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Analytic singularities and geodesic completeness. I (*)

by

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ABSTRACT. — The concept of an analytic singularity is defined in General Relativity. Although there are space-times with non-analytic singularities it is shown every Uniform Stationary space-time has only analytic singularities. As a result every periodic Uniform Stationary space-time is singularity free.

1. INTRODUCTION

Recently Hawking has proven a series of singularity theorems which show that singularities occur under very general circumstances in General Relativity [*1*]. However these theorems do not exhaust all the physically interesting models since they assume that the cosmological constant is zero. In fact the Hawking theorems can be viewed as a generalization of the circumstances that occur in one of three types of Robertson-Walker metric [*2*]. The three types are those in which $R(t)$ is not defined for all real t, those which are singular and $R(t)$ is defined for all t, and those which are nonsingular. These types contain respectively the big bang universes, the steady state model, and the periodic universes (which can be made locally the same as any Robertson Walker universe). The question arises whether there are singularity theorems which generalize the situations in these other types. In [*2*] we have given a singularity theorem which generalizes the circumstances in the second type of Robertson Walker metric. We will now try to find some nonsingularity theorems that correspond to the third type.

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In this third class we have the metrics with a periodic function $R(t)$. These space-times are periodic as will be defined later, and thus are singular if and only if a corresponding compact space-time is singular. If the space-time metric were positive definite then we could immediately see why these space-times are complete. However there are space-times which are compact but are incomplete [*3*]. Thus one problem with singularities in relativity is that there are singularities which do not correspond to the metric going bad somewhere. We define the concept of an analytic singularity which will exclude the bad types of singularities. Then any theorem which can state that in a certain circumstance every singularity must be analytic will give us as a corollary a nonsingularity theorem that periodic space-times in this circumstance are complete. Thus such theorems would start to provide theorems generalizing the third type of Robertson-Walker metrics. These theorems will also provide a very good start towards answering the question when a singular space-time must have some invariant that « blows up ».

In this paper we first define the concept of an analytic singularity. We then define a special class of space-times, the Uniform Stationary space-times, which can be easily dealt with using classical techniques. We finish with a theorem showing every singularity is analytic in a Uniform Stationary space-time. Periodic space-times are defined and the nonsingularity theorem is given as a corollary.

2. ANALYTIC SINGULARITIES

We try now to give a precise meaning to our idea of « analytic singularity ». By a singularity in a space-time M we mean a causal geodesic $\gamma : [0, \alpha) \to M$ which is incomplete (one can include timelike paths of bounded acceleration if one wishes [*4*]). We define an analytic singularity † as follows:

2.1. DEFINITION. — A singularity γ is analytic iff $Im\ \gamma$ is not contained in any compact subset of M.

We see that even though a space-time may be compact and have a singularity, no compact space-time has analytic singularities. Let us now try to relate this definition of an analytic singularity to our idea of what an analytic singularity should be. We think intuitively that γ should represent an analytic singularity if γ runs into some point where the metric is not defined. Of course, the whole difficulty with this idea is that in our space-time any points at which the metric is not defined are left out. But let us suppose we have an analytic singularity in this sense; that is, we suppose our space-time M is a submanifold of a larger manifold and that we have a singularity which runs into one of the points p of the larger manifold which is not a point of M. We will show γ is analytic in the sense of our definition.

First give the larger manifold any positive definite metric. There is a sequence $\gamma(t_\iota)$ which converges to p since γ runs into p. $\gamma(t_\iota)$ is thus Cauchy in our metric and so is Cauchy in the induced metric on M. Suppose γ is not analytic; that is, suppose the image of γ is contained in a compact subset of M. Then $\gamma(t_\iota)$ is a Cauchy sequence in a complete space since a compact set is complete in any metric. Thus $\gamma(t_\iota)$ converges in M showing that p has to be in the space-time, a contradiction.

Suppose now that we have an analytic singularity in our precise sense. Where is the « bad » point of our space-time. It is ∞ in the one point compactification of $M(M_\infty)$. Our condition for an analytic singularity implies that $\gamma(t)$ will have ∞ as a limit point as t approaches α. This, of course, is not intended to have any precise physical meaning since M_∞ is not in general even a manifold, but it does enable us to get a feeling of a « place » where things « go bad » for the geodesic. We should remark that our definition does not imply that some invariant must « blow up » for an analytic singularity. However if some invariant does go to infinity along a geodesic, then the singularity must be analytic since no invariant can go to infinity on a compact set. We should also observe that all inessential singularities are analytic. A singularity γ in a space-time M is inessential if there is another space-time M′ with M an open sub-space-time of M′ in which γ can be extended. Suppose we have such an inessential singularity which is not analytic, that is, suppose Im γ is contained in K compact. Suppose γ is defined on $[0, \alpha)$ in M and can be considered a continuous map from $[0, \alpha + t]$ into M′. Now $\gamma^{-1}(K)$ is closed, so it must contain α. $\gamma^{-1}(M)$ is open so it must contain $[0, \alpha + \lambda)$ since it contains α. So γ is actually extendable in M giving a contradiction.

Now that we have defined analytic singularities we will define Uniform Stationary space-times, and find some of their properties.

3. UNIFORM STATIONARY SPACE-TIMES

We start with the following definitions:

3.1. DEFINITIONS. — *a*) A *physical timelike Killing vector field* is a timelike Killing vector field of constant length.

b) A *special Killing vector field* is a vector field v together with a timelike vector field w such that

i) v is a Killing vector,
ii) $(v, v) =$ constant,
iii) $D_w v = 0$.

Note that a physical timelike Killing vector (PTKV) is also a special

Killing vector (SKV). For, if v satisfies *i*) and *ii*) above, then given any vector field x we have

$$(x, D_v v) = -(x, A_x v) = (A_v x, v) = -(D_x v, v) = -\frac{1}{2} x(v, v) = 0$$

So $D_v v = 0$ (here $A_x y = -D_y x$) and *iii*) is satisfied with $w = v$.

The following Lemma will give us a hint how to define a Uniform Stationary space-time:

3.2. LEMMA. — Suppose M has a SKV then

a) if Ric $(v, v) \geq 0$ then v is totally geodesic, *i. e.*

$$D_x v = 0 \qquad \forall x$$

and consequently

b) Ric $(v, v) \leq 0$ at every pt.

Proof. — We have (see [5])

$$\text{Ric}(v, v) + \text{trace}(A_v A_v) = 0.$$

Now

$$\text{trace } A_v A_v = \Sigma(x_\iota, A_v A_v x_\iota)\varepsilon(\iota) = -\Sigma(A_v x_\iota, A_v x_\iota)\varepsilon(\iota),$$

where $\varepsilon(\iota) = 1$ if $\iota \neq 0$, $\varepsilon(0) = -1$. Here $\{ x_\iota \}$ is an orthonormal basis of the tangent space at a point. So we need only show that $A_v(x_\iota)$ is spacelike; then both sides of the above equality will be positive, and thus 0. Then

$$\Sigma(A_v x_\iota, A_v x_\iota) = 0$$

and each $A_v x_\iota$ is spacelike, thus

$$0 = A_v x_\iota = -D_{x_\iota} v$$

and we have shown part *a*). *b*) Will follow trivially since Ric $(v, v) > 0$ at a pt will contradict trace $(A_v A_v) \geq 0$. So we need only show $A_v x_\iota$ is spacelike. We will show $A_v x_\iota$ is perpendicular to the timelike vector w

$$(w, A_v x_\iota) = -(A_v w, x_\iota) = (D_w v, x_\iota) = 0.$$

3.3. COROLLARY. — Suppose M has a PTKV then

a) if Ric $(w, w) \geq 0$ for all w timelike then M has a totally geodesic timelike vector field,

and consequently

b) Ric $(w, w) \leq 0$ for some timelike vector w.

The corollary eliminates the possibility of a spacetime with a PTKV which satisfies the energy condition with $\Lambda = 0$. Part *a*) says that if we relax the energy condition and allow equality with 0 for nonempty space, we still will get some models with a PTKV. This condition is compatible with any $\Lambda \neq 0$. Since Λ should be very small the condition that $\mathrm{Ric}(w, w) \geq 0$ for w timelike is perhaps not unreasonable. Since it is the only one that mathematically is of use we will make the following definition:

3.4. DEFINITION. — A space-time M is a Uniform Stationary space-time if

a) it has a locally defined PTKV in a neighbourhood of each point,
b) it satisfies the weak energy condition $\mathrm{Ric}(w, w) \geq 0$ for w timelike.

We give here some examples of Uniform Stationary space-times. The first example is perhaps one of the earliest cosmological models and it motivated Einstein's introduction of the cosmological constant Λ. It is $\mathbf{R} \times S^3$ as a manifold with metric $(dt)^2 - (d\sigma)^2$ where $d\sigma^2$ is the metric of the 3 sphere S^3 induced from its embedding in $\mathbf{R}^4$.

The second class of examples are the Gödel universes. The metric is given by

$$ds^2 = a(dx^1)^2 + 2be^{\psi(x^4)}dx^1dx^2 + ce^{2\phi(x^4)}(dx^2)^2 + g_{33}(dx^3)^2 + g_{44}(dx^4)^2$$

where a, b, c, g_{33}, g_{44} are constants [*6*]. These space-times are Uniform Stationary if $\frac{\partial}{\partial x_3}$ is timelike.

We now prove an easy theorem that classifies the Uniform Stationary space-times.

3.5. THEOREM. — Let M be a space-time. The following are equivalent:

a) M is Uniform Stationary,

b) M is locally isometric at each point to $M' \times \mathbf{R}$, M' with a negative definite metric and $\mathbf{R}$ with the flat metric dt^2,

c) Φ^*, the local holonomy group of M, is compact at each point,

d) M has a timelike totally geodesic vector field v in a neighbourhood of each point.

Proof. — *a)* $\Rightarrow$ *d)* is 3.3. *d)* $\Rightarrow$ *a)* since v is the physically timelike Killing vector. *d)* $\Leftrightarrow$ *b)* is de Rham's decomposition theorem. *d)* $\Rightarrow$ *c)* since any subgroup of the Lorentz group that fixes a timelike vector is compact. For *c)* $\Rightarrow$ *d)* we assume the connection is analytic. Thus there is a neighbourhood U such that Φ^* is the holonomy group of the connection restricted to U. Now a compact connected subgroup of the Lorentz group must fix a timelike vector. We define the vector field v in U by parallel transport which is well defined since Φ^* fixes v. v is totally geodesic and thus we are done.

3.6. Corollary. — Let M be simply connected and complete. Then M is a Uniform Stationary space-time $\Leftrightarrow$

$$M \simeq \mathbf{R} \times M'$$

where **R** is the flat metric and M' has a negative definite metric.

Proof. — « $\Leftarrow$ » is obvious. « $\Rightarrow$ » follows from *a*) $\Rightarrow$ *b*) of the theorem where $\pi_1(M) = 0$ and completeness makes the De Rham theorem global.

4. SINGULARITY THEOREMS

We are now ready to prove the theorem relating Uniform Stationary space-times to analytic singularities. First we need a definition.

4.1. Definition. — A Uniform Stationary space-time is special if it has no locally defined spacelike totally geodesic vector fields.

A special Uniform Stationary space-time (S. U. S. space-time) is one without too much flatness. If a Uniform Stationary space time has a unique timelike hypersurface orthogonal Killing vector field then it is S. U. S. Thus the « special » condition means there is a « unique time ».

4.2. Theorem. — Let $\pi_1(M)$ be finitely generated and let M be a special Uniform Stationary space-time. Then γ is a singularity $\Leftrightarrow \gamma$ is an analytic singularity.

Proof. — Let α_ι be a set of generators of $\pi_1(M)$. Let $(\overline{M}, \pi)$ be the covering space of M such that $\pi_1(\overline{M})$ is generated by $\bar{\alpha}_1$ where

$$\pi_*(\bar{\alpha}_\iota) = 2\alpha_\iota.$$

$\overline{M}$ has a natural space-time structure which makes it a S. U. S. space-time. Let γ be a path in M and $\bar{\gamma}$ its lift to $\overline{M}$. $\bar{\gamma}$ is singular if and only if γ is. Also $\bar{\gamma}$ is an analytic singularity if and only if γ is since π^{-1} of a compact set is compact (π is a finite covering). Thus we may confine ourselves to showing the theorem for $\overline{M}$. First we define a global totally geodesic timelike vector field v. Since $\overline{M}$ is Uniform Stationary v is locally defined. To define v globally we need only verify that parallel transport around $\bar{\alpha}_\iota$ leaves v fixed. We claim that transport around α_ι in M of $\pi_*(v) = v_1$ gives $\pm\, v_1$, which shows $\bar{\alpha}_\iota$ fixes v. Suppose α_ι does not take v_1 into $\pm\, v_1$. Then the result of transport around α_ι of v_1, v_2 say, must be independent of v_1. By using parallel transport locally one can extend v_2 to a totally geodesic vector field. By taking an appropriate linear combination of v_1 and v_2 one can thus find a locally defined spacelike totally geodesic vector field, giving a contradiction.

Thus we have a globally defined totally geodesic vector field v. Now define the following positive definite metric (normalizing v so that $g(v, v) = -1$)

$$\bar{g}(v_1, v_2) = g(v_1, v_2) - 2g(v_1, v)g(v_2, v).$$

Since g and $\bar{g}$ have the same Levi-Citiva connection, a geodesic in one is a geodesic in the other. Since every singularity is analytic in a positive definite metric we are finished.

4.3. COROLLARY. — Suppose M is a special Uniform Stationary space-time in some open neighborhood U and $\pi_1(M)$ is finitely generated. Assume M is real analytic. Then M has only analytic singularities.

Proof. — Using the above argument and Theorem 3.5 we find the identity component of the group to be $O(n-1)$. This was all that was used in the proof of our theorem.

Although S. U. S. space-times are special indeed (because they are Uniform Stationary) Corollary 4.3 is of physical interest because of its form. All the conditions (except the negligable technical condition that $\pi_1(M)$ be finitely generated) are local, yet the result is global. Since observational information about the universe can only be gathered from a small portion of it, only this type of theorem can give an observational bases for a global property.

4.3. DEFINITION. — A space time M is periodic if there is a compact space-time $\tilde{M}$ and a map π

$$\pi : M \to \tilde{M}$$

which is a covering projection and a local isometry.

The most common examples of periodic space-times are the Robertson Walker metrics with spherical spacelike sections and a periodic time function $R(t)$. $\tilde{M}$ in this case will be $S^3 \times S^1$. Some periodic Uniform Stationary space-times are the Gödel universes with periodic $\psi(x^4)$. $\tilde{M}$ in this case is T^4, the 4 dimensional torus.

4.4. THEOREM. — If M is a periodic special Uniform Stationary space-time then M is complete.

Proof. — Let $\pi : M \to \tilde{M}$ with $\tilde{M}$ a compact space-time. $\tilde{M}$ is also a special Uniform Stationary space-time since π is a local isometry. $\tilde{M}$ has a finitely generated Poincaré group because it is a compact manifold. Since $\tilde{M}$ is compact 4.2 says it must be singularity free. Thus M is also.

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(*) *Note added in proof:* A similar concept seems to have been also considered by Shepley and Misner (cf. *Ann. Phys.* t. **48**, 1968, p. 526). The theorems given here, we believe, provide new insights.