case only, j_Ω will be denoted by j in the sequel. With this in mind, we are ready to show the following:

PROPOSITION 1. — The duality holds for $\mathcal{O}$ if and only if the locality is true for $\mathcal{O}$ and $\mathfrak{A}(\mathcal{O}')^-$ is j-implementable.

Proof. — From the duality, $\mathfrak{A}(\mathcal{O}')^- = \mathfrak{A}(\mathcal{O})'$, the locality follows obviously. As $\mathfrak{A}(\mathcal{O})$ has the vacuum Ω as a separating and cyclic vector, $j[\mathfrak{A}(\mathcal{O})] = \mathfrak{A}(\mathcal{O})'$. Hence, by the duality, $j[\mathfrak{A}(\mathcal{O}')^-] = j[\mathfrak{A}(\mathcal{O})'] = \mathfrak{A}(\mathcal{O}) = \mathfrak{A}(\mathcal{O}')'$. Conversely, from the locality, $\mathfrak{A}(\mathcal{O}')^- \subseteq \mathfrak{A}(\mathcal{O})'$, we have $j[\mathfrak{A}(\mathcal{O}')^-] \subseteq j[\mathfrak{A}(\mathcal{O})']$. As $\mathfrak{A}(\mathcal{O}')^-$ is j-implementable,

$$\mathfrak{A}(\mathcal{O}')' = j[\mathfrak{A}(\mathcal{O}')^-] \subseteq j[\mathfrak{A}(\mathcal{O})'] = \mathfrak{A}(\mathcal{O}),$$

hence, $\mathfrak{A}(\mathcal{O}')^- \supseteq \mathfrak{A}(\mathcal{O})'$. Compared with the locality, we have the duality for $\mathcal{O}$, $\mathfrak{A}(\mathcal{O}')^- = \mathfrak{A}(\mathcal{O})'$.

Remark 1. — We note that if Ω is also cyclic and separating for $\mathfrak{A}(\mathcal{O}')^-$ and the locality holds, then $\mathfrak{A}(\mathcal{O}')^-$ is j-implementable. Furthermore, we will see later that the j-implementability also plays a crucial role in the other notions of dualities.

Remark 2. — In the free Boson fields [2], by means of the Fock representation, it has been computed explicitely the modular involution J restricted on the one particle subspace of the Fock space. It turns out that

$j[\mathfrak{A}(\mathcal{O})] \subseteq \mathfrak{A}(\mathcal{O}')^-$. Hence, by the locality, the duality holds for the free Boson fields. Therefore, from the above proposition, $\mathfrak{A}(\mathcal{O}')^-$ of free Boson fields is j-implementable.

Instead of starting with observable algebras, Doplicher, Haag and Roberts [*4*] have studied the field algebra with a compact gauge group of first kind, and then defined the observable algebra as a gauge invariant subalgebra of field algebra. We will investigate the duality for observable algebra defined in this way and the other notions of dualities for field algebras in this setting.

We give some of their assumptions and notations which we need for our study:

To each region $\mathcal{O}$ in the space-time, there is a von Newmann algebra $\mathcal{F}(\mathcal{O})$, the *field algebra* of the region $\mathcal{O}$, acting on a Hilbert space $\mathcal{H}$. The total field algebra $\mathcal{F}$ is defined as the norm-closure of the union of all $\mathcal{F}(\mathcal{O})$:

$$\mathcal{F} = \overline{\cup \mathcal{F}(\mathcal{O})}.$$

Furthermore, it is assumed that $\mathcal{F}^- = \mathcal{F}'' = \mathcal{B}(\mathcal{H})$, the algebra of all bounded operators on $\mathcal{H}$.

Given a group G, let $g \in \mathrm{G} \to \mathrm{U}(g) \in \mathcal{B}(\mathcal{H})$ be a unitary representation of G on $\mathcal{H}$ such that it induces a group of automorphisms of $\mathcal{F}$:

$$\alpha_g(\mathrm{F}) = \mathrm{U}(g)\mathrm{F}\mathrm{U}(g)^{-1} \qquad \text{for each } \mathrm{F} \in \mathcal{F}.$$

And, α_g acts locally on $\mathcal{F}(\mathcal{O})$, *i. e.*

$$\alpha_g(\mathcal{F}(\mathcal{O})) = \mathcal{F}(\mathcal{O}).$$

The observable algebra $\mathfrak{A}(\mathcal{O})$ of the region $\mathcal{O}$ is then defined as a subalgebra of $\mathcal{F}(\mathcal{O})$ invariant under $\mathrm{U}(g)$:

$$\mathfrak{A}(\mathcal{O}) = \mathcal{F}(\mathcal{O}) \cap \mathcal{U}(\mathrm{G})'.$$

Here $\mathcal{U}(\mathrm{G})$ is the image of G under the mapping $g \to \mathrm{U}(g)$. The total observable algebra $\mathfrak{A}$ is defined similarly to the total field algebra; the norm-closure of the union of all $\mathfrak{A}(\mathcal{O})$:

$$\mathfrak{A} = \overline{\cup \mathfrak{A}(\mathcal{O})}.$$

Moreover, we assume that

$$\mathfrak{A}(\mathcal{O}')^- = \mathcal{F}(\mathcal{O})' \cap \mathcal{U}(\mathrm{G})',$$

where $\mathfrak{A}(\mathcal{O}')$, as before, denotes the C*-algebra generated by all $\mathfrak{A}(\mathcal{O}_i)$ with $\mathcal{O}_i$ totally space-like to $\mathcal{O}$.

The relative commutant $\mathfrak{A}^c(\mathcal{O})$ is given by

$$\mathfrak{A}^c(\mathcal{O}) = \mathfrak{A}(\mathcal{O})' \cap \mathfrak{A}.$$

Final assumption: $\mathcal{F}(\mathcal{O})$ has a cyclic and separating vector Ω, which is corresponding to an analytic vector for the energy operator, such that $U(g)\Omega = \Omega$ for all $g \in G$.

Let E_0 be the projection onto the subspace of all G-invariant vectors in $\mathcal{H}$, *i. e.* it is the projection on $\mathcal{H}$ with range $= \{\xi \in \mathcal{H}; U(g)\xi = \xi \text{ for all } g \in G\}$. Then, from Lemma 2.2 in [5], there exists two normal G-invariant projection maps Φ from $\mathcal{F}(\mathcal{O})$ onto $\mathfrak{A}(\mathcal{O})$ and Φ' from $\mathcal{F}(\mathcal{O})'$ onto $\mathfrak{A}(\mathcal{O}')^-$ respectively. We note that by the given assumption, $j[\mathcal{F}(\mathcal{O})] = \mathcal{F}(\mathcal{O})'$; and $j \circ \Phi = \Phi' \circ j$ if $E_0 J = J E_0$.

PROPOSITION 2. — With the above notations, assume $j \circ \Phi = \Phi' \circ j$, then

(*i*) $j[\mathfrak{A}(\mathcal{O})] = \mathfrak{A}(\mathcal{O}')^-$.

(*ii*) $j[\mathfrak{A}(\mathcal{O})]E_0 = E_0\mathfrak{A}(\mathcal{O})'E_0, j[\mathfrak{A}(\mathcal{O}')^-]E_0 = E_0\mathfrak{A}(\mathcal{O}')'E_0$.

(*iii*) The following statements are equivalent:

(*a*) $\mathfrak{A}(\mathcal{O})$ is j-implementable,

(*b*) the duality holds for $\mathcal{O}$, and

(*c*) $\mathfrak{A}(\mathcal{O}')^-$ is j-implementable.

Proof.

(*i*) $j[\mathfrak{A}(\mathcal{O})] = j[\Phi(\mathcal{F}(\mathcal{O}))] = \Phi'[j(\mathcal{F}(\mathcal{O}))] = \Phi'[\mathcal{F}(\mathcal{O})'] = \mathfrak{A}(\mathcal{O}')^-$.

(*ii*) From Corollary 2.3 in [5], we have

$\mathfrak{A}(\mathcal{O}')^-E_0 = E_0\mathfrak{A}(\mathcal{O})'E_0$,

$\mathfrak{A}(\mathcal{O})E_0 = E_0\mathfrak{A}(\mathcal{O}')'E_0$.

Hence (*ii*) follows immediately from (*i*).

(*iii*) (*a*) $\Rightarrow$ (*b*): $j[\mathfrak{A}(\mathcal{O})] = \mathfrak{A}(\mathcal{O})' = \mathfrak{A}(\mathcal{O}')^-$, by (*i*).

(*b*) $\Rightarrow$ (*c*): $\mathfrak{A}(\mathcal{O}')' = \mathfrak{A}(\mathcal{O})$, hence, $j[\mathfrak{A}(\mathcal{O}')'] = j[\mathfrak{A}(\mathcal{O})] = \mathfrak{A}(\mathcal{O}')^-$, again by (*i*).

(*c*) $\Rightarrow$ (*b*): From (*i*), $j[\mathfrak{A}(\mathcal{O})] = \mathfrak{A}(\mathcal{O}')^-$, and $\mathfrak{A}(\mathcal{O}')^- = j[\mathfrak{A}(\mathcal{O}')']$ by (*c*), we have $j[\mathfrak{A}(\mathcal{O})] = j[\mathfrak{A}(\mathcal{O}')']$, as $j^2 = 1$, $\mathfrak{A}(\mathcal{O}) = \mathfrak{A}(\mathcal{O}')'$.

(*b*) $\Rightarrow$ (*a*): An immediate consequence of (*i*) and (*b*).

Remark 3. — From (*ii*) in Proposition 2, if $E_0 = I$, then $\mathfrak{A}(\mathcal{O})$ and $\mathfrak{A}(\mathcal{O}')^-$ are j-implementable. Indeed, from Lemma 2.2 in [5], $E_0 = [\mathfrak{A}(\mathcal{O})\Omega] = [\mathfrak{A}(\mathcal{O}')^-\Omega]$; $E_0 = I$ implies that Ω is a cyclic vector for $\mathfrak{A}(\mathcal{O})$ and $\mathfrak{A}(\mathcal{O}')^-$. As Ω is a separating vector for $\mathcal{F}(\mathcal{O})$ and $\mathcal{F}(\mathcal{O})'$, it is also separating for $\mathfrak{A}(\mathcal{O})$ and $\mathfrak{A}(\mathcal{O}')^-$. Hence, Ω is a separating and cyclic vector for $\mathfrak{A}(\mathcal{O})$ and $\mathfrak{A}(\mathcal{O}')^-$. Therefore, $\mathfrak{A}(\mathcal{O})$ and $\mathfrak{A}(\mathcal{O}')^-$ are j-implementable.

Remark 4. — As $\mathfrak{A}(\mathcal{O}) \subseteq \mathcal{F}(\mathcal{O})$, $\mathfrak{A}(\mathcal{O}')^- \subseteq \mathcal{F}(\mathcal{O})'$, the j-implementability of $\mathfrak{A}(\mathcal{O})$ and $\mathfrak{A}(\mathcal{O})^-$ implies that $\mathfrak{A}(\mathcal{O}) = \mathcal{F}(\mathcal{O})$ and $\mathfrak{A}(\mathcal{O}')^- = \mathcal{F}(\mathcal{O})'$ respectively, hence the duality is a trivial consequence.

In contrast to the observable algebra, there is no commutation relation for field algebra, hence no « locality » for $\mathcal{F}(\mathcal{O})$. Moreover, from Proposition 2 and Remark 4, the duality holds only in a very special case. Hence, a specialization of commutation structure of field algebra is considered in [4], so that a new notion of locality and duality for $\mathscr{F}(\mathcal{O})$ are introduced. This is why a « twisting » operation is defined as follows [4]:

Let M be a von Newmann algebra on $\mathscr{H}$, and stable under the action of the group G. An operation t on M, t: $\mathrm{M} \to t[\mathrm{M}] = \mathrm{M}^t$ (and t on M', t: $\mathrm{M}' \to t[\mathrm{M}'] = \mathrm{M}^{t'}$), is called *twisting* if

$$\Phi(\mathrm{M}^t) = \Phi(\mathrm{M}),$$

$$\Phi'(\mathrm{M}^{t'}) = \Phi'(\mathrm{M}').$$

$\mathscr{F}^t(\mathcal{O})'$ is called the *twisted commutant* of $\mathscr{F}(\mathcal{O})$, and $\mathscr{F}$ is said to satisfy the *twisted locality* if $\mathscr{F}(\mathcal{O}')^- \subseteq \mathscr{F}^t(\mathcal{O})'$ for each $\mathcal{O}$. Moreover, $\mathscr{F}$ is said to satisfy the *twist duality* for $\mathcal{O}$ if

$$\mathscr{F}(\mathcal{O}')^- = \mathscr{F}(\mathcal{O})'.$$

The duality of observable algebra $\mathfrak{A}(\mathcal{O})$ is a special case of the twisted duality $\mathfrak{A}^t(\mathcal{O}) = \mathfrak{A}(\mathcal{O})$. And, the assumption of $\Phi'(\mathscr{F}(\mathcal{O})') = \mathfrak{A}(\mathcal{O}')^-$ follows immediately from the twist duality [4].

As $j[\mathscr{F}(\mathcal{O})] = \mathscr{F}(\mathcal{O})'$, we have also a characterization for j-implementability of $\mathscr{F}^t(\mathcal{O})$ in the following:

LEMMA 3. — $\mathscr{F}^t(\mathcal{O})$ is j-implementable if and only if $j \circ t = t \circ j$.

Since $j[\mathscr{F}^t(\mathcal{O})] = t[j(\mathscr{F}(\mathcal{O}))] = t[\mathscr{F}(\mathcal{O})'] = \mathscr{F}^t(\mathcal{O})'$.

Conversely, $(j \circ t)[\mathscr{F}(\mathcal{O})] = j[\mathscr{F}^t(\mathcal{O})] = \mathscr{F}^t(\mathcal{O})' = t[\mathscr{F}(\mathcal{O})'] = t[j(\mathscr{F}(\mathcal{O}))] = (t \circ j)[\mathscr{F}(\mathcal{O})]$.

We characterize the twisted duality for $\mathcal{O}$ as follows:

PROPOSITION 4. — Assume $j \circ t = t \circ j$, then the twisted duality is true for $\mathcal{O}$ if and only if the twisted locality holds and $\mathscr{F}(\mathcal{O}')^-$ is j-implementable.

Proof. — Suppose $\mathscr{F}(\mathcal{O}')^- = \mathscr{F}^t(\mathcal{O})'$, then the twisted locality follows trivially. We note that $\mathscr{F}^t(\mathcal{O})'' = \mathscr{F}^t(\mathcal{O})$ from the definition of twisted operation. Hence,

$$j[\mathscr{F}(\mathcal{O}')^-] = j[\mathscr{F}^t(\mathcal{O})'] = \mathscr{F}^t(\mathcal{O}) = \mathscr{F}^t(\mathcal{O}')',$$

where the second equality is due to Lemma 3.

Conversely, from the twisted locality, we have $j[\mathscr{F}(\mathcal{O}')^-] \subseteq j[\mathscr{F}^t(\mathcal{O})']$.

From the hypotheses, $j[\mathscr{F}(\mathcal{O}')^-] = \mathscr{F}(\mathcal{O}')'$ and $j[\mathscr{F}^t(\mathcal{O})'] = \mathscr{F}^t(\mathcal{O})$, thus we can obtain the twisted duality by using the same arguments given in the proof of Proposition 1.

Similar to the relative commutant $\mathfrak{A}^c(\mathcal{O})$ for observable algebra, the *relative twisted commutant* $\mathscr{F}^{tc}(\mathcal{O})$ is defined for field algebra:

$$\mathscr{F}^{tc}(\mathcal{O}) = \mathscr{F}^t(\mathcal{O})' \cap \mathscr{F}.$$

$\mathscr{F}$ is said to satisfy *weak twisted duality* for $\mathcal{O}$ if

$$\mathscr{F}^{tc}(\mathcal{O})^- = \mathscr{F}^t(\mathcal{O})'.$$

PROPOSITION 5. — Suppose $j \circ t = t \circ j$. Then the weak twisted duality holds for $\mathcal{O}$ if and only if $\mathscr{F}^{tc}(\mathcal{O})^-$ is j-implementable.

Proof. — If $\mathscr{F}^{tc}(\mathcal{O})^- = \mathscr{F}^t(\mathcal{O})'$, then

$$j[\mathscr{F}^{tc}(\mathcal{O})^-] = j[\mathscr{F}^t(\mathcal{O})'] = \mathscr{F}^t(\mathcal{O}) = \mathscr{F}^{tc}(\mathcal{O})',$$

where the second equality is due to Lemma 3 and the last equality from the weak twisted duality. Conversely, as

$$\mathscr{F}^{tc}(\mathcal{O}) = \mathscr{F}^t(\mathcal{O})' \cap \mathscr{F},\ \mathscr{F}^{tc}(\mathcal{O})^- \subseteq \mathscr{F}^t(\mathcal{O})'.$$

Hence, $j[\mathscr{F}^{tc}(\mathcal{O})^-] \subseteq j[\mathscr{F}^t(\mathcal{O})']$, and by the given assumptions,

$$j[\mathscr{F}^{tc}(\mathcal{O})^-] = \mathscr{F}^{tc}(\mathcal{O})' \quad \text{and} \quad j[\mathscr{F}^t(\mathcal{O})'] = \mathscr{F}^t(\mathcal{O}),$$

and using the same arguments in the proof of Proposition 1, we have

$$\mathscr{F}^{tc}(\mathcal{O})^- = \mathscr{F}^t(\mathcal{O})'.$$

In fact, the weak twisted duality is introduced in [4] to show the *weak duality* for $\mathcal{O}$:

$$\mathfrak{A}^c(\mathcal{O})^- = \mathfrak{A}(\mathcal{O})' \cap \mathscr{U}^-,$$

where $\mathfrak{A}^c(\mathcal{O})$ is the relative commutant of $\mathfrak{A}(\mathcal{O})$. It has been shown in [4] that the weak duality follows from the weak twisted duality and some additional assumptions on the unitary representation of guage group (*i. e.* Assumption 8 in [4]). We will show in the next proposition that the j-implementability of relevant local algebras will also imply the weak duality, without using any assumption of weak twisted duality. However, we need some properties of a G-invariant, positive normal projection map m on $\mathscr{B}(\mathscr{H})$ as follows [4]:

Let G be a compact guage group. Then there is a G-invariant, positive normal projection m from $\mathscr{B}(\mathscr{H})$ onto $\mathscr{U}(\mathrm{G})'$:

$$m(\mathrm{A}) = \int_{\mathrm{G}} \alpha_g(\mathrm{A}) d\mu(g) \quad \text{for} \quad \mathrm{A} \in \mathscr{B}(\mathscr{H}),$$

where μ is a normalized Haar measure and the integral is taken in the weak-operator topology. Moreover, m is local and normal as a map on $\mathscr{F}$:

$$m[\mathscr{F}(\mathcal{O})] = \mathscr{F}(\mathcal{O}) \cap \mathscr{U}(\mathrm{G})' = \mathfrak{A}(\mathcal{O}),$$

$$m[\mathscr{F}] = \mathscr{F} \cap \mathscr{U}(\mathrm{G})' = \mathfrak{A}, \quad \text{and} \quad \mathfrak{A}^- = m[\mathscr{F}]^- = m[\mathscr{F}^-] = \mathscr{U}(\mathrm{G})',$$

since $\mathscr{F}^- = \mathscr{B}(\mathscr{H})$ by our assumption. We note that m is a special case of Φ and Φ' in [5], indeed, $m = \Phi = \Phi'$.

PROPOSITION 6. — Let $\mathfrak{A}(\mathcal{O})$ be j-implementable. Consider the conditions:

(*i*) $\{ \mathfrak{A}(\mathcal{O})' \cap \mathcal{F} \}^-$ is j-implementable;

(*ii*) $\{ \mathfrak{A}(\mathcal{O})' \cap \mathcal{F} \}^- = \mathfrak{A}(\mathcal{O})'$;

(*iii*) $\mathfrak{A}^c(\mathcal{O})^- = \mathfrak{A}(\mathcal{O})' \cap \mathfrak{A}^-$, *i. e.*, the weak duality for $\mathcal{O}$.

Then (*i*) $\Leftrightarrow$ (*ii*) $\Rightarrow$ (*iii*).

Proof. — (*i*) $\Rightarrow$ (*ii*): Since $\mathfrak{A}(\mathcal{O})' \supseteq \mathfrak{A}(\mathcal{O})' \cap \mathcal{F}$, so that

$$\mathfrak{A}(\mathcal{O})' \supseteq \{ \mathfrak{A}(\mathcal{O})' \cap \mathcal{F} \}^- \tag{1}$$

Hence, $$j[\mathfrak{A}(\mathcal{O})'] \supseteq j[\{ \mathfrak{A}(\mathcal{O})' \cap \}^-].$$

From (*i*) and the hypotheses,

$$\mathfrak{A}(\mathcal{O}) \supseteq \{ \mathfrak{A}(\mathcal{O})' \cap \mathcal{F} \}'.$$

Thus, $$\mathfrak{A}(\mathcal{O})' \subseteq \{ \mathfrak{A}(\mathcal{O})' \cap \mathcal{F} \}^-. \tag{2}$$

Compare (1) and (2):

$$\mathfrak{A}(\mathcal{O})' = \{ \mathfrak{A}(\mathcal{O})' \cap \mathcal{F} \}^-.$$

(*ii*) $\Rightarrow$ (*i*). From (*ii*),

$$j[\{ \mathfrak{A}(\mathcal{O})' \cap \mathcal{F} \}^-] = j[\mathfrak{A}(\mathcal{O})'] = \mathfrak{A}(\mathcal{O}),$$

where the last equality is due to the given assumption. Again by (*ii*);

$$\mathfrak{A}(\mathcal{O}) = \{ \mathfrak{A}(\mathcal{O})' \cap \mathcal{F} \}'.$$

Therefore, $$j[\{ \mathfrak{A}(\mathcal{O})' \cap \mathcal{F} \}^-] = \{ \mathfrak{A}(\mathcal{O})' \cap \mathcal{F} \}'.$$

(*ii*) $\Rightarrow$ (*iii*) is given in [*4*]. However, we show more explicitely as follows. From (*ii*),

$$m[\{ \mathfrak{A}(\mathcal{O})' \cap \mathcal{F} \}^-] = m[\mathfrak{A}(\mathcal{O})'].$$

However,

$$\mathfrak{A}^c(\mathcal{O}) = \mathfrak{A}(\mathcal{O})' \cap \mathfrak{A} = \mathfrak{A}(\mathcal{O})' \cap (\mathcal{F} \cap \mathcal{U}(\mathrm{G})') = m[\mathfrak{A}(\mathcal{O})' \cap \mathcal{F}].$$

Hence,

$$\begin{aligned}\mathfrak{A}^c(\mathcal{O})^- = m[\mathfrak{A}(\mathcal{O})' \cap \mathcal{F}]^- = m[\{ \mathfrak{A}(\mathcal{O})' \cap \mathcal{F} \}^-] &= m[\mathfrak{A}(\mathcal{O})'] \\ &= \mathfrak{A}(\mathcal{O})' \cap \mathcal{U}(\mathrm{G})' = \mathfrak{A}(\mathcal{O})' \cap \mathfrak{A}^-.\end{aligned}$$

Remark 5. — From the above proposition and the proof given in Theorem 5.3 in [*4*], we note that their assumptions will imply ours. Indeed, their assumptions on the gauge group and the weak twisted duality imply (*ii*) of the above proposition.

Remark 6. — Let $m[\mathcal{F}(\mathcal{O})'] = \Phi'[\mathcal{F}(\mathcal{O})'] = \mathfrak{A}(\mathcal{O}')^-$. Suppose $j \circ t = t \circ j$, then the weak twisted duality and the j-implementability of $\{ \mathfrak{A}(\mathcal{O})' \cap \mathcal{F} \}^-$ imply $\mathfrak{A}(\mathcal{O}')^- = \mathfrak{A}^c(\mathcal{O})^-$. In fact, as $\mathcal{F}^{tc}(\mathcal{O})^- \subseteq \{ \mathfrak{A}(\mathcal{O})' \cap \mathcal{F} \}^-$, and by Proposition 5 and hypotheses, $\mathcal{F}^{tc}(\mathcal{O})^- = \{ \mathfrak{A}(\mathcal{O})' \cap \mathcal{F} \}^-$. Hence

$$\mathcal{F}^t(\mathcal{O})' = \{ \mathfrak{A}(\mathcal{O})' \cap \mathcal{F} \}^-,$$

by the weak twisted duality. Thus

$$\mathfrak{A}(\mathcal{O}')^- = m[\mathcal{F}(\mathcal{O})'] = m[\mathcal{F}^t(\mathcal{O})'] = m[\{\mathfrak{A}(\mathcal{O})' \cap \mathcal{F}\}^-] = \mathfrak{A}^c(\mathcal{O})^-.$$

The last follows from the proof in Proposition 5. This result would never be possible under the assumptions in [*4*].

Remark 7. — In fact, the *j*-implementability of $\mathfrak{A}(\mathcal{O})$ implies $\mathfrak{A}(\mathcal{O}) = \mathcal{F}(\mathcal{O})$ from remark 4, hence Proposition 6 follows immediately. This is the case of free Boson fields.

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(Manuscrit reçu le 20 juin 1974)