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On the existence of a class of stationary quantum stochastic processes

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ABSTRACT. — Given a classical stationary stochastic process we construct a corresponding quantum stochastic process. As an example we use the Ornstein-Uhlenbeck process to construct the quantum process whose existence was suggested by the work of Ford, Kac and Mazur.

RÉSUMÉ. — Donné un processus aléatoire stationnaire classique, on construit un processus aléatoire quantique correspondant. Par exemple on se sert du processus Ornstein-Uhlenbeck pour construire le processus quantique dont l'existence a été proposée par Ford, Kac et Mazur.

1. INTRODUCTION

The work of Ford, Kac and Mazur [*1*] suggests the existence of a quantum-mechanical analogue of the Ornstein-Uhlenbeck stochastic process [*2*]. In this paper we prove a theorem which establishes the existence of a class of such quantum stochastic processes.

Recall that a classical stochastic process $\{ X_t : -\infty < t < \infty \}$ is a family of random variables on a probability space Σ with a probability measure $\mathbb{P}$ given on a field $\mathscr{B}$ of subsets of Σ; the expectation $\mathbb{E}Y$ of a random variable Y on Σ is given by

$$\mathbb{E}Y = \int_\Sigma Y(\sigma)\mathbb{P}(d\sigma). \tag{1}$$

We shall be interested in processes for which for all t

$$\mathbb{E}X_t = 0 \quad , \quad \mathbb{E}X_t^2 < \infty \tag{2}$$

so that $t \to X_t$ can be regarded as a curve in the real Hilbert space $L^2(\Sigma, \mathbb{P})$ and such a process will be called stationary if the correlation $\mathbb{E}X_tX_{t+\tau}$ depends only on the time-difference τ. The history $\mathfrak{H}^X$ of the process $t \mapsto X_t$ is the closed subspace of $L^2(\Sigma, \mathbb{P})$ spanned by the X_t :

$$\mathfrak{H}^X = \bigvee \{ X_t : -\infty < t < \infty \}. \tag{3}$$

It follows that for a stationary process $t \mapsto X_t$ there exists a group $\{ U_t : -\infty < t < \infty \}$ of unitary operators on its history $\mathfrak{H}^X$ such that for all t, τ

$$X_{t+\tau} = U_\tau X_t. \tag{4}$$

Thus a classical process involves three objects: an underlying space Σ, a probability measure $\mathbb{P}$ and a family $\{ X_t : -\infty < t < \infty \}$ of random variables. If in addition the process is stationary, its time-development is given by a family of operators which preserve expectations. This suggests the following tentative definition of a quantum stochastic process:

A quantum stochastic process is a family of self-adjoint operators $\{ Q_t : -\infty < t < \infty \}$ on a Hilbert space $\mathfrak{H}$ with a state vector $\Omega(\| \Omega \| = 1)$. The expectations are given by

$$\mathbb{E}Q_t = \langle \Omega, Q_t\Omega \rangle, \tag{5}$$

$$\mathbb{E}Q_t \circ Q_s = \frac{1}{2} \langle \Omega, (Q_tQ_s + Q_sQ_t)\Omega \rangle. \tag{6}$$

The symmetrized expectation (6) is taken because in general Q_t does not commute with Q_s and so Q_tQ_s is not self-adjoint. For a discussion of quantum correlations see [*1*] and [*3*].

A quantum stochastic process $t \mapsto Q_t$ is said to be stationary if there exists a one-parameter group $\{ V_t : -\infty < t < \infty \}$ of unitary operators on $\mathfrak{H}$ such that for all t, τ

$$Q_{t+\tau} = V_\tau Q_t V_\tau^{-1} \tag{7}$$

and

$$V_\tau\Omega = \Omega. \tag{8}$$

It then follows from (5) and (6) that for all t, τ

$$\mathbb{E}Q_t = \mathbb{E}Q_0 \tag{9}$$

and

$$\mathbb{E}Q_t \circ Q_{t+\tau} = \mathbb{E}Q_0 \circ Q_\tau \tag{10}$$

so that the correlation depends only on the time-difference τ.

2. DESCRIPTION OF A CLASS OF QUANTUM PROCESSES

It is well-known (for example Doob [*4*]) that if $\tau \mapsto \gamma(\tau)$ is the correlation of a stationary stochastic process $t \mapsto X_t$ so that

$$\gamma(\tau) = \mathbb{E}X_t X_{t+\tau} \tag{11}$$

then γ is a function of positive type, and that conversely if γ is a function of positive type there exists a stochastic process $(X_t, \Sigma, \mathbb{P})$ such that (11) holds. Given a correlation function γ we define for each $\mu > 0$ a quantum modification γ_μ such that for all τ

$$\lim_{\mu \to 0+} \gamma_\mu(\tau) = \gamma(\tau), \tag{12}$$

and we prove the existence of a stationary quantum stochastic process $(Q_t, \mathfrak{H}, \Omega)$ such that

$$\gamma_\mu(\tau) = \mathbb{E}Q_t \circ Q_{t+\tau}. \tag{13}$$

One motivation for this is that in the case in which γ is the correlation function of the Ornstein-Uhlenbeck x-process, the quantum modification $\gamma_\hbar$ is the correlation function obtained by Ford, Kac and Mazur. The Ornstein-Uhlenbeck process arises as the stationary solution of a Langevin equation which describes the motion of an oscillator of frequency ω_0 and frictional constant f coupled to a heat-bath at inverse temperature β. Then

$$\gamma(\tau) = \frac{e^{-f|\tau|/2}}{\beta\omega_0^2}\left(\cos \tilde{\omega}\tau + \frac{f}{2\omega_0}\sin \tilde{\omega}\,|\,\tau\,|\right). \tag{14}$$

Since the process X_t is stationary the oscillator is in thermal equilibrium with the heat-bath and so we can regard the inverse temperature β as describing the stationary state of the process. Indeed the joint probability distribution of the position and momentum of the oscillator is just the Maxwell-Boltzmann equilibrium distribution at inverse temperature β. A quantum-analogue of the Maxwell-Boltzmann condition is the Kubo-Martin-Schwinger boundary condition; we use it as formulated by Haag, Hugenholtz and Winnink [*5*]. The pair (A, B) satisfy the KMS-boundary condition at inverse temperature β if for every pair (A, B) of bounded self-adjoint operators on $\mathfrak{H}$ there exists a function $z \mapsto F_{AB}(z)$ analytic and uniformly bounded on the strip $-\beta\mu \leqslant \mathrm{Im}z \leqslant 0$ such that for all t

$$\left.\begin{aligned} F_{AB}(t) &= \langle \Omega, A_t B\Omega \rangle \\ F_{AB}(t - i\beta\mu) &= \langle \Omega, BA_t\Omega \rangle \end{aligned}\right\} \tag{15}$$

where

$$A_t = V_t A V_{-t} \tag{16}$$

Suppose $\tau \mapsto \gamma(\tau)$ is a positive definite function given by

$$\gamma(\tau) = \gamma(0) \int_0^\infty \cos \omega\tau d\mathrm{G}(\omega). \tag{17}$$

Then for $\mu > 0$ the quantum modification at inverse temperature β is defined to be

$$\gamma_\mu(\tau) = \gamma(0) \int_0^\infty b_\mu(\omega) \cos \omega\tau d\mathrm{G}(\omega) \tag{18}$$

where

$$b_\mu(\omega) = \frac{\beta\mu\omega}{2} \coth \frac{\beta\mu\omega}{2}. \tag{19}$$

Notice that, because of the inequality $x \coth x < 1 + x$ for $x > 0$, the function γ_μ is bounded provided $\int_0^\infty \lambda \, d\mathrm{G}(\lambda) < \infty$.

THEOREM. — *Let* $(\mathrm{X}_t, \Sigma, \mathbb{P})$ *be a real stationary stochastic process with correlation function*

$$\gamma(\tau) = \gamma(0) \int_0^\infty \cos \omega\tau d\mathrm{G}(\omega)$$

such that $\Delta\mathrm{G}(0) = 0$ *and* $\int_0^\infty \omega d\mathrm{G}(\omega) > \infty$.

Then there exists a stationary quantum stochastic process $(\mathrm{Q}_t, \mathfrak{H}, \Omega)$ *with* $\mathrm{Q}_t = \mathrm{V}_t\mathrm{Q}_0\mathrm{V}_t^{-1}$ *such that*

(i)
$$\langle \Omega, \exp\left\{ i \sum_{j=1}^n k_j \mathrm{Q}_{t_j} \right\} \Omega \rangle = \exp\left\{ -\frac{1}{2} \sum_{j,l=1}^n k_j k_l \gamma_\mu(t_j - t_l) \right\} \tag{20}$$

where γ_μ *is the quantum modification* (18) *of* γ *at inverse temperature* β.

(ii)
$$\exp(i\mathrm{Q}_s) \exp(i\mathrm{Q}_t) = \exp\{ i\mu\beta\gamma'(t-s) \} \exp(i\mathrm{Q}_t) \exp(i\mathrm{Q}_s). \tag{21}$$

(iii) The pair (V_t, Ω) *satisfies the* KMS *boundary condition* (15) *at inverse temperature* β.

Remark. — The properties *(i)* and *(ii)* are precise formulations of the more familiar looking

(i)'
$$\mathbb{E}\mathrm{Q}_s \circ \mathrm{Q}_t = \gamma_\mu(t-s), \tag{22}$$

(ii)'
$$[\mathrm{Q}_s, \mathrm{Q}_t] = i\mu\beta\gamma'(s-t)1. \tag{23}$$

The theorem is proved by constructing a process with the claimed properties.

3. THE CONSTRUCTION OF THE QUANTUM PROCESS

The construction proceeds in three parts. First we prove a lemma about classical processes which states that a real process can be complexified in such a way that time-translation acts as a unitary group with positive generator. Next we use the Araki-Woods construction [6], [7] to construct a quantum process and finally we verify the properties claimed for the process.

LEMMA. — *Let* $(X_t, \Sigma, \mathbb{P})$ *be the real process of the theorem. Then there exists a complex process* $(\xi_t, \Sigma, \mathbb{P})$ *and a one-parameter unitary group*

$$U_t = \int_{\mathbb{R}} e^{it\lambda} E(d\lambda) \tag{24}$$

on the complex Hilbert space $\mathfrak{H}^\xi$ *such that*

(i) $$X_t = \sqrt{2}\mathcal{R}e\xi_t, \tag{25}$$

(ii) $$\xi_t = U_t \xi_0, \tag{26}$$

(iii) $$U_t = \exp(iC^2 t), \tag{27}$$

with $C \geqslant 0$,

(iv) $$\| E(d\lambda)\xi_0 \|^2 = 2\gamma(0)dG(\lambda). \tag{28}$$

Proof. — By Bôchner's theorem there exists a distribution function F such that

$$\gamma(\tau) = \gamma(0) \int_{-\infty}^{\infty} e^{it\omega} dF(\omega). \tag{29}$$

For $\omega \geqslant 0$ define $G(\omega)$ by

$$G(\omega) = \begin{cases} F(\omega) - F(-\omega - 0) & , \quad \omega > 0, \\ 0 & , \quad \omega = 0 \end{cases}. \tag{30}$$

Then

$$\gamma(\tau) = \gamma(0) \int_0^\infty \cos \omega\tau dG(\omega) \tag{31}$$

since γ is real. Further, X_t has a representation

$$X_t = \int_{-\infty}^{\infty} e^{it\lambda} \zeta(d\lambda) \tag{32}$$

as the Fourier transform of a stochastic process ξ_λ with orthogonal increments such that

$$\mathbb{E} \mid \zeta(d\lambda) \mid^2 = \gamma(0)dF(\lambda). \tag{33}$$

Let $\tilde{X}_t$ be the Hilbert transform of X_t :

$$\tilde{X}_t = \int_{-\infty}^{\infty} e^{it\lambda} g(\lambda)\zeta(d\lambda) \tag{34}$$

where

$$g(\lambda) = \begin{cases} i & , \quad \lambda < 0, \\ 0 & , \quad \lambda = 0, \\ -i & , \quad \lambda > 0. \end{cases} \tag{35}$$

Then we have

$$\mathbb{E}\tilde{X}_t\tilde{X}_{t+\tau} = \gamma(\tau), \tag{36}$$

$$\mathbb{E}X_t\tilde{X}_{t+\tau} = \tilde{\gamma}(\tau) = \gamma(0)\int_0^\infty \sin \lambda\tau dG(\lambda). \tag{37}$$

Put

$$\xi_t = \frac{1}{2^{1/2}}(X_t + i\tilde{X}_t) \tag{38}$$

so that

$$\mathbb{E}\bar{\xi}_t\xi_{t+\tau} = \gamma(\tau) + i\tilde{\gamma}(\tau). \tag{39}$$

Then there exists a unitary operator

$$U_t = \int_{-\infty}^{\infty} e^{it\lambda}E(d\lambda) \tag{40}$$

on $\mathfrak{H}^\xi$ such that

$$\xi_t = U_t\xi_0 \tag{41}$$

and so

$$\int_{-\infty}^{\infty} e^{it\lambda} \| E(d\lambda)\xi_0 \|^2 = \gamma(0)\int_0^\infty e^{it\lambda}dG(\lambda). \tag{42}$$

Thus the spectrum of the generator of U_t is positive and

$$\| E(d\lambda)\xi_0 \|^2 = \gamma(0)dG(\lambda). \tag{43}$$

Put

$$C = \int_0^\infty \lambda^{1/2}E(d\lambda) \tag{44}$$

then

$$U_t = e^{itC^2} \quad , \quad C \geqslant 0. \tag{45}$$

For the Araki-Woods construction we follow Chaiken [7]. A Weyl system over a complex vector space $\mathscr{V}$ is a map W from $\mathscr{V}$ to the unitary operators $\mathscr{U}(\mathfrak{H})$ on a Hilbert space $\mathfrak{H}$ together with a vector Ω in $\mathfrak{H}$ such that

$$\vee \{ W(v)\Omega : v \in \mathscr{V} \} = \mathfrak{H},$$

which satisfies

(i)
$$W(v_1)W(v_2) = \exp\left\{ -\frac{i}{2}\mathscr{I}m \langle v_1, v_2 \rangle \right\} W(v_1 + v_2), \tag{46}$$

(ii) for all $v \in \mathscr{V}$ the map $\lambda \mapsto W(\lambda v)$ is continuous from $\mathbb{C}$ to $\mathscr{U}(\mathfrak{H})$ in the strong operator topology.

It follows from *(ii)* by Stone's theorem that for each $v \in \mathscr{V}$ there exists a self-adjoint operator $R(v)$ such that

$$W(v) = \exp\{ iR(v) \}. \tag{47}$$

A Weyl system is determined up to equivalence by its generating functional

$$\mu(v) = \langle \Omega, W(v)\Omega \rangle. \tag{48}$$

THEOREM (Chaiken [7]). — *Let $\mathscr{V}$ be a complex Hilbert space. Let* T *be a self-adjoint operator on $\mathscr{V}$ with domain $\mathscr{D}$(T) and such that* Y $\geqslant$ 1. *Then there exists a Weyl system* (W, $\mathfrak{H}$, Ω) *over $\mathscr{D}$(T) with generating functional*

$$\mu(v) = \exp\left(-\frac{1}{4}\| Tv \|^2\right). \tag{49}$$

Proof. — Let $A = \frac{1}{2}(T^2 - 1)$ and let J be a conjugation on $\mathscr{V}$ which commutes with A. Let $\mathscr{M}$ be the closure of the range of $A^{1/2}$; let (W_1, $\mathscr{I}(\mathscr{V})$, Ω_1) be the Fock-Cook Weyl system over $\mathscr{M}$ and let (W_2, $\mathscr{I}(\mathscr{M})$, Ω_2) be the Fock-Cook Weyl system over $\mathscr{M}$. For $v \in \mathscr{D}(T)$ put

$$W(v) = W_1((1 + A)^{1/2}v) \otimes W_2(A^{1/2}Jv) \tag{50}$$

on

$$\mathfrak{H} = \mathscr{I}(\mathscr{V}) \otimes \mathscr{I}(\mathscr{M})$$

with

$$\Omega = \Omega_1 \otimes \Omega_2.$$

Then direct calculation shows that

$$\langle \Omega, W(v)\Omega \rangle = \exp\left(-\frac{1}{4}\| Tv \|^2\right). \tag{51}$$

The methods of Araki-Woods [6] show that Ω is cyclic for W and that W is a factor representation; it is irreducible if and only if A = 0.

With the notation of the lemma take, in the Araki-Woods construction, $\mathscr{V} = \mathfrak{H}^{\xi}$, $v_t = (\mu\beta)^{1/2}CU_t\zeta_0$, and let T be the unique positive operator such that

$$T^2 = \coth\frac{\mu\beta}{2}C^2. \tag{52}$$

Then $T^2 > 1$ since $\coth x < 1$ for $x > 0$, and $C\zeta_0 \in \mathscr{D}(T)$ since

$$\begin{aligned}\| TC\zeta_0 \|^2 &= \int_0^\infty \lambda\coth\frac{\mu\beta\lambda}{2}\| E(d\lambda)\zeta_0 \|^2 \\ &= \gamma(0)\int_0^\infty \lambda\coth\frac{\mu\beta\lambda}{2}dG(\lambda) = \frac{2}{\mu\beta}\gamma_\mu(0) < \infty.\end{aligned} \tag{53}$$

A similar calculation gives

$$\mathscr{R}e\langle Tv_s, Tv_t \rangle = 2\gamma_\rho(t - s), \tag{54}$$

and

$$\mathscr{I}m\langle v_s, v_t \rangle = -\mu\beta\gamma'(t - s). \tag{55}$$

Putting

$$Q_t = R(v_t) \tag{56}$$

we check that (20) and (21) follow from (46) and (49).

Let (W, $\mathscr{I}(\mathscr{K})$, Ω) be the Fock-Cook Weyl system over a Hilbert space $\mathscr{K}$;

then to each self-adjoint operator B on $\mathscr{K}$ there corresponds a self-adjoint operator $\Gamma(B)$ on $\mathscr{I}(\mathscr{K})$ such that

$$W(e^{iBt}v) = e^{i\Gamma(B)t}W(v)e^{-i\Gamma(B)t} \tag{57}$$

for all t and all v in $\mathscr{K}$. It is straightforward to check that

$$V_t \exp(i\lambda Q_s)V_t^{-1} = \exp(i\lambda Q_{s+t}) \tag{58}$$

and

$$V_t\Omega = \Omega \tag{59}$$

where

$$V_t = e^{i\Gamma(C^2)} \otimes e^{-i\Gamma(C^2)} \tag{60}$$

so that the conditions (7) and (8) are satisfied. It remains to check that (V_t, Ω) satisfy the KMS boundary condition at inverse temperature β. It is enough to consider the function defined for v, $w \in \mathscr{D}(T)$ by

$$F_{vw}(t + iy) = \exp\left[-\frac{1}{4}\left\{\left\|T(U_te^{-yC^2}v + w)\right\|^2 + \langle U_te^{-yC^2}v, w\rangle - \langle w, U_te^{-yC^2}v\rangle\right\}\right] \tag{61}$$

which is analytic and bounded in the strip $-\beta\mu \leqslant y \leqslant 0$.

Using (52), we find that

$$T^2(e^{\mu\beta C^2} - 1) = (e^{\mu\beta C^2} + 1) \tag{62}$$

which together with (46) and (49) gives

$$F_{vw}(t) = \langle \Omega, W(U_tv)W(w)\Omega \rangle, \tag{63}$$

$$F_{vw}(t - i\mu\beta) = \langle \Omega, W(w)W(U_tv)\Omega \rangle, \tag{64}$$

so that the KMS boundary condition is satisfied and the proof of the theorem is complete.

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