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Analyticity properties in Ising models

by

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ABSTRACT. — We investigate domains in which the pressure depends analytically on the coupling constants and on the temperature (high or low enough temperature). Some consequences are deduced.

0. INTRODUCTION

The purpose of this paper is to derive in appropriate domains the analytic dependence of the pressure on each complex variable $z_{A_i} = e^{-2\beta J_{A_i}}$ where β is the inverse temperature, J_{A_i} is the coupling constant which measures the interaction of the particles within the subset A_i of Z^ν. This is a generalization of the famous Lee Yang theorem [*1*]. Two basic tools are needed: the Asano contraction process [*2*] and a theorem of Ruelle [*3*]; which were already used in [*4*] in some special cases.

In Section I, we use the Asano contraction to build the partition function in two different ways which correspond to low and high temperatures; at the end of this section the partition function is expressed by means of a generalized « contour formula ».

In Section II, using a theorem of Ruelle, we look for domains in the product of complex planes of the variables $\{ z_{A_1}, z_{A_2} \dots z_{A_n} \}$ within which the partition function does not vanish.

Section III is devoted to applications. The analyticity of the pressure is derived. We exhibit the so called « Hamiltonian algebra » $\mathscr{A}_H$ generated by the bonds of H and show the existence of a unique translation invariant equilibrium state on this algebra. In some cases the uniqueness of the equilibrium states is proved, from which an ergodic decomposition follows.

In Section IV, these properties are derived in the Ising nearest neighbor model with an external magnetic field both in the ferromagnetic and in the antiferromagnetic cases.

In Section V, high spin systems and hard core conditions are studied.

I. BUILDING UP THE PARTITION FUNCTION

1. Generalities

We consider a spin $\frac{1}{2}$ lattice system interacting through many-body Ising interactions. The third Pauli matrix σ_i^3, acting on a two-dimensional space $\mathscr{H}_i$, is associated with each vertex i. We shall use $\mathscr{H}_\Lambda$ to denote the tensor-product of such $\mathscr{H}_i$'s, $i \in \Lambda$, with Λ some finite set belonging to Z^ν. Let $\mathscr{A}$ be the commutative C*-algebra generated by linear combinations of products of σ_i's. We shall deal with the following hamiltonian:

$$H_\Lambda = \sum_{A \in B_\Lambda} J_A(\sigma_A - 1) \tag{1.1}$$

where $\sigma_A = \bigotimes_{i \in A} \sigma_i^3$, B_Λ being the set of interacting bonds included in Λ; this set will be ordered:

$$B_\Lambda = \{ A_1^{(\Lambda)}, A_2^{(\Lambda)}, \ldots, A_j^{(\Lambda)} \ldots A_{N(\Lambda)}^{(\Lambda)} \}$$

$N(\Lambda)$ being the number of bonds in Λ.

The object of this section is to compute the partition function of models defined by (1.1), using the contraction process [2]; two such processes will be used, corresponding either to the low temperature expansion (L. T. E.) or to the high temperature expansion (H. T. E.).

We first recall the definition of « topological cycles » as elements of $\{0, 1\}^{N(\Lambda)}$ [5]. The set T_Λ of topological cycles spans a vector space on the field $\{0, 1\}$ through the component wise addition modulo 2 denoted (Δ). We also use the component wise product denoted ($\cap$). We shall also need the mixed operation:

$$X \cup X' = (X \Delta X') \cup (X \cap X')$$

The number of non zero components of a topological cycle X will be called its length [X]. One can define in a standard way [5] a basis of the vector space T_Λ.

1.1. *The low temperature topological cycles.* — Let us define the « low temperature topological cycles » denoted L. T. T. C. as the topological cycles associated with a subset of bonds $J \subset B_\Lambda$, satisfying:

$$(i) \qquad \prod_{A \in J} \sigma_A = 1$$

$$(ii) \qquad \nexists J' \subset J \quad \text{with} \quad \prod_{A \in J'} \sigma_A = 1$$

The L. T. T. C. generate an additive group L_Λ with composition law Δ.

1.2. *The high temperature topological cycles.* — We shall also consider the « high temperature topological cycles » denoted H. T. T. C. : they are the topological cycles associated with each point i of the lattice, their non zero components being the bonds issued from the point i. There are thus exactly as many H. T. T. C. as points in the lattice Λ. The H. T. T. C. also generate an additive group « K_Λ » with respect to the addition Δ.

2. Building up the partition function with Asano contractions

The basic idea developed in this part is the following one: first we compute the partition function of elementary sublattices which will be chosen to be L. T. T. C. in the low temperature expansion case, H. T. T. C. in the high temperature expansion case. Then we use the Asano contraction to construct the general partition function by means of these « elementary partition functions ».

2.1 Let $\Gamma_{i_1}, \Gamma_{i_2}, \ldots, \Gamma_{i_p}$ be L. T. T. C. belonging to a basis of T_Λ: we define

$$\Xi(\Gamma_{i_1}) = \operatorname*{Tr}_{\mathscr{H}_{\Gamma_{i_1}}} e^{-\beta H_{\Gamma_{i_1}}} \qquad \text{with} \qquad H_{\Gamma_{i_1}} = \sum_{A \in \Gamma_{i_1}} J_A(\sigma_A - 1)$$

$\mathscr{H}_{\Gamma_{i_1}}$ being the Hilbert space associated with the set of spins occurring in the interaction bonds of Γ_{i_1}. We define similarly:

$$\Xi(\Gamma_{i_1} \cup \Gamma_{i_2} \cup \ldots \Gamma_{i_p}) = \operatorname*{Tr}_{\mathscr{H}_{\Gamma_{i_1} \cup \Gamma_{i_2} \ldots \cup \Gamma_{i_p}}} e^{-\beta H_{\Gamma_{i_1} \cup \Gamma_{i_2} \ldots \cup \Gamma_{i_p}}}$$

2.2 Let $\theta_{j_1}, \theta_{j_2}, \ldots, \theta_{j_p}$ be H. T. T. C. associated with the points $j_1, j_2, \ldots, j_p$, we shall use:

$$\Xi(\theta_{j_1}) = \operatorname*{Tr}_{\substack{\theta_{j_1} = \pm 1 \\ \sigma_i = +1 \text{ if } i \neq j_1}} e^{-\beta H_{\theta_{j_1}}} \qquad \text{with} \qquad H_{\theta_{j_1}} = \sum_{A \in \theta_{j_1}} J_A(\sigma_A - 1)$$

and similarly:

$$\Xi(\theta_{j_1} \cup \theta_{j_2} \ldots \cup \theta_{j_p}) = \operatorname*{Tr}_{\substack{\sigma_j = \pm 1 \ldots \sigma_{j_p} = \pm 1 \\ \sigma_i = +1 \text{ if } i \neq j_1, \ldots, j_p}} e^{-\beta H_{\theta_{j_1} \cup \theta_{j_2} \ldots \cup \theta_{j_p}}}$$

2.3 We shall define a homomorphism from the additive group K_Λ

onto the set of restricted partition functions Ξ_θ, equipped with the law « $*$ » as follows:

$$\theta \in K_\Lambda \to Z_\theta = \sum_{A\in\theta} z_A \qquad (\text{with } z_A = e^{-2\beta J_A}\,;\, Z_\phi = 1)$$

$$Z_\theta * Z_{\theta'} = Z_{\theta\Delta\theta'} \qquad \forall\theta, \theta' \in K_\Lambda$$

Similarly in the high temperature situation we shall define a homomorphism from the group L onto the set of restricted partition functions Ξ_θ, equipped with the law « $*$ »:

$$\Gamma \in L_\Lambda \to \zeta_\Gamma = \prod_{A\in\Gamma} \zeta_A \qquad (\text{with } \zeta_A = \text{th } \beta J_A,\, \zeta_\phi = 1)$$

$$\zeta_\Gamma * \zeta_{\Gamma'} = \zeta_{\Gamma'\Delta\Gamma} \qquad \forall\Gamma, \Gamma' \in L_\Lambda$$

These $*$ laws are distributive with respect to the usual addition of functions.

Now we can derive the following results.

2.4 LEMMA. — The following formula holds:

$$(i) \qquad \Xi(\Gamma_i) \simeq 1 + \zeta_{\Gamma_i} \simeq \prod_{A\in\Gamma_i}(1 + z_A) + \prod_{A\in\Gamma_i}(1 - z_A)$$

$$(ii) \qquad \Xi(\theta_j) \simeq 1 + Z_{\theta_j} \simeq \prod_{A\in\theta_j}(1 + \zeta_A) + \prod_{A\in\theta_j}(1 - \zeta_A)$$

$$(iii)\ \Xi(\Gamma_{i_1} \cup \Gamma_{i_2}) \simeq (1 + \zeta_{\Gamma_{i_1}}) * (1 + \zeta_{\Gamma_{i_2}}) \simeq \prod_{A_i\in(\Gamma_{i_1}\cap\Gamma_{i_2})} D(z_{A_{i_1}}, z_{A_{i_2}}) \{ \Xi(\Gamma_{i_1}).\Xi(\Gamma_{i_2}) \}$$

$$(iv)\ \Xi(\theta_{j_1} \cup \theta_{j_2}) \simeq (1 + Z_{\theta_{j_1}}) * (1 + Z_{\theta_{j_2}}) \simeq \prod_{A_j\in\{\theta_{j_1}\cap\theta_{j_2}\}} D(\zeta_{A_{j_1}}, \zeta_{A_{j_2}}) \{ \Xi(\theta_{j_1}).\Xi(\theta_{j_2}) \}$$

where D(.,.) denotes the contraction operator (cf. Appendix A1) and $\simeq$ means that unessential factors are omitted.

Proof. — (*i*) is obtained by taking the trace over $H_{\Gamma_{i_1}}$ and making use of the irreducibility property of L. T. T. C. 's.

(*ii*) is obtained by taking the trace over the space associated with the site *i*.

(*iii*) describes the contraction process in terms of the variables z_A, taking into account the irreducibility property of L. T. T. C. 's.

$$\Xi(\Gamma_{i_1} \cup \Gamma_{i_2}) \simeq 1 + \prod_{A\in\Gamma_{i_1}} \zeta_A + \prod_{A'\in\Gamma_{i_2}} \zeta_{A'} + \prod_{A''\in\Gamma_{i_1}\Delta\Gamma_{i_2}} \zeta_{A''}$$

Then we replace ζ_A by $\dfrac{1 - z_A}{1 + z_A}$.

(*iv*) is derived by the same trick upon evaluating

$$\Xi(\theta_{j_1} \cup \theta_{j_2}) \simeq 1 + \prod_{A \in \theta_{j_1}} z_A + \prod_{A' \in \theta_{j_2}} z_{A'} + \prod_{A'' \in \theta_{j_1} \Delta \theta_{j_2}} z_{A''}$$

we replace z_A by $\dfrac{1 - \zeta_A}{1 + \zeta_A}$.

We are now able to construct the general partition function associated with the box Λ. We choose a basis $\mathscr{B}_\Lambda$ of the vector-space of L. T. T. C. 's. Each A_i belongs to $k(i)$ such L. T. T. C. 's chosen among that basis, allowing us to define:

$$D_{z_{A_i}} = \prod_{j \neq j' = k(1)}^{k(i)} D\left(z_{A_i}^{\Gamma_j}, z_{A_i}^{\Gamma_{j'}}\right)$$

and similarly for $D_{z_{A_i}}$.

2.5. LEMMA. — The general partition function is written as:

$$(i) \qquad \Xi_\Lambda^{\text{L. T.}} \simeq \prod_{i \in \mathscr{B}_\Lambda}^{*} (1 + \zeta_{\Gamma_i}) \simeq \prod_{A_i} D_{z_{A_i}} \{ \Xi(\Gamma_1)\Xi(\Gamma_2) \dots \Xi(\Gamma_{|\mathscr{B}_\Lambda|}) \}$$

$|\mathscr{B}_\Lambda|$ is the number of L. T. T. C. 's in $\mathscr{B}_\Lambda$ in the low temperature case,

$$(ii) \qquad \Xi_\Lambda^{\text{H. T.}} \simeq \prod_{i \in \Lambda}^{*} (1 + Z_{\theta_i}) \simeq \prod_{A_i} D_{z_A} \{ \Xi(\theta_1)\Xi(\theta_2) \dots \Xi(\theta_{|\Lambda|}) \}$$

in the high temperature case. Here $\prod^{*}$ denotes the $*$ law product extended by distributivity. In (1) the product is taken over the L. T. T. C. 's belonging to the basis.

Proof. — We shall only prove (1) since the proof of (2) is quite similar. Let us first consider the right hand side:

$$\prod_{i \in |\mathscr{B}_\Lambda|}^{*} (1 + \zeta_{\Gamma_i}) \simeq \sum_{\{\Gamma_{i1}, \dots, \Gamma_{ip}\}} \zeta_{\Gamma_{i_1}} * \zeta_{\Gamma_{i_2}} \dots * \zeta_{\Gamma_{ip}}$$

where $\{ \Gamma_{i1}, \dots, \Gamma_{ip} \}$ are up to a permutation all possible distinct subsets of $\{ \Gamma_1, \dots, \Gamma_{|\mathscr{B}_\Lambda|} \}$.

Thus:

$$\prod_{i \in |\mathscr{B}_\Lambda|} (1 + \zeta_{\Gamma_i}) = \sum_{\{\Gamma_{i1}, \dots, \Gamma_{ip}\}} \zeta_{\Gamma_{i_1} \Delta \Gamma_{i_2} \dots \Delta \Gamma_{ip}}$$

Thus:

$$\prod_{i \in |\mathscr{B}_\Lambda|} (1 + \zeta_{\Gamma_i}) = \sum_\alpha \zeta_{\Gamma_\alpha}$$

where α runs over the group L_Λ of L. T. T. C., so that we have just obtained the partition function.

The second part of *i*) then follows by induction from Lemma 2.4 (*iii*).

3. A generalization of contour formulæ

The partition function just derived can be expressed in a very simple and compact form. Let us consider for simplicity ferromagnetic bonds

$$\Xi(\Gamma_i) \simeq \sum_{X \subset \Gamma_i} C_{\Gamma_i}(X) Z^{|X|} \qquad \forall \Gamma_i \in L_\Lambda$$

with

$$Z^{|X|} = \prod_{A \in X \subset \Gamma_i} z_A \qquad C_{\Gamma_i}(X) = \frac{1 + (-1)^{[X \Delta \Gamma_i]}}{2}$$

$[X \Delta \Gamma_i]$ denotes the number of non zero components of $X \Delta \Gamma_i$. If we choose a basis B_Λ of L. T. T. C., $\Xi_\Lambda^{L.T.}$ can be written as:

$$\Xi_\Lambda^{L.T.} \simeq \sum_{X \subset \Lambda} Z^X \prod_{\Gamma_i \in \mathscr{B}_\Lambda} C_{\Gamma_i}(X)$$

where the sum extends over all subsets of bonds in Λ.

In a similar way we derive in the high temperature case:

$$\Xi(\theta_j) \simeq \sum_{Y \subset \theta_j} \zeta^{|Y|} C_{\theta_j}(Y) \qquad \forall \theta_j \in K_\Lambda$$

$$\Xi_\Lambda^{H.T.} \simeq \sum_{Y \subset \Lambda} \zeta^{|Y|} \prod_{\theta_j \in \Lambda} C_{\theta_j}(Y)$$

Remark 1. — This formula can be used to obtain contour equations generalizing those of [*6*].

Remark 2. — Up to now $\Xi_\Lambda^{L.T.}$ refers to free boundary conditions, while $\Xi_\Lambda^{H.T.}$ refers to « + » boundary conditions. In fact one could obtain similar formulæ whatever the boundary conditions may be.

Remark 3. — Formula 3.1 depends on the choice of the basis.

II. ON THE LOCALIZATION OF ZEROS OF THE PARTITION FUNCTION

1. Preliminary

In this part we look for domains of the complex plane where the partition function does not vanish. We need to study the following polynomials, which appear in low and high temperature expansions:

$$Q(X) = Q(x_1, \ldots, x_p) = \prod_{i=1}^{p}(1 + x_i) + \prod_{i=1}^{p}(1 - x^i) \qquad \forall X \in \mathbb{C}^p$$

More precisely we are interested in domains where Q(X) does not vanish; the problem is reduced to find these domains for:

$$Q(X) \neq 0 \qquad\qquad | x_i | \leqslant 1$$

and

$$Q'(X) = \prod_{i=1}^{p}(1 + x_i) - \prod_{i=1}^{p}(1 - x_i) \neq 0 \qquad | x_i | \leqslant 1$$

This follows from the invariance of Q(X) (up to unessential factors) under the mapping:

$$(x_i, x_j) \to (x_i^{-1}, x_j^{-1}) \qquad x_i \neq 0$$

1.1. Lemma

(i) A set of real numbers M_i can be found, such that:

$$\forall | x_i | < K_i \qquad Q(X) \neq 0$$

(ii) For any x_j, x_i a set of real numbers L_{ij} can be found, such that for $| x_j | > L_{ij} | x_i |$, $Q'(X) \neq 0$.

The proof of (i) and (ii) is trivial. We refer to [*4*] for a detailed geometry of the domains.

Remark. — The argument is valid for any finite union of cycles. For antiferromagnets the existence of such domains is connected with the existence of « well-defined ground states », see Domb [7].

2. The partition function for L. T. E. and H. T. E.

We consider an ordered set $\{ B_\Lambda \}_\Lambda$ of compatible basis, i. e. for any $\Lambda \subset \Lambda'$, B_Λ is a sub-basis of $B_{\Lambda'}$. We impose the following condition: *condition* H. *There exists a sequence of basis of L. T. T. C.* $\{ V_\Lambda \}_\Lambda$ *such that the length of any L. T. T. C. in any basis of that sequence is bounded by some number k independent of* Λ.

For simplicity we shall suppose that the Hamiltonian is translationally invariant.

First we investigate the case of *finite range interactions*, i. e. when there is a finite set of different variables.

2.1. LEMMA. — The partition function $\Xi_\Lambda(z_1, z_2, \ldots, z_p)$ does not vanish whenever z_i belongs to the domains described below:

(i) *The Lee-Yang case:* a basis of topological cycles can be found, with L. T. T. C., resp. H. T. T. C., such that in each L. T. T. C. resp. H. T. T. C. only two variables appear in the contraction process (the others being taken real), see figures 1, 2.

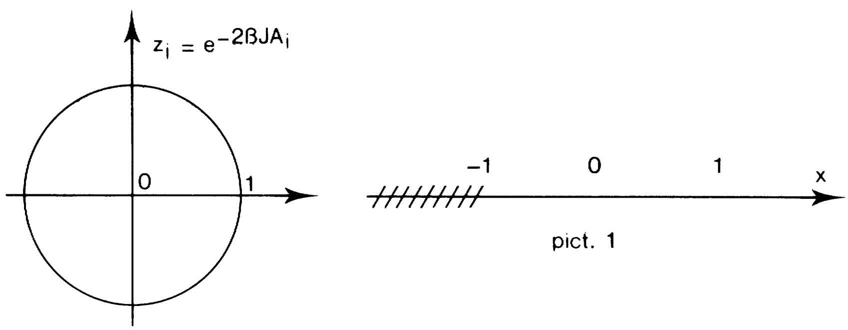


FIG. 1.

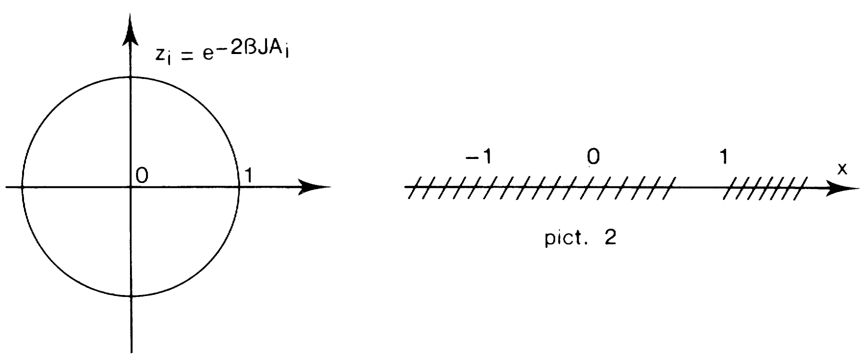


FIG. 2.

(ii) *The general case:* the domains (1) and (3) of figure 3 are always obtained. The domains (2) and (4) of figure 3 appear whenever Lemma II.1.1 is valid for the corresponding topological cycles of the basis.

Proof. — Lemma II.1.1 applied to any basis of L. T. T. C., resp. H. T. T. C., provides domains in which the restricted partition to such a cycle does not vanish. Then we build the partition function in both cases

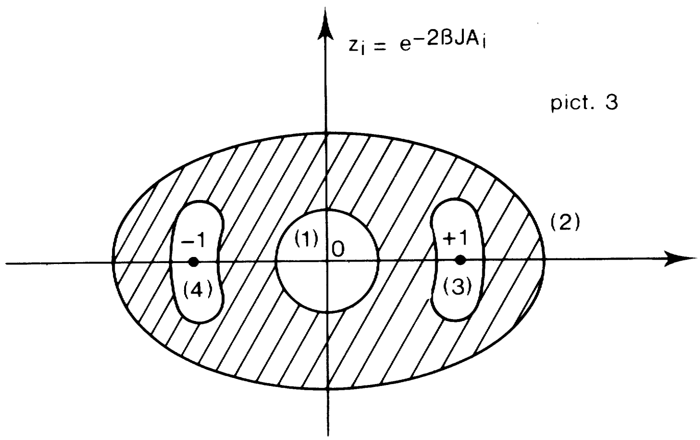


FIG. 3.

with Asano contraction as in § 1. Finally a lemma of Ruelle [3] provides domains in which the general partition does not vanish depending on the structure of the model as described in Lemma 2.1.

One can notice that in the high temperature case the domains are obtained, whatever the boundary conditions may be. This remains true in the low temperature case for free and cyclic boundary conditions as well as those corresponding to the ground states (cf. Appendix A2).

Now we investigate the case of *infinite range interactions.*

We assume for the sake of simplicity in the low temperature case that the coupling constants are ferromagnetic.

2.2. LEMMA. — The partition function does not vanish under the following conditions:

(*i*) High temperature case:

(*a*) $$\sum_{0 \in A \subset Z^\nu} \mathrm{Re}\,(\beta J_A) < k_0$$

(*ii*) Low temperature case:

(*b*) $$\sum_{0 \in A \subset Z^\nu} |J_A|^{\frac{2}{k-2}} < k_1 \,; \qquad \mathrm{Re}\,(\beta) \geqslant \beta_1$$

where the constants k_0, k_1, β_1 depend on the geometry of the model.

Proof. — (*i*) With any H. T. T. C. θ_i we associate the partition function $\Xi_\Lambda(\theta_i)$:

$$\Xi_\Lambda(\theta_i) = 1 + \prod_{i \in A \subset \Lambda} z_A$$

Since only finite body interactions are considered a finite number of contractions occur in any variable z_{A_i} then Ruelle's lemma [3] gives condition (*i*).

(*ii*) We recall that the condition H imposes the boundedness of the length of any L. T. T. C. uniformly in Λ. Clearly the number of contractions to perform in each variable z_{A_i} increases with Λ. Let us consider all the L. T. T. C. $\{ \Gamma_i \}_i$ of the given basis B_Λ, satisfying condition H, which contains a fixed interacting bond. It is necessary in our case that the interacting coupling constants decrease to zero when the range of the interaction increases to infinity. More precisely, for trivial geometric reasons, we have only to consider L. T. T. C. 's in which at least two coupling constants decrease when the range of the interaction increases. Then:

$$\Xi(\Gamma_i) \simeq \prod_{\substack{j \neq 0,1 \\ A_{ij} \in \Gamma_i}} (1 + z_{A_{i,j}}) + \beta^2 J_{A_{i0}} J_{A_{i1}} \prod_{\substack{j \neq 0,1 \\ A_{ij} \in \Gamma_i}} (1 - z_{A_{ij}})$$

does not vanish when:

$$| z_{A_{ij}} + 1 | \geqslant | \beta^2 J_{A_{i0}} J_{A_{i1}} |^{\frac{1}{k-2}} \qquad k \text{ is defined by condition H}$$

It suffices then to apply Ruelle's lemma [3].

One should note that conditions (*b*) depend on the geometry of the system.

III. APPLICATIONS

In Section II we have found domains in which the partition function does not vanish. We shall now derive some consequences.

1. Analyticity of the pressure

The pressure is defined as:

$$P = \lim P_\Lambda = \lim_{\Lambda \to \infty} \frac{1}{\Lambda} \log \Xi_\Lambda$$

1.1. THEOREM

(*i*) Short range interaction:

We have a finite number of different variables $\{ z_i \}_i$ $i \in \{ 1, \ldots, n \}$ (i. e. those corresponding to different bonds). The pressure $\underline{P}(z_1, z_2, \ldots, z_n)$ extends over analytic functions in the domains described by Lemma II.2.1) (fig. 1, 2). In particular the pressure is analytic at high temperature and, provided condition H is satisfied and Lemma II.2.1) is available (*), at low temperature.

(*) We recall that Lemma II.2.1 is always available in the ferromagnetic case.

(*ii*) Longe range interactions:
The pressure $P(\beta)$ extends over analytic functions when alternatively

a)
$$\sum_{0 \in A} \mathrm{Re}\,(\beta J_A) < K_0$$

b) In the ferromagnetic case

$$\sum_{0 \in A} |J_A|^{\frac{2}{k-2}} < K_1 \,; \qquad \mathrm{Re}\,(\beta) > \beta_1$$

iii) The Lee, Yang case.
In the low temperature we derive the usual Lee, Yang theorem for an arbitrary ferromagnetic law interaction as soon as the limit of the pressure exists.
In the high temperature case we consider at each point i of the lattice a real positive magnetic field h_i and only two interactions $\{ J_{A_{i,j}} \}_{j \in \{1,2\}}$. The pressure extends over analytic functions in when

$$- \mathrm{Re}\,(\beta J_{A_{ij}}) > 0 \qquad \beta h_i \geqslant 0$$
$$- \mathrm{Re}\,(\beta J_{A_{ij}}) < 0 \qquad \beta h_i \geqslant 0$$

Proof. — We derive these results in the standard way $\Xi_\Lambda^{\frac{1}{|\Lambda|}}$ is uniformly bounded with respect to Λ and analytic in the domains given in Lemma II.1.1).
Applying the theorems of Hurwitz and Vitali, all three results follow.

2. The Hamiltonian algebra

Let us consider the $*$-algebra generated by linear combinations of products of interaction bonds involved in the hamiltonian and let us call its strong closure $\mathscr{A}_H$ « the Hamiltonian algebra ». In the classical case $\mathscr{A}_H$ is a commutative C*-subalgebra of the commutative C*-algebra $\mathscr{A}$ associated with the lattice. We shall use the set of averageable interactions § [9]. Then assuming that the pressure is analytic the following results hold:

2.1. Theorem

(i) *Low temperature:* the restrictions to $\mathscr{A}_H$ of translation invariant equilibrium states coincide; and this common restriction is extremal in the restriction to $\mathscr{A}_H$ of equilibrium states.
(ii) *High temperature:* there exists a unique translation invariant equilibrium state.
(iii) *In both cases:* when $\sum_S J(r + s + S)\sigma_S \geqslant 0$ (r, s fixed) (i. e. when

F. K. G. holds [*10*] the uniqueness described in (*i*) and (*ii*) extends to non necessarily translation invariant states).

Proof. — We need to prove the following lemma which asserts that the pressure remains analytic under a perturbation of the averageable type [9]. Let us denote $\mathscr{B}_{\mathscr{A}_H}$ (respectively $\mathscr{B}_{\mathscr{A}}$) the algebra of averageable interactions generated by $\mathscr{A}_H$ (resp. $\mathscr{A}$) relevant in the low temperature (resp. high temperature) case.

2.2. Lemma. — The partition function $\Xi_\Lambda^\varepsilon = \mathrm{Tr}\, e^{-\beta H_\Lambda + \varepsilon U_\Lambda(\sigma_{A_i} \dots \sigma_{A_{|\Lambda|}})}$ where $U_\Lambda(\sigma_{A_1}, \dots, \sigma_{A_{|\Lambda|}})$ is a potential associated with the averageable interactions belonging to $\mathscr{B}_{\mathscr{A}_H}$ (resp. $\mathscr{B}_{\mathscr{A}}$) relevant to the low temperature (resp. high temperature) case. Then the partition function Ξ_Λ^ε is different from zero for small ε in domains close to those of Figures 1, 2, 3 under the conditions of Lemmas II.2.1, II.2.2.

Proof. — In the low temperature case one wishes to include some new L. T. T. C., a finite number of which contains *each* interacting bond. The partition function of such a new L. T. T. C. Γ^ε takes the form

$$\Xi(\Gamma^\varepsilon) \simeq \prod_i (1 + z_i) + \varepsilon\beta \prod_i (1 - z_i)$$

$\Xi(\Gamma^\varepsilon)$ does not vanish when any z_i belongs to the complement of the ball $\beta(-1, \varepsilon'(\varepsilon))$ centered at -1 with radius $\varepsilon'(\varepsilon)$ ($\varepsilon'(\varepsilon) \to 0$ when $\varepsilon \to 0$). Performing a bounded number of new contraction independent of Λ yields Lemma 2.2 in the low temperature case.

In the high temperature case an analogous process is used; the averageable interactions can be chosen in $\mathscr{B}_{\mathscr{A}}$ since the domains associated with the variables $\zeta_A = \mathrm{th}\, \beta J_A$ are neighborhoods of the origin.

We now proceed to the proof of Theorem 2.1; (*i*) and (*ii*) are consequences of Lemma 2.2; (*iii*) is an easy generalization of the theorem of the appendix in [9].

2.3. Remark. — Part (*i*) of Theorem 2.1 does not show that *on a subalgebra* of $\mathscr{A}$ the absence of symmetry breakdown in the sense given in [9] implies the existence of a unique equilibrium state on this subalgebra; in fact one knows that there exist non translation invariant equilibrium states in the two body interaction Ising model in three dimensions, as shown by R. L. Dobrushin [*11*].

3. About ergodic decomposition of states (ferromagnetic case)

An important consequence of the preceding theorem is the ergodic decomposition of states. Below the critical temperature of the two dimensional nearest neighbor Ising model it was shown in [*12*] that there exists a unique even translation invariant correlation function. From this result together with a « simplex argument » due to Ruelle, one can derive the ergodic decomposition of translation invariant equilibrium states.

$$\rho(\sigma_A) = \alpha\rho^+(\sigma_A) + (1-\alpha)\rho^-(\sigma_A) \qquad \text{with } 0 \leqslant \alpha \leqslant 1$$

This simplex property had been used for a more general class of interactions by J. Slawny [*13*]; we include a further generalization.

Let us consider a closed subalgebra $\tilde{\mathscr{A}}$ of $\mathscr{A}$. We associate with $\tilde{\mathscr{A}}$ the following closed compact additive group $G_{\tilde{\mathscr{A}}}$ equipped with the law Δ

$$G_{\tilde{\mathscr{A}}} = \{ g \in \{-1, 1\}^{Z^V} \mid \sigma_A(g) = 1 \quad \forall \sigma_A \in \tilde{\mathscr{A}} \}$$

Let dg be the Haar measure associated with $G_{\tilde{\mathscr{A}}}$.

Let $\mathscr{S}$ be the simplex of the equilibrium states [*14*] [*15*] associated with a given hamiltonian H which generates the algebra $\tilde{\mathscr{A}}$. We define $S_{\tilde{\rho}}$:

$$S_{\tilde{\rho}} = \{ \rho \in \mathscr{S} \mid \rho/\tilde{\mathscr{A}} = \tilde{\rho}/\tilde{\mathscr{A}} \}$$

$\mathscr{S}_{\tilde{\rho}}$ is a subsimplex of $\mathscr{S}$ ($\tilde{\rho}$ can be chosen extremal in $\mathscr{S}$).

3.1. THEOREM :

(*i*) There is a unique state $\rho_G \in \mathscr{S}_{\tilde{\rho}}$ invariant under the automorphism group defined in Appendix A2

$$\rho_G(\cdot) = \int \tilde{\rho} \circ g(\cdot) dg$$

(*ii*) Every $\rho \in \mathscr{S}_\rho$ can be written as:

$$\rho(\cdot) = \int \tilde{\rho} \circ g(\cdot) d\mu(g)$$

where $\mu(g)$ is a probability measure on $G_{\tilde{\mathscr{A}}}$.

Proof. — (*i*) In appendix A2 we prove that, if $\tilde{\rho}$ is an equilibrium state $\tilde{\rho}.g$ is an equilibrium state. Then

$$\rho_G(\sigma_A) = \tilde{\rho}(\sigma_A) \int \sigma_A(g) dg \qquad \forall \sigma_A \in \mathscr{A}$$

By classical harmonic analysis we derive the following properties of:

$$\alpha \begin{cases} \rho_G(\sigma_A) = \tilde{\rho}(\sigma_A) & \forall \sigma_A \in \tilde{\mathscr{A}} \\ \rho_G(\sigma_A) = 0 & \forall \sigma_A \in \mathscr{A}\,; \quad \sigma_A \notin \tilde{\mathscr{A}} \end{cases}$$

We want to show that the set of extremal points of $\mathscr{S}_{\tilde{\rho}}$ is of the form $\{ \tilde{\rho} \circ g \}_g$. If not, there would exist another extremal point $\tilde{\tilde{\rho}}$; let us define ρ'_G as:

$$\rho'_G(\sigma_A) = \int \tilde{\tilde{\rho}} \circ g(\sigma_A) dg \qquad \forall \sigma_A \in \mathscr{A}$$

then ρ'_G verifies properties α; that is impossible if $\tilde{\tilde{\rho}}/\tilde{\mathscr{A}}$ is different from $\tilde{\rho}/\tilde{\mathscr{A}}$ since $\mathscr{S}_{\tilde{\rho}}$ is a simplex.

We can now apply this result to the situation described in Theorem III.2.1: at low temperature we take $\tilde{\mathscr{A}} = \mathscr{A}_H$ and $\tilde{\rho} = \rho^+$, the unique restriction to $\mathscr{A}_H$ of translation invariant equilibrium states; $\mathscr{S}_{\rho^+}$ is the corresponding simplex.

3.2. THEOREM. — When Theorem III.2.1 is available we rewrite Theorem III.3.1:

(*i*)
$$\rho_G(.) = \int \rho^+ \circ g(.)dg$$

(*ii*)
$$\rho_\mu(.) = \int \rho^+ \circ g(.)d\mu(g);$$

$\mu(g)$ being a probability measure on G_H.

(*iii*) In some cases strong cluster properties can be deduced.

(*iv*) The states of $\mathscr{S}_{\rho^+}$ obey a variational principle and the entropy of such states can be defined.

Proof. — ρ^+ is extremal in $\mathscr{S}$ by GKS; (*i*) and (*ii*) is then derived. The existence of a derivative of the pressure implies (*iv*). (*iii*) is deduced from arguments of Lebowitz and Penrose [*22*].

3.3. REMARK. — The states $\{ \rho^+ \circ g \}_g$ correspond to the pure phases of the model.

IV. TWO TYPICAL EXAMPLES : 1) THE ISING MODEL WITH 2 BODY INTERACTIONS AND MAGNETIC FIELD

For the sake of simplicity we consider the nearest neighbor interaction where
$$H_\Lambda = \Sigma J_{ij}\sigma_i\sigma_j + h(\sigma_i + \sigma_j) + \eta(\sigma_i - \sigma_j)$$

1·1· BUILDING THE TOPOLOGICAL CYCLES. — *At low temperature two different L. T. T. C.* must be considered (we give an explicit form in the two dimensional case, the generalization being trivial).

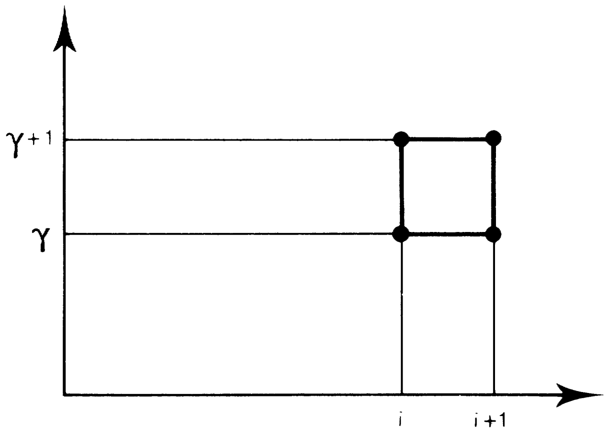
a) $h = \eta = 0$ (*), the L. T. T. C. are associated to

$$\{ \sigma_{ij}\sigma_{i+i,j} ; \sigma_{i+i,j}\sigma_{i+i,j+1} ; \sigma_{i+i,j+1}\sigma_{i,j+1} ; \sigma_{ij+1}\sigma_{ij} \}$$

b) Otherwise:

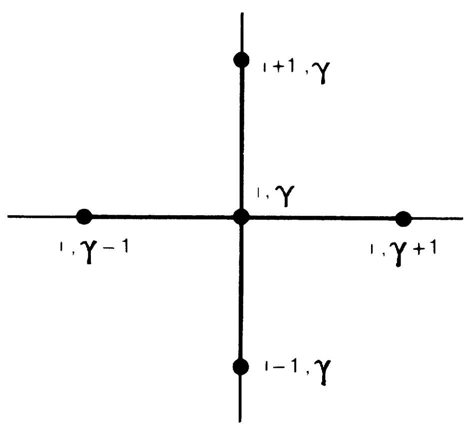
$$\{ \sigma_{ij} ; \sigma_{i,j+1} ; \sigma_{ij}\sigma_{i,j+1} \} \underset{(i,j)\qquad (i,j+1)}{\circ\!\!-\!\!-\!\!-\!\!-\!\!-\!\!\circ}$$

(*) In this case the original idea to use the contraction process is due to S-Miracle-Sole.



At high temperature two different H. T. T. C. are used:

a) $h = \eta = 0 \{ \sigma_{ij}\sigma_{ij+1} ; \sigma_{ij}\sigma_{ij-1} ; \sigma_{ij}\sigma_{i+1,j} ; \sigma_{ij}\sigma_{i-1,j} \}$



b) Otherwise:

$$\{ \sigma_{ij} ; \sigma_{ij}\sigma_{ij+1} ; \sigma_{ij}\sigma_{ij-1} ; \sigma_{ij}\sigma_{i+1,j} ; \sigma_{ij}\sigma_{i-1,j} \}$$

Remark. — For low temperature in the case *a*), formula I.3.1 [*15*]:

$$\Xi_\Lambda = \sum_{X \subset \Lambda} Z^X \prod_\alpha \left(\frac{1 + (-1)^{[\Gamma_\alpha \cap X]}}{2} \right)$$

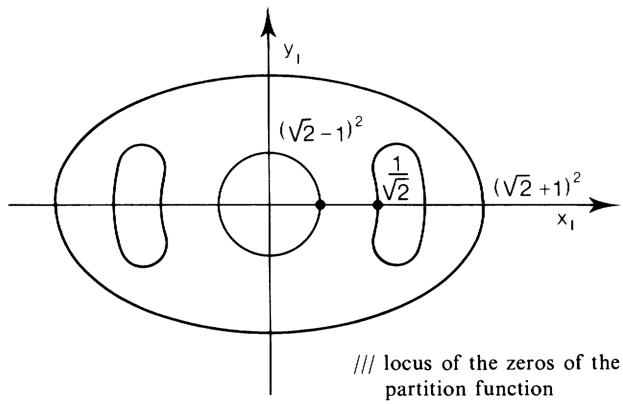
It is the usual « contour expression » of the partition function, i. e. from each point an even number of vertices is issued.

1.2. We consider the case $h = \eta = 0$.

Then the partition function associated to the L. T. T. C. is

$$\Xi(\square) = \prod_{i=1}^{4}(1 + z_i) + \prod_{i=1}^{4}(1 - z_i)$$

The pressure extends to an analytic function in the domain described in the figure below. We refer to [4] for a detailed description.



The group G of internal symmetry of the Hamiltonian, has two elements; the associated Haar measure is $\left(\frac{1}{2}, \frac{1}{2}\right)$, so that the state invariant under $G_H \circ \rho_G$ is:

$$\rho_G(.) = \frac{1}{2}\rho^+(.) + \frac{1}{2}\rho^-(.)$$

And the ergodic decomposition of any translation invariant state is:

$$\rho(.) = \alpha\rho^+(.) + (1 - \alpha)\rho^-(.) \qquad 0 \leqslant \alpha \leqslant 1$$

In the high temperature case, one can deduce the uniqueness of translation invariant equilibrium state.

1.3. We consider the case of a non-zero magnetic field. Then, the elementary function of a L. T. T. C. is:

$$\Xi(\circ\!\!-\!\!\circ) = (1 + z_1)(1 + z_2)(1 + x) + (1 - z_1)(1 - z_2)(1 - x)$$

It does not vanish in the ferromagnetic case ($x \leqslant 1$) when either $|x| \leqslant 1$

$$|z_1| > 1, \qquad |z_2| > 1$$

or $|x| \leqslant 1$

$$|z_1| < 1, \qquad |z_2| < 1$$

It does not vanish in the antiferromagnetic case ($x > 1$) when either $x \geqslant 1$ and

$$| z_1 | > 1, \qquad | z_2 | < 1$$

or $x \geqslant 1$ and

$$| z_1 | < 1, \qquad | z_2 | > 1$$

We use Theorem III.1.1. The famous Lee, Yang theorem [*1*] is derived, for ferromagnet in the magnetic field; for antiferromagnet in the staggered field. At high temperature a domain around $+1$ is obtained (but not around -1 in general).

Theorem III.2.1 is reduced to the well-known result [9] (See also [*17*] for a different proof): there is an unique equilibrium state when the magnetic field is different from zero (and positive), and when, $h = \eta = 0$, the temperature is high enough.

We are now interested in case of an antiferromagnet with a magnetic field and a staggered field (equivalently a ferromagnet with a molecular field and a magnetic field):

a) $\eta \neq 0$. We can apply the theorems of III: a low temperature domain and a high temperature domain are obtained. We obtain a unique equilibrium state in this domain.

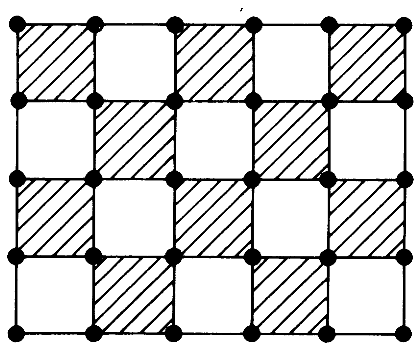
b) $\eta = 0$; $h > \mathrm{V}$. It is similar to 1).

c) $\eta = 0$; $b < \mathrm{V}$. Only a high temperature domain is obtained. It is well-known that the pressure is not analytic at low temperature with respect to *each variable* associated to the magnetic field, since two different states can be exhibited [*19*]. Theorem III.1.1 is in this sense maximal. However analyticity can be proved with respect to some lot of variables [*18*].

An exemple of the Lee, Yang theorem in the high temperature case.

Let us consider a four-body interaction Ising model with a positive non-uniform magnetic field, as described below.

The shaded squares correspond to four-body interactions



Let

$$\mathrm{H} = \sum_{i \subset \Lambda} h_i \sigma_i + \Sigma \mathrm{J}_{ijk\rho} \sigma_i \sigma_j \sigma_k \sigma_\rho ;$$

Then the pressure depends analytically on the fields h_i and on the coupling constants, in the domains:

$$\operatorname{Re} J_{ijk\rho} > 0, \quad h_i \geqslant 0 \qquad \text{and} \qquad \operatorname{Re}(J_{ijk\rho}) < 0 \quad h_i \geqslant 0.$$

This model appears as « dual » of the usual Ising two-dimensional model, with magnetic field.

V. SOME GENERALIZATIONS

First we notice that in case of hard core on a lattice the analyticity properties are deduced easily, at low temperature. Let us consider higher spin models. We shall study the case of spin $\frac{p}{2}$. The basic tool in this case is a paper of Griffith [*20*]. Let S_i such a spin taking the values p, $p-1 \dots -p$; we write S_i as:

$$S_i = \sigma_i^1 + \sigma_i^2 + \dots + \sigma_i^p$$

where σ_i^j are the usual Pauli matrices. We consider the general ferromagnetic Hamiltonians $H_\Lambda(S_1 \dots S_n)$. The following theorem holds.

1.1. THEOREM. — With slight modifications sketched in the proof one can derive theorems III.1.1; III.2.1; III.3.1.

Proof. — With each Hamiltonian $H(S_1 \dots S_n)$ we associate the so-called « Analog Hamiltonian » [*20*]: $\tilde{H}$

$$\begin{aligned}\tilde{H}_\Lambda(\sigma_1^1, \dots, \sigma_1^p; \dots; \sigma_n^1, \dots, \sigma_n^p) \\ = H_\Lambda(S_i(\sigma_1^1, \dots, \sigma_1^p) \dots S_n(\sigma_n^1, \dots, \sigma_n^p)) - \beta^{-1} \sum_j \log W_j^\Lambda(\sigma_1^j) \dots\end{aligned}$$

W_j^Λ being the so-called « ferromagnetic pair weight function » defined in [*20*]. We can now use the method developed in this paper to find analyticity properties and the related results, taking into account the existence of the GKS and FKG inequalities for such spins. But there is an important difference in the results: *the Hamiltonian* $\tilde{H}_\Lambda$ *depends on the temperature* in a specific way; then the domains must be modified in a trivial way.

1.2. REMARK. — We notice that for high spin the method is not always adequate except in the Lee, Yang case because we introduce a large number of new parameters. It would be interesting to generalize *directly* the Asano contractions and Ruelle's lemma to this case, as well as to more general spin distributions [*20*], [*21*] or to the quantum case.

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APPENDIC A1

TWO BASIC TOOLS

1. The Asano contraction

We recall the definition of Asano contractions [2]. Let us consider the affine polynomial in two complex variables:

$$P(x_1, x_1) = ax_1x'_1 + bx_1 + cx'_1 + d \qquad a, b, c, d \in \mathbb{C}$$

$D(x_1, x'_1)$ is the contraction operator:

$$D(x_1, x'_1)P(x_1, x'_1) = ax_1 + d$$

The Asano contraction is a commutative operation, i. e. if we have an affine function of several variables the contraction can be done in an arbitrary order. We shall use the lemma given by D. Ruelle [3].

2. Lemma

Let A and B closed subsets of $\mathbb{C}$ which do not contain O. Suppose the complex polynomial $P(x_1, x'_1)$ can vanish only when $x_1 \in A$, or $x_2 \in B$.

Then $D(x_1, x'_1)P(x_1, x'_1)$ can vanish only when $x_1 \in -(A.B)$.

$$A.B = \{ y.y' \mid y \in A ; y' \in B \}.$$

APPENDIX A2

We are going to introduce the usual group structure associated with the model we consider (see for example [24]). We consider Hamiltonians such that in part I, i. e. finite body interactions.

Let $K = [-1, +1]^{Z^\nu}$ the configuration space. We shall identify an element X of K with a set of occupied point by 1. A natural group structure can be put on K (with respect to the symmetric difference). We define on K a subgroup G_H for each Hamiltonian

$$G_H = \{ g \in [-1, +1]^{Z^\nu} \mid \sigma_A(g) = +1 \quad \forall A \in B_H \}$$

(B_H is the set of bonds of the Hamiltonian). G_H is a closed compact group.

We consider an equilibrium state ρ_Λ for the interaction H (i. e. a state satisfying the D. L. R. equations)

$$\rho_\Lambda(\sigma_A) = \frac{\sum_{X \subset \Lambda} e^{-\beta H_\Lambda(X)} \sigma_A(X)}{\sum_{X \subset \Lambda} e^{-\beta H_\Lambda(X)}} \qquad \forall \sigma_A \in \mathscr{A}_\Lambda$$

For each $g \in G_H$, we put

$$\rho_\Lambda \circ g(\sigma_A) = \frac{\sum_{X \subset \Lambda} e^{-\beta H_\Lambda(X)} \sigma_A(g \circ X)}{\sum_{X \subset \Lambda} e^{-\beta H_\Lambda(X)}} = \sigma_A(g) \rho_\Lambda(\sigma_A) .$$

It follows that the $\{ \rho_\Lambda \circ g \}$ satisfy the D. L. R. equations.

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