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Integrable Cylinder Functionals for an Integral of Feynman-Type

by

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ABSTRACT. — Integrability of cylinder functionals is investigated, for the Feynman-type integral as defined by Itô. Several classes of unbounded integrable cylinder functionals are specified. An inductive property of the integral is established.

RÉSUMÉ. — L'intégrabilité des fonctionnelles cylindriques est étudiée pour l'intégrale de type Feynman, telle qu'elle soit définie par Itô. Quelques classes de fonctionnelles cylindriques intégrables non bornées sont spécifiées. Une propriété inductive de l'intégrale est établie.

1. INTRODUCTION

In this note we establish the integrability of several classes of cylinder functionals, for an integral of Feynman-type. These classes are characterized in terms of order of analytic functions, or in terms of polynomial bounds. The classes are of interest for the following reason: they are of a general character, and include several families of unbounded integrable functionals. The unbounded functionals which were known previously to be integrable are primarily those which could be integrated in closed form, like cylinder polynomial functionals.

The subject of Feynman-type integrals and of their physical applications

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was reviewed recently in [1]. The results that follow relate to Itô's definition of the integral [2], and are only a modest contribution.

We recall that a cylinder functional on a linear space is one satisfying $f(\eta) = f(P\eta)$ where P is a finite-dimensional projection. We will first discuss the finite-dimensional integrals, corresponding to dim P = n, and will turn to the problem of enlarging the space of integration in sec. 3.

For finite-dimensional integrals, we found it useful to modify slightly Itô's definition. The following definition of $\mathscr{A}_n$ is in part more general than Itô's (cf. sec. 3), since it allows distributions, but in part seems to be more restrictive, since it requires a limit for a wider family of operators. Furthermore, it is convenient for us to refer to the space $\mathscr{S}^{\frac{1}{2}}_{\frac{1}{2}}(\mathrm{R}^n)$ of [3]. This space is invariant under Fourier transformation, and includes the Gaussian functions $e^{-\frac{1}{2}\langle x, \mathrm{B}x\rangle}$ if Re B > 0. Its dual is larger than $\mathscr{S}'$, and contains the various functions and distributions of sec. 2.

DÉFINITION 1. — *Let $\mathscr{A}_n$ be the class of distributions f in $\mathscr{S}^{\frac{1}{2}}_{\frac{1}{2}}(\mathrm{R}^n)'$ for which*

$$\lim_{\mathrm{B}\to 0} \int d^n x e^{-\frac{1}{2}\langle x-a, \mathrm{B}(x-a)\rangle + \frac{1}{2}i\langle x,x\rangle} f(x) \qquad (a \in \mathrm{R}^n) \tag{1a}$$

exists and does not depend on a. The n × n matrices B *are restricted by the conditions*

$$\mathrm{B} = \mathrm{B}^{\mathrm{T}}, \qquad \mathrm{Re}\,\mathrm{B} > 0\,. \tag{1b}$$

2. INTEGRABILITY FOR FINITE DIMENSIONS

We start with two simple lemmas. The proof of the first is omitted.

LEMMA 2. — Let A satisfy (1*b*). Then

i) A* and A^{-1} also satisfy (1*b*).

ii) The quadratic form defined by A can be diagonalized by a real (nonsingular) transformation.

iii) The mapping $\mathrm{A} \to (\mathrm{A} - i)^{-1} - i$ maps the operators satisfying (1*b*) into themselves and is continuous. In particular, if $\|\mathrm{A}\| \leqslant \mathrm{K} < 1$ then

$$(1 + \mathrm{K})^{-1}\|\mathrm{A}\| \leqslant \|(\mathrm{A} - i)^{-1} - i\| \leqslant (1 - \mathrm{K})^{-1}\|\mathrm{A}\|\,. \tag{2}$$

LEMMA 3. — If $f^* \in \mathscr{A}_n$, then $\tilde{f} \in \mathscr{A}_n$ ($\tilde{f}$ being the Fourier transform of f).

Proof. — An easy calculation gives

$$\left(c_{\mathrm{B}} \int d^n x f^*(x) e^{\frac{1}{2}i\langle x,x\rangle - \frac{1}{2}\langle x-a, \mathrm{B}(x-a)\rangle}\right)^*$$

$$= (2\pi)^{-n/2} \int d^n p \tilde{f}(p) e^{\frac{1}{2}i\langle p,p\rangle - \frac{1}{2}\langle p-a, [(\mathrm{B}-i)^{-1}-i]^*(p-a)\rangle}, \tag{3}$$

where c_B is defined by the condition that $f = 1$ integrate to one when $a = 0$. Now, the limits $B \to 0$ and $[(B - i)^{-1} - i]^* \to 0$ are equivalent, and therefore $f^* \in \mathscr{A}_n$ implies $\tilde{f} \in \mathscr{A}_n$.

We next introduce the following classes of complex-valued functions and distributions on R^n.

(1) The first class consists of entire functions of order less than two, i. e. satisfying

$$| g(z) | \leqslant C \exp (M | z |^{2-\varepsilon}) \tag{4}$$

for some constants C, M, and $\varepsilon > 0$.

(2) The second class consists of two subclasses:

(2*a*) the functions and distributions f of the form

$$f(x) = Dh(x), \tag{5}$$

where D is a differential operator of order m with constant coefficients, and where $(1 + | x |)^m h(x)$ is integrable;

(2*b*) the functions f whose derivatives satisfy (Δ being the Laplacian in R^n)

$$(1 - \Delta)^N f(x) = (1 + x^2)^M l(x), \tag{6}$$

where $l \in L_1$, and N, M are nonnegative integers such that $N - M > \frac{1}{2} n$.

We conjecture that the classes (2*a*) and (2*b*) contain the spaces $\mathcal{O}'_M$ and $\mathcal{O}_C$ respectively (cf. e. g. [4]).

PROPOSITION 4. — *The classes* (1) *and* (2*a-b*) *are included in* $\mathscr{A}_n$. *Moreover, the Fourier transform of a function in the class* (2*b*) *is an element of the class* (2*a*).

Proof. — For class (1) we rewrite the integral (1*a*) in the form

$$\int d^n x e^{-\frac{1}{2}\langle x,(B-i)x\rangle} f_a(x), \tag{7}$$

where $f_a(x) = e^{\frac{1}{2}i\langle a,a\rangle + i\langle x,a\rangle} f(x + a)$. Since $f_a(x)$ is again an element of class (1), we can proceed by diagonalizing the quadratic form $\langle z, (B - i)z \rangle$, and then by rotating the contours of integration into the complex planes. The details are straightforward.

For the assertion about Fourier transforms, we start with

$$\tilde{f}(p) = (1 + p^2)^{-N}(1 - \Delta_p)^M \tilde{l}(p), \tag{8}$$

where $\tilde{l}(p)$ is continuous and decreasing. The r. h. s. can easily by rearranged to yield a sum of terms, each as in (5).

In view of Lemma 3, we now only need to show that class (2*a*) is included in $\mathscr{A}_n$. Let $(\bar{m})$ be an n-index quantity with $|(\bar{m})| = m$. Let $D = \partial^{(\bar{m})}$ and let $\| B \| \leqslant K$. Then it is obvious that

$$| \partial^{(\bar{m})} e^{\frac{1}{2}i\langle x,x\rangle - \frac{1}{2}\langle x-a,B(x-a)\rangle} | \leqslant c_m(a)(1 + | x |)^m, \tag{9}$$

where $c_m(a)$ depends only on m, a and K. Now

$$\int d^n x f(x) e^{\frac{1}{2}i\langle x,x\rangle}(e^{-\frac{1}{2}\langle x-a,B(x-a)\rangle} - e^{-\frac{1}{2}\langle x-a',B'(x-a')\rangle})$$
$$= (-1)^m \int d^n x h(x) \partial^{(\bar{m})} e^{\frac{1}{2}i\langle x,x\rangle}(e^{-\frac{1}{2}\langle x-a,B(x-a)\rangle} - e^{-\frac{1}{2}\langle x-a',B'(x-a')\rangle}), \quad (10a)$$

and the integrand of the r. h. s. is bounded by

$$(c_m(a) + c_m(a'))\,|\,h(x)\,|\,(1 + |\,x\,|)^m. \quad (10b)$$

An application of the bounded convergence theorem now completes the proof.

We mention two other properties of $\mathscr{A}_n$.

By the bounded convergence theorem, complex Borel measures on R^n of bounded total absolute variation belong to $\mathscr{A}_n$. Hence their Fourier transforms also belong to $\mathscr{A}_n$ (cf. [2]).

Finally, one can easily show that the approximation procedure suggested by Friedrichs and Shapiro for integrals over a Hilbert space [5] can be adapted to functions in $\mathscr{A}_n$. We forego a formal statement, as it would be rather awkward. [Extend f from R^n to $\mathscr{H}$, use increasing projections P_m, and integrate $f(P_m\cdot)$ with the normalized Gaussian factor $N_m e^{\frac{1}{2}[i-(\sigma/m)]\langle u,u\rangle}$, where σ is fixed. As $m \to \infty$, one recovers the limit of (1a).]

3. ENLARGEMENT OF THE SPACE OF INTEGRATION

We now summarize Itô's definition (see e. g. [1] [2]). Let $\mathscr{H}$ be a real, separable Hilbert space, and let $d\mu_{T,\alpha}$ be the Gaussian measure on $\mathscr{H}$ with the covariance operator T and the mean vector $\alpha \in \mathscr{H}$. The operator $T : \mathscr{H} \to \mathscr{H}$ must satisfy

$$\operatorname{tr} T < \infty, \qquad T = T^*, \qquad T > 0. \quad (11)$$

The integral is now defined by

$$\hat{I}(f) = \lim_{T\to\infty} c_T^{-1} \int_{\mathscr{H}} d\mu_{T,\alpha}(\eta) e^{\frac{1}{2}i\langle\eta,\eta\rangle} f(\eta), \quad (12a)$$

where

$$c_T = \int_{\mathscr{H}} d\mu_{T,0}(\eta) e^{\frac{1}{2}i\langle\eta,\eta\rangle}. \quad (12b)$$

The limit is to be taken by following the partially ordered set of the T's (where $T' \geqslant T'' \Leftrightarrow T' - T'' \geqslant 0$). The limit must also be independent of α.

It is obvious that if $\dim \mathscr{H} = n < \infty$, if $f \in \mathscr{A}_n$, and if f is a locally integrable function, then $\hat{I}(f)$ exists. (We did not include the analogue to c_T in Definition 1, since such factors would not affect the existence of the limit in the finite-dimensional case.)

If dim $\mathscr{H} > n$, perhaps infinite, the elements of $\mathscr{A}_n$ remain integrable in the following way.

PROPOSITION 5. — *Consider two Hilbert spaces, $\mathscr{H}_1 \subset \mathscr{H}$. Let $f_1 : \mathscr{H}_1 \to C^1$ be a functional which is integrable (in the sense of Itô). Then the functional $f : \mathscr{H} \to C^1$ which is the result of extending f_1,*

$$f(\eta_1 + \eta_2) = f_1(\eta_1), \qquad \eta_1 \in \mathscr{H}_1, \qquad \eta_2 \in \mathscr{H}_1^{\perp}, \tag{13a}$$

is integrable over $\mathscr{H}$. The respective integrals are equal,

$$\hat{I}(f) = \hat{I}_1(f_1). \tag{13b}$$

Proof. — We start with a measure $d\mu_{T,\alpha}$ as above, and we introduce the following quantities, whose meaning is obvious:

$$\mathscr{H} = \mathscr{H}_1 + \mathscr{H}_2 = P_1\mathscr{H} + P_2\mathscr{H}; \tag{14a}$$

$$\alpha = \alpha_1 + \alpha_2, \qquad \eta = \eta_1 + \eta_2 \qquad (\alpha, \eta \in \mathscr{H}), \tag{14b}$$

$$T_j = P_jTP_j/\mathscr{H}_j, \qquad j = 1, 2. \tag{14c}$$

We next recall a theorem on decomposition of measures [6] [7]. This theorem also applies to complex measures, if the total absolute variation is finite. For the case at hand, it implies the existence of measures $d\lambda_{\eta_1}(\eta_2)$, depending on T and α, such that

$$c_T^{-1} \int_{\mathscr{H}} d\mu_{T,\alpha}(\eta) e^{\frac{1}{2}i\langle \eta,\eta\rangle} f(\eta) = c_{T_1}^{-1} \int_{\mathscr{H}_1} d\mu_{T_1,\alpha_1}(\eta_1) \times e^{\frac{1}{2}i\langle \eta_1,\eta_1\rangle} f_1(\eta_1) \int_{\mathscr{H}_2} d\lambda_{\eta_1}(\eta_2). \tag{15}$$

The measure $d\lambda_{\eta_1}$ here is normalized so that

$$\lambda_{\eta_1}(\mathscr{H}_2) = 1 \qquad \text{for} \qquad \mu_{T_1,\alpha_1} - \text{almost all } \eta_1, \tag{16}$$

and it includes the effect of the factors $e^{\frac{1}{2}i\langle \eta_2,\eta_2\rangle}$ and c_{T_1}/c_T. Now, as one considers lim ($T \to \infty$) on the l. h. s., this entails lim ($T_1 \to \infty$) on the r. h. s., and by hypothesis, the integral over $\mathscr{H}_1$ tends to the limit $\hat{I}_1(f_1)$, independently of α_1. The integrability over $\mathscr{H}$ and the equality follow.

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