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Approximate symmetries and their spontaneous breakdown

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ABSTRACT. — We propose a rigorous definition of approximate symmetry and of its spontaneous breakdown in connection with a phase transition. We show that this definition is of interest in the case of the Ising model with a small three-spin perturbation.

1. INTRODUCTION

A big progres has been recently achieved in the theory of phase transitions by the rigorous proof that the Ising model on ν-dimensional square lattice Z^ν ($\nu \geqslant 2$) with a formal Hamiltonian of the type

$$H(\underline{\sigma}) = -J\sum_{\langle i,j\rangle}\sigma_i\sigma_j - J'\sum_{\langle i,j,k\rangle}\sigma_i\sigma_j\sigma_k - h\sum_i \sigma_i \qquad (1.1)$$

shows a phase transition at low temperature if J' is small [*1*]. Here $\underline{\sigma}$ is a spin configuration, $J > 0$, h is the external field; $\sum_{\langle i,j\rangle}$ means the sum over the couples of nearest neighbours sites $i, j \in Z^\nu$; $\sum_{\langle i,j,k\rangle}$ means the sum over the triples of sites in which two of them are nearest neighbours of the third and lie on a triangle.

Actually the type of lattice and the form of the perturbation term could be taken quite arbitrary, but we shall limit ourselves to the above choices, and also to $\nu = 2$, to avoid uninteresting complications. Clearly, if $|J'|/J \ll 1$ the above system is, in some sense « very close » to the usual Ising model ($J' = 0$), so that we expect it to show a phase transition which is « very similar » to the one that takes place in the usual Ising model.

It is well known that the phase transition in the ordinary Ising model takes place at $h = 0$, and can be interpreted as a spontaneous breakdown of the spin reversal symmetry shown by the Hamiltonian (see, for instance, ref. [*2*]); other systems with a similar behaviour are described in ref. [*3*], [*4*]. It is easy to see that if $J' \neq 0$ there is no value of h which gives rise to an invariant Hamiltonian under spin reversal symmetry; still we would like to interpret the phase transition associated with the above model, which we shall denote with the simbol (J, J'), as a spontaneous breakdown of some symmetry of $H(\underline{\sigma})$.

In this paper we investigate in what sense this can be really done. Our aim is to propose some rigorous and « reasonable » definitions of « approximate symmetry » and of its spontaneous breakdown, and to show that the phase transition of the model (J, J') can be interpreted as a spontaneous breakdown of an approximate symmetry. The main idea underlying our study is that, when a system is enclosed in a large enough box, only a very limited number of « typical » configurations are possible (*i. e.* have non-negligeable probability), and the typical configurations of each pure phase are, in some sense, approximately symmetric of each other.

2. SOME NOTATIONS

Let Λ be a finite square box (centered at the origin); we denote by $|\Lambda|$ the number of sites in Λ; on each site i of Λ we put a spin $\sigma_i = \pm 1$ and define

$$H_\Lambda(\underline{\sigma}) = -J\sum_{\langle i,j\rangle}\sigma_i\sigma_j - J'\sum_{\langle i,j,k\rangle}\sigma_i\sigma_j\sigma_k - h\sum_i\sigma_i \tag{2.1}$$

where $\underline{\sigma}$ is a spin configuration in the box Λ (*i. e.* $\underline{\sigma} \equiv (\sigma_1 \ldots \sigma_{|\Lambda|})$), and the sums have the same interpretation as in (1.1) except that the labels are restricted to be sites in Λ. We denote $\partial\Lambda$ the set of sites on the boundary of Λ; for each $i \in \partial\Lambda$ we define $\tau_i = \pm 1$; let $\underline{\tau}$ be the set of such numbers. Throughout this paper J will be considered fixed once and for all.

We consider the ensembles of spin configurations $\mathscr{U}^\tau(\Lambda) = \{$ spin configurations $\underline{\sigma}$ in Λ such that $\sigma_i = \tau_i$ if $i \in \partial\Lambda\}$ and consider the Boltzmann distribution over $\mathscr{U}^{\underline{\tau}}(\Lambda)$ with respect to the Hamiltonian (2.1). We shall then define

$$P_{\underline{\tau},\Lambda}(\underline{\sigma}) = \frac{e^{-\beta H_\Lambda(\underline{\sigma})}}{\sum_{\underline{\sigma}' \in \mathscr{U}^{\underline{\tau}}(\Lambda)} e^{-\beta H_\Lambda(\underline{\sigma}')}} \qquad \underline{\sigma} \in \mathscr{U}^{\underline{\tau}}(\Lambda) \tag{2.2}$$

and denote $Z(\underline{\tau}, \Lambda)$ the normalization factor in the above formula; here β is the inverse of the temperature.

The correlation functions are defined, for $x_1 \ldots x_n \in \Lambda$, by

$$\langle \sigma_{x_1} \ldots \sigma_{x_n} \rangle_{\underline{\tau},\Lambda} = \sum_{\underline{\sigma} \in \mathscr{U}^{\underline{\tau}}(\Lambda)} P_{\underline{\tau},\Lambda}(\underline{\sigma})(\sigma_{x_1} \ldots \sigma_{x_n})$$

Consider now a sequence $\{ \underline{\tau}_\Lambda \}$ with $\{ \Lambda \}$ ordered by inclusion, and assume that the limits

$$\langle \sigma_{x_1} \ldots \sigma_{x_n} \rangle = \lim_{\Lambda \to \infty} \langle \sigma_{x_1} \ldots \sigma_{x_n} \rangle_{\underline{\tau}_\Lambda,\Lambda} \tag{2.3}$$

exists for all $x_1 \ldots x_n \in Z^\nu$, all $n = 1, 2, \ldots$; we consider only sequences $\{ \underline{\tau}_\Lambda \}$ such that the above limits exist and are invariant under simultaneous translations of the lattice points $x_1 \ldots x_n$. We call such sequences « boundary conditions ». The value of the limits may of course depend on the choice of $\{ \underline{\tau}_\Lambda \}$ and, when necessary, we shall add further labels to the l. h. s. to take this fact into account. In particular, if $(\underline{\tau}_\Lambda)_i \equiv +1 \ \forall i \in \partial\Lambda$ for all Λ, we shall use the label $+$, and if $(\underline{\tau}_\Lambda)_i \equiv -1 \ \forall i \in \partial\Lambda$ for all Λ we shall use the label $-$ (provided the limits exist). It is well-known that the existence of the above limits (2.3) defines a probability measure μ on the set of the infinite spin configurations (*i. e.* on the space K of the sequence $\{ \sigma_i \}_{i \in Z^\nu}$): this measure is, in some sense, the limit of the Boltzmann measures $P_{\underline{\tau},\Lambda}$ on the finite volume configurations.

The σ-field Σ of the sets of the space K of sequences $\{ \sigma_i \}_{i \in Z^\nu}$ over which the measure turns out to be naturally defined is the one generated by the sets of spin configurations of the form

$$E_{x_1 \ldots x_n, \sigma_{x_1} \ldots \sigma_{x_n}} = \left\{ \{ \sigma_i \}_{i \in Z^\nu} \,\middle|\, \sigma'_{x_1} = \sigma_{x_1}, \ldots, \sigma'_{x_n} = \sigma_{x_n} \right\}$$

for all $x_1 \ldots x_n \in Z^\nu$ and all values of $\sigma_{x_1} \ldots \sigma_{x_n}$.

For convenience we extend it in the natural way to the σ-field Σ_μ which contains also the sets which have outer μ-measure 0. By our assumptions the measure will be translationally invariant in the sense that if $E \in \Sigma_\mu$ is a measurable set of spin configurations,

$$\mu(E) = \mu(T_\xi E) \qquad \forall \xi \in Z^\nu$$

where $T_\xi E$ is the set of configurations obtained by a ξ translation of the configurations in E, *i. e.* $\underline{\sigma} \in T_\xi E$ if exists $\underline{\sigma}' \in E$ such that $\sigma_i = \sigma'_{i-\xi} \ \forall i \in Z^\nu$.

In the language of ergodic theory the triple (K, μ, T) forms a dynamical system.

The spin configurations $\underline{\sigma} \in \mathcal{U}^{\tau}(\Lambda)$ can be simply described by means of the lines which separate neighbouring sites i, j with $\sigma_i = +1$, $\sigma_j = -1$ or *vice versa*: we draw these lines as chains of unit segments each of which is perpendicular to a bond (i, j) with opposite spins lying on its extremes. The set of lines thus obtained will join into connected components, called contours, $\lambda_1 \ldots \lambda_s, \gamma_1 \ldots \gamma_k$ where $\lambda_1 \ldots \lambda_s$ denote the open contours and $\gamma_1 \ldots \gamma_K$ denote the closed contours. The name contour comes from the geometrical appearence of the above lines which look like paths of self-avoiding random walks (a contour may intersect itself, but at each of its vertices always meet two or four segments belonging to the contour; the open contours have two vertices, outside of Λ of course, into which an odd number of lines meet).

The + or − boundary conditions are privileged in the above contour description because the associated contours must necessarily be all closed.

We denote then a spin configuration $\underline{\sigma} \in \mathcal{U}_{\tau}(\Lambda)$ by the set of its contours $\gamma_1 \ldots \gamma_k, \lambda_1 \ldots \lambda_s$; notice that different spin configurations correspond to the same set of contours if we consider $\underline{\sigma} \in \mathcal{U}^{+}(\Lambda)$ or $\underline{\sigma} \in \mathcal{U}^{-}(\Lambda)$.

3. PHASE TRANSITION

In the case of the model (J, 0) it is known that, if β is large enough, the limits (2.3) are unique (*i. e.* boundary conditions independent) if $h \neq 0$ (ref. [5]); if $h = 0$ the limits (2.3) exist for both the b. c. + and − and are given in the general case by

$$\langle \sigma_{x_1} \ldots \sigma_{x_n} \rangle_{\underline{\tau}} = \alpha \langle \sigma_{x_1} \ldots \sigma_{x_n} \rangle_{+} + (1-\alpha) \langle \sigma_{x_1} \ldots \sigma_{x_n} \rangle_{-} \quad (3.1)$$

where $0 \leqslant \alpha \leqslant 1$ is independent on n and $x_1 \ldots x_n$ [6]; furthermore

$$\langle \sigma_{x_1} \ldots \sigma_{x_n} \rangle_{+} = (-1)^n \langle \sigma_{x_1} \ldots \sigma_{x_n} \rangle_{-} \; ; \; \langle \sigma_x \rangle_{\pm} \neq 0 \text{ [7]}. \quad (3.2)$$

If β is small enough the above statements are also true (except $\langle \sigma_x \rangle_{\pm} \neq 0$), but

$$\langle \sigma_{x_1} \ldots \sigma_{x_n} \rangle_{+} \equiv \langle \sigma_{x_1} \ldots \sigma_{x_n} \rangle_{-}$$

The above facts are, as it is well known, interpreted as a description of a phase transition.

Let us call $P_{+}^{(h)}$, $P_{-}^{(h)}$ the infinite volume Gibbs measures on K associated to the Hamiltonian (2.1) with $J' = 0$, where the b. c. is + or − and the external field is h. Then it is known that [11]

$$P_{+}^{(h)} \equiv P_{-}^{(h)} \quad \text{if} \quad h \neq 0, \quad \text{and} \quad P_{+} \equiv P_{+}^{(0)} \neq P_{-}^{(0)} \equiv P_{-}$$

if $h = 0$ and β large enough. Furthermore the spin reversal transformation

$$\Psi : K \to K \quad \text{defined by} \quad (\Psi \underline{\sigma})_i = -\sigma_i \forall i \in Z^{\nu}$$

is an automorphism of K such that

$$\Psi^* P^{(h)}(E) = P^{(-h)}(E) \quad , \quad h \neq 0$$
$$\Psi^* P_+^{(0)}(E) = P_-^{(0)}(E) \quad , \quad h = 0$$

for all $E \in \Sigma_{P^{(h)}}$, where Ψ^* is the transformation induced by Ψ on the set of the Gibbs measures, *i. e.* $\Psi^*(P^{(h)}(E)) = P^{(h)}(\Psi E)$. These results simply reflect the fact that the formal Hamiltonian (1.1) is formally invariant if $\sigma_i \to -\sigma_i$ when $h = 0$, and is changed to the formal Hamiltonian with $-h$ in place of h otherwise. The above relations, together with the result $\langle \sigma_x \rangle_\pm \neq 0$ for large β and $h = 0$, are usually referred as a spontaneous breakdown of the up-down symmetry of (1.1) and are considered as a proof of the existence of a phase transition in the Ising model (J, 0) [*11*]. Generally one says that a model (J, J′) shows a phase transition for a certain value of h and β if there are two b. c. which in formula (2.3) give rise to different limits.

4. APPROXIMATE DYNAMICAL SYMMETRIES

The main features of the above description of the phase transition in the model (J, 0) which we shall retain are the following:

1) There is a transformation Ψ which acts on the space K of the infinite spin configurations and which transforms Gibbs measures into Gibbs measures.

2) An equilibrium measure $P^{(h)}$ with $h \leqslant h_\beta$ is mapped by Ψ^* into an equilibrium measure $P^{(h')}$ with $h' \geqslant h_\beta$ (actually $h' = -h$; $h_\beta = 0$).

3) $P_+^{(h)} = P_-^{(h)}$ if $h \neq h_\beta$ (actually $h_\beta = 0$).

4) The dynamical system $(K, Z_{P_+^{(h)}}, T)$ and $(K, \Sigma_{\Psi^* P_-^{(h)}}, T)$ are isomorphic (for $h \neq h_\beta$ we can omit the index $\pm$).

Since just very few spin configurations will be realized with non-zero probability, we shall think that Ψ is defined $P^{(h)}$-almost everywhere and maps, in the measure theoretical sense $P^{(h)}$ into $P^{(h')}$.

We are thus led to the following definitions:

DEFINITION 1. — Given $\varepsilon > 0$ and two translationally invariant measures on the spin configurations, μ, μ', we say that they are ε-symmetric with respect to the spin inversion if there is a square box Λ_ε such that for *all* squares $\Lambda > \Lambda_\varepsilon$ the following happens:

i) One can find two sets of configurations $\mathscr{C}_\Lambda$ and $\mathscr{C}'_\Lambda$, respectively μ-and μ'-measurable such that

$$\mu(\mathscr{C}_\Lambda) > 1 - \varepsilon \quad , \quad \mu'(\mathscr{C}'_\Lambda) > 1 - \varepsilon$$

ii) There is a measure preserving invertible mapping $\Psi_\Lambda : (K, \mu) \to (K, \mu')$

of the measure space (K, μ) into the measure space (K, μ'), which maps $\mathscr{C}_\Lambda$ into $\mathscr{C}'_\Lambda$ and:

$$iii) \qquad \sum_{i \in \Lambda} \frac{| \sigma_i + (\Psi_\Lambda \underline{\sigma})_i |}{|\Lambda|} < \varepsilon \qquad \forall \underline{\sigma} \in \mathscr{C}_\Lambda$$

It is important, as we shall see, that *iii*) should be true for all $\Lambda \supset \Lambda_\varepsilon$: this clarifies the global character of the definition. A simpler, but more formal definition, equivalent to the above definition 1 is the following:

DEFINITION 1'. — Ergodic Gibbs distribution μ and μ' on the spin configurations are ε-symmetric if one can find a μ-typical configuration σ and a μ'-typical configuration σ' such that

$$\lim_{\Lambda \to \infty} \sum_{i \in \Lambda} \frac{| \sigma_i + \sigma'_i |}{|\Lambda|} \leqslant \varepsilon$$

where typical means here that the frequency of the translates of a finite spin configuration in σ is well defined for all finite spin configurations and is equal to the probability of the associated cylinder (almost all configurations have this property because of the ergodicity).

DEFINITION 2. — Let $h = h(\beta)$ be a line in the plane (h, β). We say that the model (1.1) is ε-symmetric around the line $h = h(\beta)$ if it is possible to define a continuous correspondence between the points on the two sides of the line and close to it in such a way that:

i) the line is mapped into itself;

ii) the equilibrium states associated with corresponding points are ε-symmetric.

After the above definitions, 1 and 2, the following is a natural definition of spontaneous breakdown of an approximate symmetry:

DEFINITION 3. — An approximate symmetry is said to be spontaneously broken on the line $h = h(\beta)$ if the following happens:

i) the model has an approximate symmetry in the sense of definition 2;

ii) the states associated with points outside the symmetry line are unique, and depend continuously on the parameters (β, h) if (β, h) is outside of the line; furthermore they have a limit when the parameters (β, h) tend to points on the line staying always on the same side.

Here limit of measures has the following sense: we say that a sequence of measures $\{ \mu_n \}$ tends to μ if

$$\lim_{n \to \infty} \langle \sigma_{x_1} \dots \sigma_{x_k} \rangle_{\mu_n} = \langle \sigma_{x_1} \dots \sigma_{x_k} \rangle_\mu \ \forall k, \forall x_1 \dots x_k \in Z^\nu$$

i. e. space of the measures is equipped with the vague topology ;

iii) the states obtained as limit of states associated with two corresponding points, when one of the two tends to the line $h = h(\beta)$ from one of its sides, are different.

Clearly the above definitions of ε-approximate symmetry and of spontaneous breakdown of an approximate symmetry can be easily generalized to much more general situations, but we do not enter into this discussion since it seems to us that the main interest of the above abstract definitions lies upon the fact that we are able to prove the following theorem:

THEOREM. — Let us consider an Ising model (J, J′) with $| J' | \ll J$ then there is a line $h = h(\beta)$ defined for β large enough, such that the model (J, J′) has an 0 (J′/J)-approximate symmetry of spin reversal around the line $h = h(\beta)$. Furthermore this symmetry is spontaneously broken.

The proof of this theorem is based on results and ideas concerning the recently proved isomorphism of Ising model with Bernoulli shift with the same entropy (ref. [*8*]).

5. THE THEOREM OF PERAGOV-SINAI

Let us consider the set of contour configurations associated with configurations in the ensemble $\mathscr{U}^+(\Lambda)(\mathscr{U}^-(\Lambda))$; we denote that ensemble $\tilde{\mathscr{U}}^+(\Lambda)$ $(\tilde{\mathscr{U}}^-(\Lambda))$.

We introduce a function $\omega(\gamma) \geqslant 0$ defined on the contours in such a way that $\omega(\gamma) = \omega(\gamma')$ if γ and γ' are translates of each other.

We introduce then on $\tilde{\mathscr{U}}^+(\Lambda)(\tilde{\mathscr{U}}^-(\Lambda))$, considered as ensemble of contours, a probability distribution which associates to $(\gamma_1 \ldots \gamma_n) \in \tilde{\mathscr{U}}^+(\Lambda)$ the probability:

$$\mathrm{P}_+(\gamma_1 \ldots \gamma_n \mid \omega) = \frac{\exp\left(-\sum_{i=1}^{n} \omega(\gamma_i)\right)}{\sum_{(\gamma'_1 \ldots \gamma'_n) \in \tilde{\mathscr{U}}^+(\Lambda)} \exp\left(-\sum_{i=1}^{n} \omega(\gamma'_i)\right)} \tag{5.1}$$

(and the analogue for $\tilde{\mathscr{U}}^-(\Lambda)$).

Denote $\tilde{\mathscr{U}}^+_{\text{ext}}(\Lambda)(\tilde{\mathscr{U}}^-_{\text{ext}}(\Lambda))$ the ensemble whose elements are the external contours of the configurations of $\tilde{\mathscr{U}}^+(\Lambda)(\tilde{\mathscr{U}}^-(\Lambda))$: the external contours of $(\gamma_1 \ldots \gamma_n) \in \tilde{\mathscr{U}}^+(\Lambda)(\tilde{\mathscr{U}}^-(\Lambda))$ are the ones that can be connected to the outside of Λ by a continuous line which does not cut other contours. The probability distribution (5.1) induces a probability distribution on $\tilde{\mathcal{U}}^+_{\text{ext}}(\Lambda)$ ($\tilde{\mathcal{U}}^-_{\text{ext}}(\Lambda)$) in a natural way, *i. e.*

$$P_+(\Gamma_1 \ldots \Gamma_s \mid \omega) = \sum^*_{(\gamma_1 \ldots \gamma_n) \in \tilde{\mathcal{U}}^+(\Lambda)} P_+(\gamma_1 \ldots \gamma_n \mid \omega) \qquad \forall (\Gamma_1 \ldots \Gamma_s) \in \tilde{\mathcal{U}}^+_{\text{ext}}(\Lambda) \tag{5.2}$$

where the Σ^* sums over the configurations $(\gamma_1 \ldots \gamma_n) \in \tilde{\mathcal{U}}^+(\Lambda)$ which have $(\Gamma_1 \ldots \Gamma_s)$ as external contours.

Consider now the model (J, J′) enclosed in a box with b. c. + (−). Let us consider the measure on the set of contours defined by $P_{+,h}(\gamma_1 \ldots \gamma_n)$ = (probability of the configuration with $(\gamma_1 \ldots \gamma_n)$ as associated contours).

As before this probability distribution defines a natural distribution on $\tilde{\mathcal{U}}^+_{\text{ext}}(\Lambda)$ which we shall also denote $P_{+,h}$.

The following theorem holds:

THEOREM (Peragov-Sinai). — Consider the model (J, J′). There is $b > 0$ such that if $\frac{|J'|}{J} < b$ there is $\beta_0 > 0$ and two smooth functions $h_{J'}(\beta)$, $e(\beta)$ analytic in $\beta_0 < \beta < \infty$ such that

a) If $0 \leqslant -h_{J'}(\beta) + h \leqslant e(\beta)$ the probability distribution $P_{+,h}(\Gamma_1 \ldots \Gamma_s)$ induced by $P_{+,h}$ on the ensemble $\tilde{\mathcal{U}}^+_{\text{ext}}(\Lambda)$ coincides with the probability $P(\Gamma_1 \ldots \Gamma_s \mid \omega_{+,h})$ induced by a suitably chosen translation invariant function $\omega_{+,h}(\gamma)$ defined on the contours.

Similarly, if $0 \leqslant h_{J'}(\beta) - h \leqslant e(\beta)$ the probability distribution on $\tilde{\mathcal{U}}^-_{\text{ext}}(\Lambda)$ induced by $P_{-,h}$ is of the form $P(\Gamma_1 \ldots \Gamma_s \mid \omega_{-,h})$ with a suitable translation invariant function $\omega_{-,h}(\gamma)$.

b)
$$\omega_{\pm,h}(\gamma) \geqslant k\beta \mid \gamma \mid \tag{5.3}$$
for a suitable $k > \frac{J}{2}$

c) The function $\omega_{+,h}$ depend continuously on h, J, J′, β for

$$0 \leqslant h - h_{J'}(\beta) < e(\beta) \quad , \quad \frac{|J'|}{J} < b$$

The function $\omega_{-,h}(\gamma)$ has the same property in the appropriate region.

d) The Gibbs equilibrium states for (J, J′) are unique if

$$0 \leqslant | h - h_{J'}(\beta) | < e(\beta)$$

e)
$$\lim_{J' \to 0} \omega_{\pm, h_{J'}(\beta)}(\gamma) = 2\beta J(\gamma) \quad \forall \gamma \tag{5.4}$$

A sketch of the proof of this theorem for the case $h = h_{J'}(\beta)$ is given for completeness in appendix A, but the reader will have to work out the details [*1*].

This theorem easily implies:

I) The correlation functions are continuous on both sides of the line $h = h_{J'}(\beta)$ and have a limit on each side of it (this follows because of the uniform bound (5.3)).

II) The entropy of the states is continuous as a function of (β, h) on both sides of the line $h = h_{J'}(\beta)$ (this immediately follows by the point c) of the above theorem after a straightforward calculation which gives the entropy in terms of $\omega_{\pm,h}(\gamma)$ as a series whose convergence is guaranted by the uniform bound (5.3)).

III) The cluster property considered in ref. [8] is true and therefore ([7]) the Gibbs states constructed above are isomorphic to Bernoulli shifts (in fact the proof in [8] depends only on the inequality (5.3).

IV) There are only two translationally invariant equilibrium states on the line $h = h_{J'}(\beta)$ (this follows as in ref. [6], see Appendix B).

V) The entropy and the correlation functions of the equilibrium states with + boundary conditions of the model (J, J′) are close, in the region $0 < \beta_0 < \beta < \beta_1$ and $0 \leqslant h - h_{J'}(\beta) \leqslant \frac{|J'|}{J} c(\beta)$ to the corresponding quantities calculated in the equilibrium state with + boundary conditions of the pure Ising model at the same temperature and $h = 0$.

More precisely the difference between corresponding quantities can be bounded by a function $0(J')$ infinitesimal when $J' \to 0$. This function $0(J')$ may depend on the interval (β_0, β_1) and on the quantities we are comparing.

A similar property holds in the region $0 \leqslant h_{J'}(\beta) - h \leqslant \frac{|J'|}{J} c(\beta)$. Here the choice of the interval $|h - h_{J'}(\beta)| \leqslant \frac{|J'|}{J} c(\beta)$ is arbitrary: any other function $f(J')$ such that $1 \geqslant f(J') \xrightarrow[J' \to 0]{} 0$ could be used to define a region $|h - h_{J'}(\beta)| < f(J')c(\beta)$ where the above statement would hold (with other functions $0(J')$, of course).

VI) Since the equilibrium states of the pure Ising model are known to be B-shifts at low enough temperature [8] it follows, from V), II) and from the fact that B-shifts are finitely determined (see [9] [8]), that the d-distance in the sense of [9] [8] between the measures μ_+, h, β, J' and $\mu_+, 0, \beta, 0$ is infinitesimal when $J' \to 0$ when $h - h_{J'}(\beta) \geqslant 0$ and a similar property holds in the region $h - h_{J'}(\beta) \leqslant 0$. Examining the definition of d-distance [9] [8] and remembering that the spin reversal transformation is an isomorphism between $\mu_+, 0, \beta, 0$ and $\mu_-, 0, \beta, 0$ it easily follows that the phase transition in the model (J, J′) can be interpreted as a spontaneous breakdown of an approximate symmetry, and this proves the theorem given at the end of the section 4.

6. CONCLUSIONS

We stress that for the above proof it is not necessary to show that $\mu_\pm$, h, β, J′ are isomorphic to B-shifts; it is however esssential the fact that the states $\mu_\pm$, 0, β, 0 are isomorphic to B-shifts.

The non-trivial character of the definitions 1, 2, 3 lies in the fact that they are of global type: if the parameter ε were allowed to depend on Λ, the statements would have been a totally trivial consequence of the continuity of the correlation functions and thermodynamic functions on the parameter J′.

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APPENDIX A

We give a sketch of the proof of the Peragov-Sinai theorem in the case $h = h_{J'}(\beta)$.

1) Easy calculations show that, if $\underline{\sigma}$ is a spin configuration in the box Λ with b. c. $+\ (-)$, and $(\gamma_1 \ldots \gamma_n)$ is the associated set of contours, then called $h + 4J' = \tilde{h}$

$$\begin{aligned} H_{\Lambda,\pm}(\underline{\sigma}) = \mp \tilde{h}\Big(|\Lambda| - 2\sum_{j=1}^{n} s_j(\underline{\gamma})\,|\vartheta(\gamma_j)|\Big) + 2J\sum_{j=1}^{n}|\gamma_j| + \\ \pm 16J'\sum_{j=1}^{n} s_j(\underline{\gamma}) - 2J\,|\Lambda| - 2J'\sum_{x\in\partial\Lambda}\sigma_x \end{aligned} \tag{A.1}$$

where $\underline{\gamma} = (\gamma_1 \ldots \gamma_n)$; $|\vartheta(\gamma_j)| =$ (number of points in the region $\vartheta(\gamma_j)$ with boundary γ_j);

$$S_j(\underline{\gamma}) = \begin{cases} 1 \text{ if around } \gamma_j \text{ there is an even number of contours in } \underline{\gamma}, \\ -1 \text{ if around } \gamma_j \text{ there is an odd number of contours in } \underline{\gamma}. \end{cases}$$

2) Given a region $\vartheta(\gamma)$ we put

$$\mathscr{I}(\gamma,\omega) = \sum_{(\gamma_1\ldots\gamma_n)\in\tilde{\mathfrak{U}}^{+}(\vartheta(\gamma))}\prod_{j=1}^{n} e^{-\omega(\gamma_j)} = \sum_{(\Gamma_1\ldots\Gamma_s)\in\tilde{\mathfrak{U}}^{+}_{\mathrm{ext}}(\vartheta(\gamma))}\prod_{i=1}^{s} e^{-\omega(\Gamma_i)}\mathscr{I}(\omega,\Gamma_i) \tag{A.2}$$

Then by ref. [10]

$$\mathscr{I}(\omega,\Gamma) = |\vartheta(\Gamma)|\,a(\omega) + \Theta(\omega,\Gamma) \tag{A.3}$$

and if $\omega(\gamma) \geqslant K\,|\gamma|$, $K > 0$

$$\sup_{\{\gamma\}}\frac{|\Theta(\omega,\Gamma)|}{|\gamma|} \leqslant e^{-\frac{k}{2}} \tag{A.4}$$

3) Using the notations of sec. 5, we assume the existence of two contour functions $\omega_+(\gamma)$ and $\omega_-(\gamma)$ such that

$$P_{\pm}(\Gamma_1 \ldots \Gamma_s) = P(\Gamma_1 \ldots \Gamma_s \mid \omega_{\pm}) \tag{A.5}$$

(the proof of their existence gives us P. S. theorem).

4) By definition of the quantities in (A.5) and using (A.3) we obtain

$$\begin{aligned} 2\beta J(\Gamma) \pm 2\beta\tilde{h}\,|\vartheta(\Gamma)| \pm 16\beta J' - a(\omega_{\mp})\,|\vartheta(\Gamma)| - \Theta(\omega_{\mp},\Gamma) \\ = \omega_{\pm}(\Gamma) - a(\omega_{\pm})\,|\vartheta(\Gamma)| - \Theta(\omega_{\pm},\Gamma) \end{aligned} \tag{A.6}$$

5) (A.6) is an equation in the unknown ω_+, ω_-; a solution is obtained if we require separately

$$\pm 2\beta(h + 4J') = a(\omega_{\mp}) - a(\omega_{\pm}) \tag{A.7}$$

$$\omega_{\pm}(\Gamma) = 2\beta J(\Gamma) \pm 16\beta J' + \Theta(\omega_{\pm},\Gamma) - \Theta(\omega_{\mp},\Gamma) \tag{A.8}$$

the (A.7) gives the values of $h = h_{J'}(\beta)$ when we know $\omega_{\pm}(\gamma)$.

6) We now prove the existence of a solution of (A.8) and so the theorem of P. S. for

$h = h_{J'}(\beta)$ by considering the Banach space $\mathscr{B}$ of the couples $(\omega_+, \omega_-) = \underline{\omega}$ of contour functions with norm

$$\| \underline{\omega} \| = \max_{+,-} \sup_{\{\gamma\}} \frac{| \omega\pm(\gamma) |}{| \gamma |} \; ;$$

on $\mathscr{B}$ (A.8) appears as

$$\underline{\omega} = \underline{a} + k\underline{\omega} \tag{A.9}$$

where

$$a\pm(\gamma) = 2\beta J \mid \gamma \mid \pm 16\beta J' \; ; \; (k\underline{\omega})\pm(\gamma) = \pm (\Theta(\omega_+, \gamma) - \Theta(\omega -, \gamma))$$

k is a nonlinear operator which is a contraction when operates on $\underline{\omega}$ such that there exist $k > 0$ and $\omega\pm(\gamma) \geqslant k(\gamma)$ (see ref. [*10*]). Then it is easily seen that a solution of (A.9) is obtained by iteration.

APPENDIX B

The proof that here are only two translationally invariant states for $h = h_{J'}(\beta)$ is essentially as in ref. [6]; we have only to show that the lemma 1 of the quoted paper is true for the model (J, J').

Lemma 1. — If $h = h_{J'}(\beta)$, $\beta > \beta_0$, $\frac{|J'|}{J} < b$, then

$$p(\underline{\tau}, L^{4/3}) = P\left\{ \underline{\sigma} \in \mathcal{U}^{\underline{\tau}}(\Lambda) \,\middle|\, \sum_j |\lambda_j| > L^{4/3} \right\} < \varepsilon(L)$$

with $\varepsilon(L)$ infinitesimal τ-independent function and $L = \sqrt{|\Lambda|}$.

Proof

$$p(\underline{\tau}, L^{4/3}) = \sum_{\substack{\sigma \in \mathcal{U}^{\underline{\tau}}(\Lambda):\\ \sum_j |\lambda_j| > L^{4/3}}} P_{\underline{\tau},\Lambda}(\underline{\sigma}) \tag{B.1}$$

Given $\underline{\tau}$ we have $(\lambda_1, \ldots, \lambda_k)$ as open contours in a spin configuration σ; fixed $(\lambda_1, \ldots \lambda_k)$ we have thus a number of region $\Lambda_1, \ldots, \Lambda_s$ with homogeneous b. c. + or −; so we get

$$p(\underline{\tau}, L^{4/3}) = \frac{1}{Z(\Lambda, \underline{\tau})} \sum_{\substack{(\lambda_1 \ldots \lambda_k):\\ \sum_j |\lambda_j| > L^{4/3}}} e^{-\beta \sum_{j=1}^{k} (2J|\lambda_j| - J'\delta(\lambda_j))} \prod_{p=1}^{s} Z(\Lambda_p, \pm) \tag{B.2}$$

where $\delta(\lambda_j)$ is the contribution to the energy due to the J'-term, which is bounded $|\delta(\lambda_j)| < 16$; the sign $\pm$ is to be chosen according to the boundary of Λ_p; $Z(\Lambda, \underline{\tau})$ is the normalization factor introduced in sec. 2.

The lemma follows by the estimates

a) $$Z(\Lambda, \underline{\tau}) \geqslant \exp\left[-8\beta L(J - 4|J'|)\right] Z(\Lambda^*, -)$$

where Λ^* is the box concentric to Λ with side L-2;

b) $$Z(\Lambda_p, +) \leqslant Z(\Lambda_p, -) \exp\left\{ 2e^{-\frac{\beta k}{4}} |\partial\Lambda_p| \right\}$$

c) $$Z(\Lambda, -) \leqslant Z(\Lambda^*, -) \exp\left\{ \left[\frac{\left(3e^{-\frac{\beta k}{2}}\right)^4}{L - 3e^{-\frac{\beta k}{2}}} + \frac{1}{2} \right] 8L \right\}$$

d) $$\sum_{\substack{(\lambda_1 \ldots \lambda_k):\\ \Sigma |\lambda_j| > L^{4/3}}} e^{-B \sum_{j=1}^{k} (\lambda_j)} \leqslant \left(\frac{4}{L - 3e^{-B}} \right)^{2L} \max_{k \leqslant 2L} \left(L_k^{4/3} \right) (3e^{-\beta}) L^{4/3}$$

$e)$
$$\prod_{p=1}^{s} Z(\Lambda_p, -) \leqslant Z(\Lambda, -)$$

The proof of $a)$ is the same as in ref. [6] if one takes into account the contribution $\delta(\lambda_j)$ which is bounded by $-32L$;

$f)$ is new and follows by the observation that P.-S. theorem implies

$$Z(\Lambda_p, \pm) = \mathscr{I}(\Lambda_p, \omega_{\pm, h_{J'}(\beta)}) e^{\pm\beta(h+4J')|\Lambda_p|} \tag{B.3}$$

then, using (A.3), (A.7) and the bound (A.4) one get $b)$.

The other estimates are the same as in ref. [6].

We can so bound (B.2) with these estimates: we use $b)$ to have only terms of the type $Z(\Lambda_p, -)$; so doing appears a term $e^{-\sum_p^* |\partial\Lambda_p| 2e^{-\frac{\beta k}{p}}}$ where Σ^* means sum over the regions with positive boundary; since $\sum_p^* |\partial\Lambda_p| \in \sum_{j=1}^{k} |\lambda_j| + 4L$ we obtain the lemma if we take in $d)$

$$B = 2\beta J - 2e^{-\frac{\beta k}{4}} \quad \text{and} \quad 3e^{-B} < 1.$$

The results of this appendix are extracted from the thesis of one of us (R. E.).

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