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Complete Positivity and Asymptotic Abelianness

by

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ABSTRACT. — Consider a C*-algebra A with a group of *-automorphisms. We are concerned with the structure of invariant completely positive maps on A, when the system possesses some particular properties known under the general name of asymptotic abelianness.

RÉSUMÉ. — On considère une C*-algèbre A avec un groupe de *-automorphismes. Nous examinons la structure des formes invariantes et complètement positives sur A, lorsque le système montre une particularité qu'on appelle en général un état abélien asymptotique.

1. INTRODUCTION

Let A be a C*-algebra, G a group and $g \to \sigma_g$ a representation of G by *-automorphisms of A. We are concerned with the structure of G-invariant completely positive linear maps on A, and we discuss in some detail various cases in which the system possesses some particular properties known under the general name of asymptotic abelianness.

Asymptotic abelianness and complete positivity have natural physical interpretations. Asymptotic abelianness [2] puts in various precise forms the heuristic argument that two observables in A tend to commute as one is removed far away from the other. As pointed out by G. Lindblad [5], completely positive maps arise physically with the study of operations on systems which remain positive when interacting with finite quantum systems.

2. NOTATION AND PRELIMINARIES

Let $\mathscr{A}$ be a von Neumann algebra on a hilbert space $\mathscr{K}$, G a group and $g \to \alpha_g$ a representation of G by *-automorphisms of $\mathscr{A}$ which is implemented by a unitary representation U_g of G on $\mathscr{K}$. Let E_0 be the projection on the G invariant elements of $\mathscr{K}$.

$$\text{i. e.} \qquad E_0\mathscr{K} = \{ \gamma \in \mathscr{K} : U_g\gamma = \gamma \quad \forall g \in G \}$$

Let $\mathscr{R}$ be the von Neumann algebra generated by $\mathscr{A}$ and U_G. Then note that $E_0 \in$ strong closure $\{ \text{conv } U_G \}$, and in particular $E_0 \in (U_G)''$.

Throughout this paper A will denote a C*-algebra, H a hilbert space, and CP(A ; H) the completely positive (CP), linear maps from A into B(H), the bounded linear operators on H. We recall [*1*] [*3*] [*7*] that if $w \in$ CP(A ; H) then there exists a hilbert space $\mathscr{K}_w$, a representation π_w of A on $\mathscr{K}_w$, and a linear mapping Λ_w from the algebraic tensor product A $\odot$ H, into a dense subspace of $\mathscr{K}_w$ such that

i) $\langle w(x^*y)f, h \rangle = \langle \Lambda_w y \otimes f, \Lambda_w x \otimes h \rangle \qquad \forall x, y \in A, \quad f, h \in H$

ii) $\pi_w(x)\Lambda_w y \otimes f = \Lambda_w xy \otimes f \qquad \forall x, y \in A, \quad f \in H$

Moreover, there exists an unique $V_w \in B(H, \mathscr{K}_w)$ such that

iii) $\pi_w(y)V_w f = \Lambda_w y \otimes f \qquad \forall y \in A, \quad f \in H.$

$g \to \sigma_g$ will always denote a representation of a group G by *-automorphisms of A. If $w \in CP_0(A ; H)$, the G invariant part of CP(A ; H), let U_g^w be the induced representation of G by unitary operators on $\mathscr{K}_w$ such that

iv) $\pi_w\sigma_g(x) = U_g^w\pi_w(x)(U_g^w)^{-1} \qquad \forall g \in G, \quad x \in A.$

v) $U_g^w\Lambda_w x \otimes f = \Lambda_w\sigma_g(x) \otimes f \qquad \forall g \in G, \quad x \in A, \quad f \in H.$

The above notation will then be used with w as a suffix for the von Neumann algebra $\mathscr{A}_w = \pi_w(A)''$, with the automorphism group $a \to U_g^w a(U_g^w)^{-1}$, $a \in \mathscr{A}_w$.

We begin with the following generalisation of [*6*, Proposition 6.3.2] for the usual state space.

PROPOSITION 1. — *If* $w \in CP_0(A ; H)$ *then*

a) $\{ \pi_w(A) \cup E_0^w \}' = \{ \pi_w(A) \cup U_G^w \}'$.

b) *The mapping* $\{ \pi_w(A) \cup U_G^w \}' \to E_0^w \{ \pi_w(A) \cup U_G^w \}'$ *is a* *-*isomorphism.*

c) $E_0^w \{ \pi_w(A) \cup U_G^w \}' = E_0^w \{ E_0^w\pi_w(A)E_0^w \}'$.

d) $E_0^w \{ E_0^w\pi_w(A)E_0^w \}''$ *is the strong closure of* $E_0^w\pi_w(A)E_0^w$.

For our main results, the following lemma will prove useful:

LEMMA 1 [2]. — *Let $\mathscr{A}$ be a von Neumann algebra acting on a hilbert space $\mathscr{K}$, and $g \to U_g$ a representation of* G *by unitary operators on $\mathscr{K}$, such that $U_g \mathscr{A} U_g^{-1} = \mathscr{A}$ $\forall g \in G$. If $E_0 \mathscr{A} E_0$ is abelian and $[AE_0] = 1$, there exists an unique normal* G*-invariant* CP *linear map* M *of $\mathscr{A}$ onto $\mathscr{R} \cap \mathscr{R}'$ such that* $M(A)E_0 = E_0 A E_0$, $\forall A \in \mathscr{A}$.

Returning to our system (A, G), it follows that if A is G-abelian, then for each w in $CP_0(A ; H)$ there is an unique normal G-invariant CP map M of $\pi_w(A)''$ onto $\mathscr{R}_w \cap \mathscr{R}'_w$ such that $M(\pi_w(A'))E_0^w = E_0^w \pi_w(A')E_0^w$ for all A′ in A. Note that if $A_1, A_2 \in A$, then

$$M(\pi_w(A_2))\pi_w(A_1)V_w = \pi_w(A_1)E_0^w \pi_w(A_2)V_w .$$

Remark. — If $n \in \mathbb{N}$, let U_n denote the C*-algebra of all $n \times n$ matrices over $\mathbb{C}$, and $A_n = A \otimes U_n$, the C*-algebra of all $n \times n$ matrices over A. We remark at this point, that if we consider G acting elementwise on A_n (i. e. $g \to \sigma_g \otimes 1_n$) and if $CP_0(A ; \mathbb{C}) \neq 0$ then A_n is never G-abelian [2, Definition 0] if $n \geqslant 2$, as simple arguments using the non-abelianness of U_n ($n \geqslant 2$) show.

3. MAIN RESULTS

The following theorem is a generalisation of [6, Ex. 6 C] from the space of invariant positive linear functionals.

THEOREM 1. — *The following conditions are equivalent, for a fixed hilbert space* H,

1) *For all w in* $CP_0(A ; H)$, *and self adjoint a in* A,

$$[\text{the weak closure of } \pi_w (\text{conv } \sigma_G(a))] \cap \pi_w(A)' \neq \emptyset .$$

2) *For all w in* $CP_0(A ; H)$, *self adjoint a in* A, *and finite sets* S, T *in* A *and* H *respectively:*

$$\inf_{a' \in \text{conv } \sigma_G(a)} \sup_{\substack{\xi,\eta \in T \\ a_1, a_2 \in S}} |\langle w(a_1[a', a_2])\xi, \eta \rangle| = 0 .$$

3) A *is* G*-abelian and for each w in* $CP_0(A ; H)$ *and self adjoint a in* A, $M\pi_w(a)$ *is in the weak closure of* π_w (conv $\sigma_G(a)$).

Proof. — 1) $\Rightarrow$ 2) Take w in $CP_0(A ; H)$ and self ajoint a in A. Choose a net a_α in conv σ_G such that $\pi_w(A_\alpha)$ converges weakly to an operator D in $\pi_w(A)'$. Then $[\pi_w(a_\alpha), \pi_w(b_j)] \to [D, \pi_w(b_j)] = 0$ weakly, whenever $b_1, \ldots, b_b$ is a finite set in A. Therefore for a given $\varepsilon > 0$ and a finite set T in H, there exists an α such that

$$|\langle w(b_i a_\alpha, b_j)\xi, \eta \rangle| = |\langle [\pi_w(a_\alpha), \pi_w(b_j)]V_w\xi, \pi_w(b_i^*)V_w\eta \rangle| < \varepsilon$$

for all $\xi, \eta \in T$, $1 \leqslant i, j \leqslant n$.

Hence

$$\inf_{a' \in \text{conv } \sigma_G(a)} \sup_{\substack{\xi,\eta \in T \\ 1 \leqslant i,j \leqslant \eta}} |\langle w(b_i[a', b_j])\xi, \eta \rangle| = 0$$

2) $\Rightarrow$ 3). Let $w \in CP_0(A ; H)$, a be a self adjoint element of A, and T a finite subset of H. Since $E_0 \in \{ \text{conv } U_G \}^-$, it follows that there exists g_i, λ_i in G, [0, 1] respectively for $1 \leqslant i \leqslant n$ with $\Sigma\lambda_i = 1$, and satisfying

$$\| E_0^w \pi_w(a) V_w \xi - \sum_{i=1}^{n} \lambda_i U_{g_i} \pi_w(a) V_w \xi \| < \varepsilon \qquad \forall \xi \in T$$

i. e.

$$\| E_0^w \pi_w(a) V_w \xi - \pi_w(a') V_w \xi \| < \varepsilon \qquad \forall \xi \in T$$

where

$$a' = \Sigma \lambda_i \sigma_{g_i}(a) \in \text{conv } \sigma_G(a) .$$

Let S be a finite subset in A, then by 2) there exists a'' in conv $\sigma_G(a')$ such that

$$| \langle \pi_w(a'') \pi_w(b_1) V\xi, \pi_w(b_2) V\eta \rangle | < \varepsilon \qquad \forall b_1, b_2 \in S, \xi, \eta \in T .$$

Suppose $a'' = \sum_{i=1}^{n} \lambda_i' \sigma_{g_i'}(a')$ where $g_i' \in G$, $\lambda_i' \geqslant 0$ and $\Sigma \lambda_i' = 1$. Hence

$$\begin{aligned}
&\langle M(\pi_w(a))\pi_w(b_1)V\xi, \pi_w(b_2)V\eta \rangle - \langle \pi_w(a'')\pi_w(b_1)V\xi, \pi_w(b_2)V\eta \rangle \\
&\quad = \langle \pi_w(b_1)E_0\pi_w(a)V\xi, \pi_w(b_2)V\eta \rangle - \langle \pi_w(a'')\pi_w(b_1)V\xi, \pi_w(b_2)V\eta \rangle \\
&\quad = \langle \pi_w(b_1)E_0\pi_w(a)V\xi, \pi_w(b_2)V\eta \rangle - \langle \pi_w(b_1)\pi_w(a'')V\xi, \pi_w(b_2)V\eta \rangle \\
&\quad + \langle \pi_w(b_1)\pi_w(a'')V\xi, \pi_w(b_2)V\eta \rangle - \langle \pi_w(a'')\pi_w(b_1)V\xi, \pi_w(b_2)V\eta \rangle \\
&\quad = \sum_{i=1}^{m} \{ \langle \pi_w(b_1)E_0\lambda_i'\pi_w(a)V\xi, \pi_w(b_2)V\eta \rangle \\
&\qquad\qquad - \lambda_i' \langle \pi_w(b_1)U_{g_i'}\pi_w(a')V\xi, \pi_w(b_2)V\eta \rangle \\
&\quad + \langle [\pi_w(b_1), \pi_w(a'')]V\xi, \pi_w(b_2)V\eta \rangle
\end{aligned}$$

Thus

$$\begin{aligned}
&| \langle M(\pi_w(a))\pi_w(b_1)V\xi, \pi_w(b_2)V\eta \rangle - \langle \pi_w(a'')\pi_w(b_1)V\xi, \pi_w(b_2)V\eta \rangle | \\
&\quad \leqslant \sum_{i=1}^{n} \| \pi_w(b_1) \| \lambda_i' \| E_0\pi_w(a)V\xi - U_{g_i'}\pi_w(a')V\xi \| \; \| \pi_w(b_2)V\eta \| \\
&\qquad\qquad + | \langle \pi_w(b_1), \pi_w(a'')V\xi, \pi_w(b_2)V\eta \rangle | \\
&\quad \leqslant \varepsilon \| \pi_w(b_1) \| \, . \, \| \pi_w(b_2)V\eta \| + \varepsilon \qquad \forall \eta, \xi \in T, \quad b_1, b_2 \in S .
\end{aligned}$$

Since $[\pi_w(A)V_wH]^- = \mathscr{K}_w$ and $\| a' \| \leqslant \| a \|$ whenever $a' \in \text{conv } \sigma_G(a)$, it follows that $M\pi_w(a)$ is in the weak closure of π_w conv $\sigma_G(a)$.

3) $\Rightarrow$ 1). This is trivial, as $\mathscr{R}_w' \subseteq \pi_w(A)'$ $\forall w \in CP_0(A ; H)$.

Note that if A satisfies conditions 1-3 of Theorem 1, then G is a large group of automorphisms for A [*2*, Definition 2]. The truth of the converse is unclear. If (A, σ) is $\mathscr{M}$-abelian [*2*, Definition 3], then the equivalent conditions 1-3 of Theorem 1 hold (which can be seen using the same argument of [*2*] when showing that if (A, σ) is $\mathscr{M}$-abelian, then G is a large group of automorphisms for A). The converse is false, using the same example in [2] which shows that (A, σ) being $\mathscr{M}$-abelian is not equivalent to G being a large group of automorphisms for A.

The next result concerns the characterisation of extremal invariant (or ergodic) CP maps, and extends [2] for states.

THEOREM 2. — *Let* B *be a* C*-*algebra acting on a hilbert space* $\mathscr{K}$, *let* $\mathscr{A}$ *denote the weak closure of* B. *Let* $g \to U_g$ *be a unitary representation of* G *on* $\mathscr{K}$, *such that* $\alpha_g a = U_g a U_g^{-1}$ *is an automorphism of* B *for each* g *in* G. *Suppose* $V \in B(H, \mathscr{K})$ *such that* $[BVH]^- = \mathscr{K}$ *and* $VH \subseteq E_0\mathscr{K}$. *Then if* χ_0 *denotes any unit vector in* E_0,

a) *The following are equivalent:*

i) $w(x) = V^*xV \quad x \in B$, *is an extremal element of* $CP_0(B ; H)$.

ii) $\mathscr{R}'$ *is the scalars.*

b) *If* $E_0\mathscr{A}E_0$ *is abelian, the conditions in* a) *are equivalent also to the following:*

iii) E_0K *is one dimensional.*

iv) $$M(a) = \langle a\chi_0, \chi_0 \rangle \qquad \forall a \in A,$$

in which case

$$w(a) = \langle a\chi_0, \chi_0 \rangle (V^*\chi_0) \otimes \overline{(V^*\chi_0)}, \qquad \forall a \in B.$$

c) *If furthermore,* $(\mathscr{A}, \alpha)$ *satisfy the hypothesis of* [2, Lemma 3], *then the above conditions are equivalent to:*

v) *If* β *is a normal positive* G-*invariant map from* $\mathscr{A}$ *into another von Neumann algebra* M, *then*

$$\beta(a) = \langle a\chi_0, \chi_0 \rangle K, \qquad \forall a \in \mathscr{A},$$

and some K *in* M_+.

COROLLARY 1. — *If* $\mathscr{A}$ *is a factor and* $E_0\mathscr{A}E_0$ *is abelian and* $\mathscr{R} \cap \mathscr{R}' \subseteq \mathscr{A}$ *then* w *is an extremal element of* $CP_0(A ; H)$ *and can be written as*

$$w(a) = \langle a\chi_0, \chi_0 \rangle (V^*\chi_0) \otimes \overline{(V^*\chi_0)} \qquad \forall a \in B.$$

The proof for the one-dimensional case [6, Ex. 6 A] can be extenaed to show the following:

PROPOSITION 2. — *Let* A *be* G-*abelian. Then if* $w \in CP_0(A ; H)$

$$\{ \beta \in CP_0(A ; H) : \beta \leqslant w \}$$

is a lattice.

Remark. — In particular if A has an identity, and H finite dimensional, then $\{ w \in CP_0(A ; H) : \text{tr } w(1) = 1 \}$ forms a Choquet simplex. In this situation, an invariant CP map can be written in an unique way as an integral (in some sense) over ergodic CP maps. We remark that if $n = \dim H$ then there is an affine bijection between $(A_n)^*_+$ and CP(A ; H) which is a homeomorphism for the weak *- and BW-topologies [4]. However, by Remark 1, we cannot deduce that

$$\{ w \in CP_0(A ; H) : \operatorname{tr} w(1) = 1 \}$$

forms a Choquet simplex from the usual state space result [6, Ex. 6 A] for A_n.

We return to the general situation, to state a generalisation of a result of Arveson [1] where G is the identity automorphism.

THEOREM 3. — *If* $K \in B(H)_+$, *then* w *is an extreme point of the convex set*

$$\{ w \in CP_0(A ; H) : V_w^* V_w = K \}$$

iff $[V_w H]^-$ *is a faithful subspace for* $\mathcal{R}'_w$.

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