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Analytic singularities and geodesic completeness. II (*)

by

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ABSTRACT. — In [*1*] we defined the concept of a periodic space-time. Here we show that a very large class of periodic space-times are complete. We also define a singular boundary for a space-time. It is shown that if a singularity is essential the potentials g_{ij} must go « bad » at this boundary.

1. INTRODUCTION

In [*2*] we defined the concept of an analytic singularity and showed that in a Uniform Stationary space-time every singularity is analytic. This gave us an immediate « nonsingularity » theorem for periodic Uniform Stationary space-times since Uniform Stationary is a local property.

Here we shall give a characterization of space-times that have a type of non-analytic singularity. Unfortunately the characterizing aspect of these space-times is a certain kind of acausality, which is a global property. This property is not preserved under covering projections, so to obtain a nonsingularity theorem about periodic space-times we must make assumptions about the causality structure of the compact space-time covered by the periodic one. Thus if our condition were just acausality, we would have no nonsingularity theorem since every compact space-time is acausal. Fortunately, the condition is somewhat stronger than acausality which very few

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space-times satisfy. Thus we do get a fairly general nonsingularity theorem which answers in part the problem posed in [*2*].

In the second section we define a new singular boundary: This boundary has the property that more than one point may correspond to a single incomplete geodesic ray. Thus it is different from both the Geroch and Schmidt boundaries. Our boundary has two advantages. Firstly, it is always smooth which the Schmidt and Geroch boundaries need not be. Secondly, there is an extension theorem which says that if the metric does not go « bad » at a singular point corresponding to an incomplete geodesic, then this geodesic is extendable in a larger manifold. This theorem answers the question whether something must go « bad » at some point when a geodesic is incomplete.

In order to keep this work relatively self-contained we remind the reader that a singularity γ (incomplete geodesic or causal path) is called analytic if its image is contained in a compact subset of the space-time. For a discussion of the definition the reader is referred to [*2*].

2. CHARACTERIZATION OF NON-ANALYTIC SINGULARITIES

Since we will make extensive use of the affine parameter of null curves we will start with the following definition:

2.1. DEFINITION. — We define the affine parameter s of a curve γ in a space-time M in the following way. Pick a real number t_0 for which $\gamma(t_0)$ is defined. Pick any positive definite metric in $T_{\gamma(t_0)}(M)$. We define a real valued function F on the domain of γ by setting

$$F(t) = (\gamma_{t,t_0} \circ \gamma_*(t) \mid \gamma_{t,t_0} \circ \gamma_*(t))^{1/2}$$

where $\gamma_*(t)$ is the tangent to γ at t, γ_{t,t_0} is the parallel transport of $\gamma_*(t)$ back to $\gamma(t_0)$ by way of γ, and $(\mid)$ is the metric in $T_{\gamma(t_0)}(M)$. Now the affine parameter s is given by

$$s = \int_{t_0}^{t} F(t)dt.$$

Observe that s is well defined to within an affine transformation of **R**. Also note that if γ is a geodesic, this is the usual affine parameter since F(t) is constant in this case when t is the usual affine parameter of γ.

Now a curve has an incomplete affine parameter if and only if it is incomplete in the sense of Schmidt [*3*]. We proceed with a theorem which is an important tool in dealing with non-analytic singularities.

2.2. THEOREM. — Let γ be a closed curve in a space-time M ($\gamma(t) = \gamma(t + \alpha)$ for all t and some fixed α).

γ's affine parameter is incomplete $\Leftrightarrow$
γ is a null curve with

$$\gamma_{t,t+\alpha} \circ \gamma_*(t) = \lambda \gamma_*(t) \qquad \lambda \neq \pm 1.$$

Proof. — « $\Leftarrow$ ». We must show that the affine parameter is incomplete. We will show

$$A = \lim_{n \to \infty} \int_{t_0}^{t_0+n\alpha} (\gamma_{t,t_0} \circ \gamma_*(t) \mid \gamma_{t,t_0}\gamma_*(t))^{1/2} dt < M_0 \qquad \text{if} \qquad |\lambda| < 1.$$

A similar proof may be used for $|\lambda| > 1$ by looking at $\lim_{n \to \infty} \int_{t_0-n\alpha}^{t_0} F(t)dt$.
Now define

$$g_{n-1} = \int_{t_0+(n-1)\alpha}^{t_0+n\alpha} (t_{t,t_0} \circ \gamma_*(t) \mid \gamma_{t,t_0} \circ \gamma_*(t))^{1/2} dt.$$

We have

$$g_{n-1} = |\lambda|^{n-1} \int_{t_0}^{t_0+\alpha} (\gamma_{t,t_0} \circ \gamma_*(t) \mid \gamma_{t,t_0} \circ \gamma_*(t))^{1/2} dt$$
$$= |\lambda|^{n-1} g_1$$

so

$$A = \sum_{n=1}^{\infty} g_{n-1} = g_1 \sum_{n-1}^{\infty} |\lambda|^n = g_1 \left[\frac{1}{1 - |\lambda|} - 1 \right] = M_0$$

and we have shown the first implication.

« $\Rightarrow$ ». Suppose γ's affine parameter is future incomplete for definiteness. That is, assume the A defined above is finite. So g_n must converge to zero. This means

$$\lim_{n \to \infty} (\gamma_{t_0+n\alpha,t_0} \circ \gamma_*(t_0 + n\alpha) \mid \gamma_{t_0+n\alpha,t_0} \circ \gamma_*(t_0 + n\alpha)) = 0.$$

Now let $L = \gamma_{t_0+\alpha,t_0}$ so $L^n = \gamma_{t_0+n\alpha,t_0}$ and Lis a Lorentz transformation. Now $\gamma_*(t_0 + n\alpha)$ equals $\gamma_*(t_0) = v$. Thus $L^n v$ must approach the 0 vector. The only vectors v for which $L^n v$ approaches zero is a null vector. Now L must fix a null ray since only vectors on the null ray fixed by the pure Lorentz part of L can contract to zero under L^n. So if L does not fix this ray, then an infinite number of $L^n v$ will be off this ray and thus bounded away from zero. Thus since L fixes a null ray and the only vectors that can go to zero under L^n are on this ray we have

$$Lv = \lambda v.$$

We observe $|\lambda| \neq 1$ for if $|\lambda| = 1$, $L^n v$ would not converge to zero. This is exactly what we had to prove since t_0 was arbitrary.

We see that incomplete closed curves must be a special type of null curve. In fact we will find these special null curves characterized space times with non-analytic singularities. Thus we give them a name in the following definition:

2.3. DEFINITION. — γ is called a trapped null curve if $\gamma(t) = \gamma(t + \alpha_0)$ for all t and

$$\gamma_{t,t+\alpha_0} \circ \gamma_*(t) = \lambda\gamma_*(t) \qquad |\lambda| \neq 1$$

where $\gamma_*(t)$ is the tangent at t to γ and $\gamma_{t,t+\alpha_0}$ denotes parallel transport along γ from $\gamma(t)$ to $\gamma(t + \alpha_0)$.

Before we state our theorem we need a definition which gives us a certain smoothness criterion.

2.4. DEFINITION. — Let γ be non-analytic singularity defined on $[0, \alpha)$. γ is called regular if there is a positive definite metric d on a neighborhood of γ together with a sequence t_ι in $[0, \alpha)$, such that

a) $t_\iota \to \alpha$, $t_{\iota+1} > t_\iota$ and $\gamma(t_\iota)$ converges.

b) $\int_{t_\iota}^{t_{\iota+1}} d(\gamma_*, \gamma_*)^{1/2} dt < \mathrm{N}$ for all ι and for some number N.

2.5. THEOREM. — M has a regular non-analytic singularity only if M has a trapped null curve.

2.6. *Corollary.* — Every causal space-time has only non-analytic singularities which are not regular.

Proof of theorem. — Note that if M has a trapped null curve then this trapped null curve is a Schmidt regular non-analytic singularity.

Assume that γ is a regular non-analytic singularity. We define

$$\gamma_\iota : [t_\iota, t_{\iota+1}] \to \mathrm{M}$$

by $\gamma_\iota(t) = \gamma(t)$ for $t \in [t_\iota, t_{\iota+1} - \varepsilon/\iota]$ and γ_ι otherwise is γ altered slightly so as to make γ_ι a smooth closed curve. We may do this so that the lengths in d of γ_ι remain bounded by $2\mathrm{N} = \mathrm{A}$. Now considering γ_ι as a function of its arclength s in d define $\bar{\gamma}_\iota$ as follows:

$$\bar{\gamma}_\iota(s) = \gamma_\iota(s\mathrm{A}_\iota/\mathrm{A} + s_\iota) \qquad \mathrm{A}_\iota = s_{\iota+1} - s_\iota \qquad s \in [0, \mathrm{A}].$$

$\bar{\gamma}_\iota$ are equicontinuous since

$$\begin{aligned} d(\bar{\gamma}_\iota(s_1), \bar{\gamma}_\iota(s_\iota)) &\leq \int_{s_1}^{s_2} d(\bar{\gamma}_{\iota*}, \bar{\gamma}_*) ds \\ &= (s_2 - s_1)/\mathrm{A}_\iota/\mathrm{A} < s_2 - s_1 . \end{aligned}$$

Since Im γ is in a compact set, $\overline{\mathrm{Im}\,\gamma}$ is compact and we can use the Ascoli theorem to find that a subsequence of the $\bar{\gamma}_\iota$ converges uniformly to $\bar{\gamma}$. Using also the Ascoli theorem and that the $\bar{\gamma}_\iota$ have bounded acceleration one can find a subsequence of this sequence which converges smoothly so that $\bar{\gamma}$ is smooth. Thus $\bar{\gamma}$ is a smooth closed curve. We will show that $\bar{\gamma}$ is a trapped null curve. Let $m \in$ Image $\bar{\gamma}$. Let $m_\iota \to m$, $m_\iota \in$ Im γ_ι. Let $\bar{\Gamma}$ (respectively Γ_ι) denote parallel transport around $\bar{\gamma}$ (respectively γ_ι) from m (respectively m_ι). We must show

$$\bar{\Gamma}\bar{\gamma}_*(m) = \lambda\bar{\gamma}_*(m) \qquad |\lambda| = 1.$$

Now since γ is geodesic and since γ_ι is the same as γ except on a small interval of affine parameter we have

$$\Gamma_\iota\gamma_*(m_\iota) = \lambda_\iota\gamma_*(m) + o(\iota).$$

Since $\bar{\Gamma}\bar{\gamma}_*(m)$ is defined we may take limits to find

$$\bar{\Gamma}\bar{\gamma}_*(m) = \lambda\bar{\gamma}_*(m)$$

$\lambda = \lim \lambda_\iota$. We need now only verify that λ_ι does not approach $\pm$ 1. If the $|\lambda_\iota|$ did approach 1, then the function $F(t)$ in Definition 2.1 would approach being periodic and thus its integral would be onto **R** which contradicts the fact that γ was incomplete. Thus $\bar{\gamma}$ is the desired trapped null curve.

Our theorem would be somewhat easier to show if we assume we had causality, that is no timelike or null curves which are closed [*4*]. However, the advantages of the stronger theorem we have shown is twofold. The first is an aesthetic advantage. We have almost a necessary and sufficient condition every singularity to be analytic, namely the non-existence of trapped null curves. This gives us an intuitive feeling how the Lorentz nature of the metric allows non-analytic singularities to occur. The second reason is practical. A theorem which states every singularity must be analytic under certain conditions will automatically give some sort of nonsingularity theorem for periodic space-times. Unfortunately, if causality were our condition, we would get no nonsingularity theorem. This is because causality is not preserved under covering projections and it is in fact known that every compact space-time is acausal [*4*]. Thus we need a weaker condition that compact space-time must satisfy. Such a condition is found in our criterion of no trapped null curves because trapped null curves occur very rarely indeed. Thus we make the following definition.

2.7. Definition. — A periodic space-time M is called finely causal if there is a covering projection π and compact space-time $\tilde{M}$ with

$$\pi : M \to \tilde{M}.$$

π is a local isometry and $\tilde{M}$ has no trapped null curves and no non-regular non-analytic singularities.

2.8. Example. — Any Robertson Walker metric with spherical spacelike section and periodic function Rt) is finely causal.

2.9. *Theorem.* — Every finely causal periodic space-time is singularity free.

Proof. — M is singularity free if and only if $\tilde{M}$ is. M is finely causal so that $\tilde{M}$ has no non-analytic singularities by 2.5. But any singularity γ is continued in $\tilde{M}$ which is compact and thus γ must be non-analytic. So $\tilde{M}$ is singularity free.

As we see in 2.8, Theorem 2.9 supplies a nonsingularity theorem which generalizes a wide class of nonsingular Robertson Walker space-times [*1*].

3. A SINGULAR BOUNDARY AND AN EXTENSION THEOREM

The most desirable theorem relating singularities (geodesic incompleteness) to space-times would be some theorem which says that to every singularity corresponds some point at which something goes bad. Of course this theorem would not apply to singularities created by removing points from otherwise complete space-times, so it should apply to essential singularities defined as follows.

3.1. DEFINITION. — A singularity $\gamma : [0, \alpha) \to M$ is not essential if there is M_1 which is a space-time with

$$M \subset M_1.$$

M open in M_1 and γ is extendable to a geodesic $\gamma : [0, \alpha + \varepsilon) \to M_1$.

Now one of the purposes of the various boundaries (Geroch boundary, Schmidt boundary, etc.) is to supply the points where things go bad. However, the topological properties of these boundaries are unknown and the topoligical problem alone of when $M \cup \partial_G M$ ($M \cup \partial M$) can be imbedded in a manifold M' is almost impossible to solve. Thus one is unable to even arrive at the analytic question which is in a certain sense the real question.

We avoid the topological problem by defining a new boundary which is topologically nice and which exists for almost every space-time.

3.2. DEFINITION. — *a*) A space-time M has a smooth infinity if the topological space M is the interior of a compact manifold with boundary $\overline{M}$. We let $\partial M = \overline{M} \sim M$.

b) M has a very smooth infinity if M has a smooth infinity as above and $\overline{M} \subset M'$ where M' is a manifold (without boundary).

Since a manifold which does not have a very smooth infinity is somewhat pathological, it would be very difficult to find a physical model that did not have this property. We give examples of $\overline{M}$ in some specific cases. If a space-time is defined on $\mathbf{R}^4$ or $S^3 \times \mathbf{R}$ (as the Robertson-Walker models) are then $\overline{M}$ will be $[0, 1] \times [0, 1] \times [0, 1] \times [0, 1]$ or $[0, 1] \times S^3$. For Schwartzschild space $\overline{M}$ is $[0, 1] \times X$ where X is $\{ x \in \mathbf{R}^3/1 \leq | x | \leq 2 \}$.

Our boundary, $\partial_c M$, will be a subset of ∂M. Thus $\partial_c M$ will be topologically nice to start off with.

3.3. DEFINITION. — *a*) A space-time is called R-causal if it is causal and has no non-regular non-analytic singularities.

b) Let M be R-causal. We define

$$\partial_c(M) \subset \partial M$$

as follows:

$$\partial_c(M) = \{ x \in \partial M / x \in (\overline{\text{Im } \gamma}) \text{ where } \gamma \text{ is any incomplete geodesic} \}.$$

We note that to each incomplete geodesic corresponds at least *one* point in $\partial_c M$. This is so because $\overline{\text{Im } \gamma}$ is closed in $\overline{M}$ and thus is compact. So if $\overline{\text{Im } \gamma} \cap \partial M = \varphi$ then $\overline{\text{Im } \gamma}$ must be in M which would mean γ is not an analytic singularity. This contradicts M's causality by Corollary 2.6. We also observe that an incomplete geodesic may correspond to many points in $\partial_c M$. Thus this boundary allows a geodesic to run back and forth between two or more boundary points which is not allowed in $\partial_G(M)$ and $\partial_S(M)$. It is this property which avoids the topological problems encountered in these other boundaries. The usefulness of this boundary is seen in the following theorem.

3.4. THEOREM. — Suppose M is a R-causal space-time with a very smooth infinity. Suppose there is a submanifold M'' of M' such that

i) $M \cup (\partial_c M \cap \overline{\text{Im } \gamma}) \subset M''$,

ii) M'' has a R-causal space-time structure, M is an open subspace-time of M'',

then γ is not an essential singularity.

Proof. — We need to extend γ in M''. We note $\overline{\text{Im } \gamma}$ is compact since it is a closed subset of the compact manifold $\overline{M}$.

$\overline{\text{Im } \gamma} \subset M''$ by definition. Thus if γ were inextendable in M'' it would be an analytic singularity in M''. But since M'' is R-causal we can use 2.6 to get our contradiction.

3.5. COROLLARY. — Suppose M is a R-causal space-time with a very smooth infinity. If there is a submanifold M'' of M' such that

i) $M \cup \partial_c M \subset M''$,

ii) M'' is an R-causal space-time having M as an open subspace-time.
Then every singularity in M is inessential.

Our theorem says that under reasonable circumstances the metric must go « bad » at some singular points in $\partial_c(M)$ for any singularities to be essential. Here going « bad » means that for some reason the metric can not be extended R-causally to a small neighbourhood of the singular points. Note that by altering the example due to Misner [5] of a 2-dimensional compact incomplete space-time, one can see that the causal part of this criterion is necessary.

Thus we have presented a partial answer to the question of when singularities are essential. The answer is only partial because there is no guarantee that the extended space-time M'' will be complete. The question when M can be extended to a complete space-time is left completely open.

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