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MARCO MODUGNO
GIANNA STEFANI

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Section A :
Physique théorique.

On gravitational and electromagnetic shock waves and their detection

by

Marco MODUGNO and Gianna STEFANI
Istituto di Matematica Applicata, Firenze

SUMMARY. — We give a global and intrinsical version of Lichnerowicz's approach to gravitational and electromagnetic shock waves and we find a detecting « deviation effect », emphasizing their transversal nature.

INTRODUCTION

Shock waves in general relativity have been studied by several authors, as A. Lichnerowicz [*1*] [*2*] [*3*], Y. Choquet-Bruhat [*4*] [*5*], A. H. Zakharov [*8*] and many others.

A general treatment of this subject is that due to A. Lichnerowicz, who studies, in terms of distributions, gravitational and electromagnetic shock waves, both of first and second kind (i. e. discontinuities of first and second derivatives of g, in the gravitational case, and discontinuities of first derivatives of F and of F itself, in the electromagnetic case). It is also well known [*1*] that first kind shock waves can be detected by the jump of relative acceleration of test particles.

Our purpose is to give a global version of Lichnerowicz's approach, by means of an intrinsical analysis of the Dirac delta, of the Hadamard formulas, of the significant potentials and of the connection and curvature distributions. Finally we prove that second kind shock waves can be detected by the jump of relative velocity of test particles, emphasizing the transversal nature of these waves.

1. THE DIFFERENTIABLE STRUCTURE OF M

We consider a paracompact C^0 n-manifold M, which will assume the meaning of space-time (§ 6 and 7).

The object of our analysis consists of tensors which present a jump through a connected C^0 $(n-1)$-submanifold

$$j : \Sigma \to M .$$

We assume on M a differentiable structure of class $C^{(h,k)}$, with

$$1 \leq h \leq k, \qquad 3 \leq k ,$$

that is of class C^h and inducing a structure of class C^k on each C^h n-submanifold N, which satisfies the condition

$$N \cap \Sigma \subset \partial N ,$$

and on Σ.

A tensor t on M is said of class $C^{(r,s)}$, with

$$-1 \leq r \leq h, \qquad r \leq s \leq k ,$$

if it is of class C^r, in the case $0 \leq r$, and if, for each C^h n-submanifold N, as above, there is a C^s tensor $\tilde{t}$ on N, such that

$$t/_{N - N \cap \Sigma} = \tilde{t}/_{N - N \cap \Sigma} .$$

If t is of class $C^{(-1,s)}$, it is said « regularly discontinuous through Σ ». We will denote by $\mathscr{C}$ the modulus of $C^{(h-1,k-1)}$ vectors.

In order to introduce globally the jump of $C^{(-1,s)}$ tensors, we suppose M and Σ to be orientable and we denote by $M^{\pm}$ and $\Sigma^{\pm}$, M and Σ with their orientations. Then, for each $x \in \Sigma$, there are two n-submanifolds $N^{\pm}$, such that

a) $N^{+} \cup N^{-}$ is a neighboroud of x,
b) $x \in N^{\pm} \cap \Sigma = \partial N^{\pm}$,
c) the orientation of $\partial N^{\pm}$, induced by $M^{\pm}$, agrees with that of $\Sigma^{\pm}$.

Hence, if t is a $C^{(-1,s)}$ tensor, it is well defined the « jump tensor » of t, that is the C^s tensor $[t]$, on Σ, given by

$$[t](x) = \tilde{t}^{+}(x) - \tilde{t}^{-}(x), \qquad \forall x \in \Sigma ,$$

where $\tilde{t}^{\pm}$ are the C^s tensors on $N^{\pm}$, as in (1).

It will be usefull a further analysis of Σ. The orientations of M^{+} and Σ^{+} determine, modulo a positive $C^{(h-1,k-1)}$ function

$$f : \Sigma \to R ,$$

a $C^{(h-1,k-1)}$ 1-form

$$l : \Sigma \to T^{*}M ,$$

orthogonal to Σ and such that

$$l(x) \neq 0, \qquad \forall x \in \Sigma .$$

Then, we can easily prove that, for each $x \in \Sigma$, there is a coordinate neighboroud Ω and a $C^{(h,k)}$ function

$$\varphi : \Omega \to R ,$$

satisfying the following conditions:

a)
$$\Omega \cap \Sigma = \varphi^{-1}(0) ,$$
$$\Omega^{\pm} \equiv \varphi^{-1}(R^{\pm} \cup \{ 0 \})$$

are connected;

b)
$$l/_{\Omega \cap \Sigma} = d\varphi/_{\Omega \cap \Sigma} . \tag{2}$$

Furthermore, we assume a Riemannian (non positive definite) structure on M, by means of a tensor g of class $C^{(0,2)}$.

Let us denote by η the unitary volume form of M^+. Then, choosen l, there is a unique volume form ω of Σ, such that locally it is written as

$$\omega = j^*\omega' ,$$

where ω' satisfies the condition

$$\eta = d\varphi \wedge \omega' \tag{3}$$

(ω agrees with the orientation induced by M^+ on $\partial\Omega^+$).

2. DISTRIBUTIONS INDUCED BY A $C^{(r,s)}$ TENSOR

A distribution [*10*] on M is said of class $C^{(r,s)}$, with

$$-1 \leq r \leq h , \qquad 0, r \leq s \leq k ,$$

if it is a continuous linear function on the space (with the natural topology) of $C^{(r,s)}$ tensors, with compact support. We will denote by $\mathscr{D}$ the modulus of 1-form distributions.

A fundamental distribution, associated with Σ, is the Dirac delta, that is the 1-form distribution, of class C^0, with support Σ, given by

$$\underline{\delta}(u) \equiv \int_\Sigma j^*(* u) \tag{4}$$

$$= \int_\Sigma \langle l, u \rangle \omega . \tag{4'}$$

This definition does not depend on the choice of l and is a little different from that given by Lichnerowicz [*2*].

If t is a tensor of class $C^{(-1,s)}$, we have a « jump distributions » $\underline{\delta} \otimes [t]$, of class C^0, with support Σ, given by

$$\underline{\delta} \otimes [t](u) \equiv \int_\Sigma j^* * (i''_{[t]} u)$$
$$= \int_\Sigma \langle l \otimes [t], u \rangle \omega . \qquad (5)$$

(where $\langle v, i''_{[t]} u \rangle \equiv \langle v \otimes [t], u \rangle$, for each 1-form v).

Furthermore, if t is a tensor of class $C^{(r,s)}$, we have an « associated distribution » $\underline{t}$, or t^D, of class $C^{(-1,0)}$, given by

$$\underline{t}(u) \equiv t^D(u) \equiv \int_M \langle t, u \rangle \eta$$
$$= \int_{M-\Sigma} \langle t, u \rangle \eta . \qquad (6)$$

3. THE INFINITESIMAL JUMP TENSOR

The fundamental way to emphasize the differentiability of a tensor is to calculate its Lie derivatives. If t is a $C^{(0,s)}$ tensor, then $L_u t$ is regularly discontinuous through Σ and the jump $[L_u t]$ is a well defined tensor on Σ. But, if M is of class $C^{(1,k)}$, $[L_u t]$ does not depend only on the differentiability of t and we are interested essentially on « the part » that, depending linearly on u, measures the differentiability of t alone. So, we obtain a generalization of Hadamard formulas, that are valable only for functions (Lichnerowicz [2]).

THEOREM 1. — *a*) If t is a $C^{(0,s)}$ covariant tensor, there is a C^{s-1} tensor $\in t$ on Σ, given by

$$j^*[L_u t] = i_u \in t , \qquad \forall C^{(0,s)} \text{ vector } u. \qquad (7)$$

Furthermore, we have

$$\in t = l \otimes t' , \qquad (7')$$

where the map

$$t \mapsto t'$$

behaves as a « derivation » of the algebra of covariant $C^{(0,s)}$ tensors of M, in the algebra of covariant C^{s-1} tensors of Σ.

b) If s is a $C^{(0,s)}$ contravariant tensor, tangent to Σ, there is a C^{s-1} tensor $\in s$ on Σ, given by

$$[L_u s] = i_u \in s , \qquad \forall C^{(0,s)} \text{ vector } u . \qquad (8)$$

Furthermore, we have

$$\in s = l \otimes s', \tag{8'}$$

where the map

$$s \mapsto s'$$

behaves as a « derivation » of the algebra of the contravariant $C^{(0,s)}$ tensors of M, tangent to Σ, in the algebra of contravariant C^{s-1} tensors on Σ.

c) The use of the same notation $\in$, in both cases, is justified by the following formula

$$\langle t, s \rangle' = \langle t', s \rangle + \langle t, s' \rangle. \tag{9}$$

Proof. — If f is a $C^{(0,s)}$ function, we have:

$$j^*[df] = j^*(df)^+ - j^*(df)^- = d(f^+ \circ j) - d(f^- \circ j) = 0.$$

Hence, there is a function f' of Σ, such that

$$[df] = lf'.$$

a) We contend that $j^*[L_u t]$ is linear respect to u. It is enough to consider tensors t of degree 1, for which we have:

$$j^*[L_{fu}t] = fj^*[L_u t] - j^*[df]\langle t, u \rangle = fj^*[L_u t].$$

Furthermore, if u is tangent to Σ, it is

$$j^*[L_u t] = L_{u/_\Sigma} j^* t^+ - L_{u/_\Sigma} j^* t^- = 0.$$

Then, there is a tensor t' of Σ, such that

$$j^*[L_u t] = l(u)t'.$$

b) We contend that $[L_u s]$ is linear respect to u. It is enough to consider tensors s of degree 1, for which we have:

$$[L_{fu}s] = f[L_u s] - u\langle [df], s \rangle = f[L_u s].$$

Furthermore, if u is tangent to Σ, it is

$$[L_u s] = [L_{u/_\Sigma} s_{/\Sigma}] = 0.$$

Then, there is a tensor s' on Σ, such that

$$[L_u s] = l(u)s'.$$

c) We have:

$$l(u)\langle t, s \rangle' = [L_u \langle t, s \rangle] = [\langle L_u t, s \rangle] + [\langle t, L_u s \rangle]$$
$$= l(u)(\langle t', s \rangle + \langle t, s' \rangle).$$

We call $\in t$ and $\in s$ the « infinitesimal jump tensors » of t and s, respectively.

We remark that, if M is of class $C^{(h,k)}$, with $2 \leq h$, then we can choose vectors u of class $C^{(1,s)}$. In this case, for each $C^{(0,s)}$ covariant or contravariant tensor t, $[L_u t]$ depends linearly on u, hence there is a tensor $\hat{\in} t$, such that

$$i_u \hat{\in} t = [L_u t] ;$$

furthermore, we have:

$$\hat{\in} t = l \otimes \hat{t}' .$$

Nevertheless, only the considered t' and s' will be significant for our purposes (§ 4).

4. THE RIEMANNIAN CONNECTION

Being the metric tensor g of class $C^{(0,2)}$, we can consider its infinitesimal jump tensor (7′)

$$\in g \equiv l \otimes b .$$

This definition of b is an intrinsical version of the « significant potentials », introduced by Lichnerowicz [2].

Such a Riemannian structure of M induces a Riemannian connection on $M - \Sigma$, which is « regularly discontinuous » through Σ. That is, if t is a tensor of class $C^{(0,s)}$, with $1 \leq s$, then ∇t may be viewed as a regularly discontinuous tensor. We are interested to give a tensorial estimation of the « connection jump ». For this purpose, we can remark that, if u is a vector tangent to Σ, then $[\nabla_u t]$ depends linearly on t. Unfortunately, there is not a canonical projection tangent to Σ, but we can get a result equivalent for our purposes, by an antisymmetrization with $\underline{\delta}$. Hence we are induced to prove the following theorem.

THEOREM 2 (see also Lichnerowicz [2]). — The map

$$\underline{H} : (\mathscr{T} \times \mathscr{T}) \times \mathscr{T} \to \mathscr{D} ,$$

given by

$$\underline{H} : (x, y ; u) \mapsto i_y \underline{\delta} \otimes [\nabla_x u] - i_x \underline{\delta} \otimes [\nabla_y u] , \tag{10}$$

is multilinear, hence it defines a tensor distribution of degree 4

$$\underline{H} : x \otimes y \otimes u \otimes v \mapsto \underline{\delta} \otimes [\nabla_x u](y, v) - \underline{\delta} \otimes [\nabla_y u](x, y) .$$

Furthermore, $\underline{H}$ is a symmetrical double 2-form distribution, of class $C^{(h-1,k-1)}$, which satisfies the « Bianchi identity »

$$\underline{H}(x, y ; u) + \underline{H}(u, x ; y) + \underline{H}(y, u ; x) = 0 .$$

More precisely, choosen l, there is a double 2-form H, on Σ, such that

$$\underline{\mathbf{H}}(x, y\,;\, u, v) = \int_\Sigma \mathrm{H}(x, y\,;\, u, v)\omega\,,$$

which is given by

$$\mathrm{H} = -\frac{1}{2}(l \otimes l) \wedge \tilde{b}\,, \tag{11}$$

where $\tilde{b}$ is any symmetrical tensor which satisfies

$$j^*\tilde{b} = b\,.$$

Proof. — Trivially $\underline{\mathbf{H}}$ is linear and alternate respect to x and y. We contend that $\underline{\mathbf{H}}$ is linear respect to u. In fact, it is

$$\underline{\mathbf{H}}(x, y\,;\, fu) = f\underline{\mathbf{H}}(x, y\,;\, u) + i_y\underline{\boldsymbol{\delta}} \otimes [xf]u - i_x\underline{\boldsymbol{\delta}} \otimes [yf]u = f\underline{\mathbf{H}}(x, y\,;\, u)\,,$$

being for (7′)

$$(i_y\underline{\boldsymbol{\delta}} \otimes [xf]u - i_x\underline{\boldsymbol{\delta}} \otimes [yf]u)(v) = \int_\Sigma (l(y)l(x) - l(x)l(y))f' \langle u, v \rangle = 0\,.$$

Then $\underline{\mathbf{H}}$ is a tensor distribution.

Furthermore, $\underline{\mathbf{H}}$ is alternate respect to u and v, being for (7′)

$$\underline{\mathbf{H}}(x, y\,;\, u, u) = \frac{1}{2}\underline{\boldsymbol{\delta}}[\nabla_y u^2](x) - \frac{1}{2}\underline{\boldsymbol{\delta}}[\nabla_x u^2](y) = 0\,.$$

Taking into account that the connection is torsion free, i. e. that [9]

$$\nabla_a b - \nabla_b a = \mathrm{L}_a b\,,$$

we get, from (8′):

$$\begin{aligned}&\underline{\mathbf{H}}(x, y\,;\, u) + \underline{\mathbf{H}}(u, x\,;\, y) + \underline{\mathbf{H}}(y, u\,;\, x)\\ &\qquad = -\, i_x\underline{\boldsymbol{\delta}} \otimes [\mathrm{L}_u y] + i_y\underline{\boldsymbol{\delta}} \otimes [\mathrm{L}_u x] + i_u\underline{\boldsymbol{\delta}} \otimes [\mathrm{L}_x y] = 0\,.\end{aligned}$$

Denoting by $||$ and $\perp$ any projection tangent and orthogonal to Σ, we have, for the algebraic properties of $\underline{\mathbf{H}}$:

$$\begin{aligned}\underline{\mathbf{H}}(x, y\,;\, u, v) &= \underline{\boldsymbol{\delta}} \otimes [\nabla_{y_{||}} u](x, v) - \underline{\boldsymbol{\delta}} \otimes [\nabla_{x_{||}} u](y, v)\,,\\ \underline{\mathbf{H}}(x, y\,;\, u, v) &= \underline{\mathbf{H}}(x, y\,;\, u^{||}, v^{\perp}) + \underline{\mathbf{H}}(x, y\,;\, u^{\perp}, v^{||}) + \underline{\mathbf{H}}(x, y\,;\, u^{||}, v^{||})\,.\end{aligned}$$

Then, taking into account that the connection is Riemannian, i. e. that [9]

$$2 \langle \nabla_a b, c \rangle = \mathrm{L}_a \langle b, c \rangle + \mathrm{L}_b \langle a, c \rangle + \langle \mathrm{L}_a b, c \rangle - (\mathrm{L}_c g)(a, b)\,,$$

we get, from (7′):

$$\begin{aligned}2 \langle [\nabla_{y_{||}} u^{||}], v^{||} \rangle &= 0\,,\\ 2 \langle [\nabla_{y_{||}} u^{||}], v \rangle &= -\,(\mathrm{L}_v g](y^{||}, u^{||}) = -\, l(v)b(y^{||}, u^{||})\,.\end{aligned}$$

Hence, we get:

$$\underline{\mathbf{H}}(x, y ; u, v) = -\frac{1}{2}\int_{\Sigma} (l(x)l(v)b(y^{||}, u^{||}) - l(y)l(v)b(x^{||}, u^{||}) - l(x)l(u)b(y^{||}, v^{||}) + l(y)l(u)b(x^{||}, v^{||})$$
$$= -\frac{1}{2}\int_{\Sigma} (l \otimes l) \wedge \tilde{b}(x, y ; u, v)\omega_{\perp}$$

We call $\underline{\mathbf{H}}$ the « connection jump tensor distribution » and we denote by

$$\underline{\boldsymbol{h}} \equiv \mathrm{C}\underline{\mathbf{H}} = -\frac{1}{2}\left(i_{\underline{\delta}}l\tilde{b} + \underline{\delta} \vee \left(\frac{1}{2}(\operatorname{tr}\tilde{b})l - i_l\tilde{b}\right)\right) \tag{12}$$

the contracted double 1-form distribution.

For its algebraic properties, $\underline{\mathbf{H}}$ resembles the curvature tensor, while $\underline{\boldsymbol{h}}$ looks like the Ricci tensor.

Finally, if $\underline{\mathrm{M}}$ is of class $\mathrm{C}^{(h,k)}$, with $2 \leq h$, we have a natural choiche of the extension $\tilde{b}$ of b, namely

$$\tilde{b} = \hat{g}' .$$

Nevertheless, the physical significant quantity is $\underline{\mathbf{H}}$, which depends essentially on b.

Let us go now to consider the action of ∇ on distributions. We can define the covariant differential $\nabla\tau$ of a distribution tensor of class $\mathrm{C}^{(-1,s)}$, with

$$0 \leq s \leq k - 2,$$

as the $\mathrm{C}^{0,s+1)}$ distribution, given by

$$\nabla\tau(u) \equiv \tau(\delta u), \tag{13}$$

where

$$\delta u \equiv -\mathrm{C}_1^1\nabla u .$$

This definition generalizes the Lichnerowicz's one.

This definition agrees with the covariant differential of a tensor t, of class $\mathrm{C}^{(0,s)}$, with $1 \leq s$, being

$$\nabla(t^{\mathrm{D}}) = (\nabla t)^{\mathrm{D}}$$

as a particular case of the following theorem, valable for $\mathrm{C}^{(-1,s)}$ tensors.

THEOREM 3 (see also Lichnerowicz [2]). — If t is a tensor of class $\mathrm{C}^{(-1,s)}$, with $1 \leq s$, we have

$$\nabla(t^{\mathrm{D}}) = (\nabla t)^{\mathrm{D}} - (-1)^n\underline{\delta} \otimes [t] . \tag{14}$$

Proof. — If u is a $\mathrm{C}^{(0,1)}$ tensor, with

$$\text{degree } u = 1 + \text{degree } t ,$$

and with compact support, we have by (13) and (6):

$$\nabla(t)^{D}(u) \equiv \int_{M} \langle t, \delta u \rangle \eta = \int_{M} \langle \nabla t, u \rangle \eta + \int_{M} \delta(i_t'' u)\eta .$$

But it is, by definition,

$$\int_{M} \langle \nabla t, u \rangle \eta = (\nabla t)^{D}(u)$$

and we contend that

$$\int_{M} \delta(i_t'' u)\eta = - \int_{M} d * i_t'' u = - (-1)^n \int_{\Sigma} j^* [* i_t'' u] = - (-1)^n \underline{\delta} \otimes [t](u) .$$

In fact, if α is a $C^{(-1,1)}$ $(n-1)$-form, with compact support U, contained in a coordinate neighboroud Ω of $x \in \Sigma$, we have, by the Stokes theorem:

$$\int_{M} d\alpha = \int_{\Omega} d\alpha = \int_{\Omega^+} d\alpha + \int_{\Omega^-} d\alpha$$
$$= (-1)^n \int_{\Omega^+ \cap \Sigma} j^*\alpha \quad - (-1)^n \int_{\Omega^- \cap \Sigma} j^*\alpha = (-1)^n \int_{\Sigma} j^*[\alpha] .$$

Moreover, we can extend, by a unity partition, this result to any $C^{(-1,1)}$ $(n-1)$-form, with compact support.

5. THE RIEMANNIAN CURVATURE

Being the metric g of class $C^{(0,2)}$, the Riemann tensor

$$R : (x, y ; u, v) \mapsto \langle \nabla_x \nabla_y u - \nabla_y \nabla_x u - \nabla_{L_{xy}} u, v \rangle$$

is defined in $M - \Sigma$ and may be viewed as a regularly discontinuous tensor of class $C^{(-1,k-3)}$.

The associated distribution R^D is given by

$$R^D : (x, y ; u, v) \mapsto (\nabla_x \nabla_y u - \nabla_y \nabla_x u - \nabla_{L_{xy}} u)^D(v) .$$

As the connection acts also on distributions, it is natural to consider the « curvature distribution » $\underline{Q}$, introduced in the following theorem.

THEOREM 4 (see also Lichnerowicz [2]). — The map

$$\underline{Q} : (\mathscr{C} \times \mathscr{C}) \times \mathscr{C} \to \mathscr{D} ,$$

given by

$$\underline{Q} : (x, y ; u) \mapsto \nabla_x(\nabla_y u)^D - \nabla_y(\nabla_x u)^D - (\nabla_{L_{xy}} u)^D \tag{15}$$

is multilinear, hence it defines a tensor distribution of degree 4.

$$\underline{Q} : x \otimes y \otimes u \otimes v \mapsto (\nabla_x(\nabla_y u)^D - \nabla_y(\nabla_x u)^D - (\nabla_{L_{xy}} u)^D)(v) .$$

Furthermore, $\underline{\mathbf{Q}}$ is a symmetrical double 2-form distribution, of class $C^{(0,2)}$, which satisfies the « Bianchi identity »

$$\underline{\mathbf{Q}}(x, y ; u) + \underline{\mathbf{Q}}(u, x ; y) + \underline{\mathbf{Q}}(y, u ; x) = 0 .$$

More precisely, it is

$$\underline{\mathbf{Q}} = \mathrm{R}^{\mathrm{D}} - (-1)^n \underline{\mathbf{H}} . \tag{16}$$

Proof. — Taking into account theorem 3, we have:

$$\underline{\mathbf{Q}}(x, y ; u) = (\nabla_x \nabla_y u - \nabla_y \nabla_x u - \nabla_{\mathrm{L}_{xy}} u)^{\mathrm{D}} - (-1)^n (i_x \underline{\boldsymbol{\delta}} \otimes [\nabla_y u] - i_y \underline{\boldsymbol{\delta}} \otimes [\nabla_x u]),$$

hence, for theorem 2, we get:

$$\underline{\mathbf{Q}} = \mathrm{R}^{\mathrm{D}} - (-1)^n \underline{\mathbf{H}} .$$

We denote by

$$r \equiv \mathrm{CR}$$

and by

$$\underline{\boldsymbol{q}} \equiv \mathrm{C}\underline{\mathbf{Q}} = r^{\mathrm{D}} - (-1)^n \underline{\boldsymbol{h}}$$

the Ricci tensor and the « Ricci distribution », respectively.

6. SHOCK WAVES IN GENERAL RELATIVITY

We are going now to apply the previous results to general relativity.

Henceforth, M is a model of space-time, that is a four dimensional, Lorentz, time oriented manifold. The metric g represents the gravitational field, hence the jump of its derivatives gets the meaning of a gravitational shock wave. We suppose that the source of the gravitational field is an electromagnetic, regularly discontinuous, field F, whose jump gets the meaning of an electromagnetic shock wave.

More precisely, we consider the weak Maxwell-Einstein equations

$$\left\{\begin{aligned} &\underline{\boldsymbol{q}} = \mathrm{T}^{\mathrm{D}} - \frac{1}{2} g \operatorname{tr} \mathrm{T}^{\mathrm{D}} \\ &d(\mathrm{F}^{\mathrm{D}}) = 0 \\ &\delta(\mathrm{F}^{\mathrm{D}}) = 0 \\ &\mathrm{T} \equiv -\frac{1}{2} \mathrm{C}(\mathrm{F} \otimes \mathrm{F} + *\mathrm{F} \otimes *\mathrm{F}) \end{aligned}\right. \tag{17}$$

on a four dimensional oriented manifold M of class $C^{(h,k)}$ (respect to Σ, as before), where the unknowns are a (time orientable) Lorentz metric g of class $C^{(0,2)}$ and a 2-form F of class $C^{(-1,1)}$.

By a purely algebraic method, these equations characterize the jump of F and of the derivatives of g and they say that the shock waves travel along geodesics (see also Lichnerowicz [*2*]), with the velocity of light.

THEOREM 5 (see also Lichnerowicz [*2*]). — Let g and F be a solution of (17). Then $\underline{\mathbf{H}}$ and [F] are, respectively, a singular double 2-form distribution and a singular 2-form, with fundamental vector l, orthogonal to the singular hypersurface Σ (support of $\underline{\mathbf{H}}$ and [F]).

More precisely, we have:

$$H = -\frac{1}{2}(l \otimes l) \wedge \tilde{b}, \tag{18}$$

$$[F] = l \wedge a, \tag{19}$$

where

$$i_l \tilde{b} = \frac{1}{2} \operatorname{tr} \tilde{b} l, \tag{20}$$

$$i_l a = 0, \tag{21}$$

and

$$l^2 = 0. \tag{22}$$

Proof. — The formula (17) gives, on $M - \Sigma$:

$$\begin{cases} r = T - \frac{1}{2} g \operatorname{tr} T \\ dF = 0 \\ \delta F = 0, \end{cases}$$

hence:

$$\begin{cases} r^D = T^D - \frac{1}{2} g \operatorname{tr} T^D \\ (dF)^D = 0 \\ (\delta F)^D = 0. \end{cases} \tag{.}$$

On other hand, taking into account theorem 3, and theorem 4, we get:

$$\begin{cases} \underline{\boldsymbol{q}} = r^D - \underline{\boldsymbol{h}} \\ d(F^D) = (dF)^D - \underline{\boldsymbol{\delta}} \wedge [F] \\ \delta(F^D) = (\delta F)^D - i_{\underline{\boldsymbol{\delta}}}[F]. \end{cases} \tag{..}$$

Hence, by comparison of (17), (.) and (..), we get:

$$\underline{\boldsymbol{h}} = 0 \tag{*}$$

$$\underline{\boldsymbol{\delta}} \wedge [F] = 0 \tag{**}$$

$$i_{\underline{\boldsymbol{\delta}}}[F] = 0. \tag{***}$$

Finally, formula (18) is exactely (11); (*) and (12) give

$$l^2 = 0 \quad \text{and} \quad i_l \tilde{b} = \frac{1}{2} \operatorname{tr} \tilde{b} l;$$

while, (**) gives

$$[F] = l \wedge a,$$

hence (***) gives

$$l^2 = 0 \quad \text{and} \quad i_l a = 0.$$

7. DETECTION OF SHOCK WAVES BY TEST PARTICLES

To detect the gravitational and electromagnetic shock waves, we consider a sheaf of particles characterized by a one-parameter local group

$$C : I \times \Omega \to \Omega,$$

of class $C^{(0,2)}$, such that the vector field

$$u : x \mapsto C'_x(0), \qquad \forall x \in \Omega,$$

is future time directed and unitary ($u^2 = -1$). (If the physical structure of the space-time manifold M admits a global one-parameter group

$$C : R \times M \to M,$$

then we can perform our considerations by such a C.) By our assumptions, $t \in I$ and u represent, respectively, the proper time and the velocity of the particles.

The choice of a vector field y, of class $C^{(0,2)}$, such that

$$L_u y = 0 \tag{23}$$

(Frobenius integrability condition) and

$$y^2 > 0$$

generates a spatial parametrization on each integral surface S of the system $\{ u, y \}$, which labels the particles on S (in fact, for (23), the one-parameter groups generated by u and y commute [*10*]).

We suppose that the sheaf has mass and charge densities μ and ρ, with $k \equiv \rho/\mu$ constant.

The Lorentz force acting on the sheaf gives the motion low:

$$\nabla_u u = k i_u F. \tag{24}$$

Now, we are able to detect the shock waves, calculating the « deviation » $[\nabla_u y]$, that measures the jump of the relative velocity of test particles.

THEOREM 6. — We have on Σ:

$$l(u)[\nabla_u y] = H(u, y ; u) + k l(y) i_u [F]. \tag{25}$$

Furthermore, calling π the unique projection orthogonal to the plane $\{ u, l \}$, we have:

$$[\nabla_u y] = (c \circ \pi)(y) + k l(y) \pi a, \tag{26}$$

where c is the symmetric, trace free, 2-tensor

$$c \equiv \frac{1}{2}(l(u)b - i_u b \otimes l - l \otimes i_u b) \tag{27}$$

(here c and $l(y)\pi a$ depend only on $\underline{H}$ and [F]).

Proof. — From (10), (23) and (24), we get:

$$H(u, y ; u) = l(u)[\nabla_y u] - l(y)[\nabla_u u] = l(u)[\nabla_u y] - kl(y)i_u[F].$$

We can decompose uniquely y in the two components $y^{||}$ and $y^{\perp}$, respectively, parallel and orthogonal to the plane $\{ u, l \}$. Then, taking into account (11) and (20), we get:

$$H(u, u ; u) = 0 = H(u, l ; u),$$

hence

$$H(u, y ; u) = H(u, y^{\perp} ; u) = l(u) \langle l(u)b - i_u b \otimes l - l \otimes i_u b, y^{\perp} \rangle .$$

Furthermore, for (19) and (21), we have:

$$i_u[F] = l(u)a - a(u)l .$$

The formula (26) gives

$$\langle [\nabla_u y], u \rangle = 0 = \langle [\nabla_u y], l \rangle,$$

that shows the « transversal » nature of shock waves, namely, it implies that the deviation $[\nabla_u y]$ has, respect to the observer u, a spatial direction orthogonal to the spatial direction of the shock waves.

Moreover, we can separate the effects of gravitational and electromagnetic shock waves. In fact, choicing y, such that

$$l(y) = 0, \quad \pi y \neq 0, \qquad \text{or} \qquad l(y) \neq 0, \quad \pi y = 0,$$

the effective contribution to the deviation is given only by the gravitational, or by the electromagnetic shock waves, respectively.

More precisely, we can choose, in $x \in \Sigma$,

$$\langle y, u \rangle = 0$$

and an orthogonal basis $\{ e_0, e_1, e_2, e_3 \}$, such that

$$e_0 \equiv u, \qquad e_1 = 1 + l(u)u, \qquad c(e_2) = \gamma e_2, \qquad c(e_3) = - \gamma e_3 .$$

Hence, we can write

$$\begin{cases} c \circ \pi = c = \gamma(e_2 \otimes e_2 - e_3 \otimes e_3), \\ \pi a = a^2 e_2 + a^3 e^3, \end{cases}$$

and

$$\begin{cases} [\nabla_u y]^0 = 0 \\ [\nabla_u y]^1 = 0 \\ [\nabla_u y]^2 = \gamma y^2 + l(y)a^2 \\ [\nabla_u y]^3 = - \gamma y^3 + l(y)a^3 . \end{cases}$$

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