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The entropy density of quasi-free states for a continuous boson system

by

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ABSTRACT. — We compute the entropy density for a gauge and euclidean invariant, quasi-free state of the continuous boson system.

1. INTRODUCTION

We compute in this paper the entropy density of a gauge and euclidean invariant quasi-free state for a continuous boson system. The result is expressed in terms of the Fourier transform of the operator associated to it's two-point function. In order to keep the notation as simple as possible we considered a 1-dimensional system but the generalization is straightforward. A lower bound for this density, which gives in fact the correct result, has been studied in [*4*] for a restricted class of states.

The method used in this paper is related to the one used in [*5*] and consists essentially in computing the entropy density for a « nice » class of states and extending afterwards the result to the general case. Instead of computing only the limit of the local densities we have formulated a more general lemma in the appendix. This lemma can then also be used to derive for instance, an analogous result for the continuous fermion system.

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2. PRELIMINARIES

We consider the C. C. R.-algebra $\mathscr{A}$ over the space $C_0^\infty(\mathbb{R})$ of the infinitely differentiable, complex-valued functions with compact support in $\mathbb{R}$. This algebra is generated by elements δ_ψ, $\psi \in C_0^\infty(\mathbb{R})$ which satisfy:

$$\delta_\psi \delta_\phi = \exp -\frac{i}{2}\sigma(\psi \mid \phi) \qquad \delta_{\phi+\psi}$$
$$\delta_\psi^* = \delta_{-\psi}$$

where $\sigma(\cdot \mid \cdot)$ is the symplectic form $\operatorname{Im} \langle \cdot \mid \cdot \rangle$ and $\langle \cdot \mid \cdot \rangle$ is the usual scalar product on $\mathscr{L}^2(\mathbb{R}, dx)$ [6].

$\mathscr{A}$ is a simple, non-separable, quasi-local C*-algebra:

$$\mathscr{A} = \overline{\bigcup_\Lambda \mathscr{A}_\Lambda}$$

where the union is taken over the compact subsets Λ of $\mathbb{R}$ and $\mathscr{A}_\Lambda$ is the C. C. R.-algebra over the space of infinitely differentiable, complex-valued functions with support contained in Λ.

The aim of this paper is to derive an explicit expression for the entropy density of a locally-normal, quasi-free state which is invariant under the action of both the gauge and the euclidean groups. The gauge automorphisms are induced by:

$$\delta_\psi \to \delta_{e^{i\theta}\psi} \qquad \psi \in C_0^\infty(\mathbb{R}),\ \theta \in [0, 2\pi[$$

and the euclidean automorphisms by:

$$\delta_\psi \to \delta_{U\psi} \qquad \psi \in C_0^\infty(\mathbb{R}),\ U \text{ euclidean}.$$

The gauge-invariant, quasi-free states on $\mathscr{A}$ are of the form:

$$\omega_A(\delta_\psi) = \omega_F(\delta_\psi) \exp -\frac{1}{2} A(\psi)$$

where

$$\omega_F(\delta_\psi) = \exp -\frac{1}{4}\langle \psi \mid \psi \rangle$$

is the Fock state and $\psi \to A(\psi)$ is a positive, quadratic form. We will assume in the following that there exists a positive self-adjoint operator A on $\mathscr{L}^2(\mathbb{R}, dx)$ such that

$$A(\psi) = \langle \psi \mid A\psi \rangle .$$

Imposing the condition that ω_A be euclidean invariant leads then to

$$A = a(-\Delta)$$

where Δ is the Laplace operator in one dimension and $a(\cdot)$ is a non-negative Borel function [7].

A state ω on $\mathscr{A}$ is said to be locally-normal if for all compacts $\Lambda \subset \mathbb{R}$, $\omega|_{\mathscr{A}_\Lambda}$ is normal with respect to $\omega_F|_{\mathscr{A}_\Lambda}$. For gauge-invariant, quasi-free states ω_B this condition amounts to asking that for all compacts $\Lambda \subset \mathbb{R}$, $E_\Lambda B E_\Lambda$ be a trace-class operator on $\mathscr{L}^2(\Lambda, dx)$, where E_Λ is the orthogonal projection from $\mathscr{L}^2(\mathbb{R}, dx)$ on $\mathscr{L}^2(\Lambda, dx)$. For the quasi-free states

$$\omega_A(\delta_\psi) = \omega_F(\delta_\psi) \exp - \frac{1}{2} \langle \psi \mid a(-\Delta)\psi \rangle$$

this condition is also equivalent to [3]

$$\int_{\mathbb{R}} dx\, a(x^2) < \infty \ [3]$$

Let Λ be a compact subset of $\mathbb{R}$ and ω a state on $\mathscr{A}$. One defines the entropy $S(\omega, \Lambda)$ of ω with respect to Λ as:

$S(\omega, \Lambda) = +\infty$ if $\omega|_{\mathscr{A}_\Lambda}$ is not normal with respect to $\omega_F|_{\mathscr{A}_\Lambda}$

$S(\omega, \Lambda) = -\operatorname{Tr}_{\mathscr{K}_F^\Lambda} \rho^\Lambda \ln \rho^\Lambda$ if $\omega|_{\mathscr{A}_\Lambda}$ is normal with respect to $\omega_F|_{\mathscr{A}_\Lambda}$.

In this last case ρ^Λ is the density matrix on $\mathscr{K}_F^\Lambda$ given by:

$$\omega(x) = \operatorname{Tr}_{\mathscr{K}_F^\Lambda} \Pi_F^\Lambda(x)\rho^\Lambda \qquad x \in \mathscr{A}$$

where $(\Pi_F^\Lambda, \mathscr{K}_F^\Lambda, \Omega_F^\Lambda)$ is the G. N. S. triplet associated to $\omega_F|_{\mathscr{A}_\Lambda}$.

Let ω be a translation-invariant state on $\mathscr{A}$. Its entropy density $s(\omega)$ is defined as:

$$s(\omega) = \lim_{l\to\infty} \frac{1}{2l} S(\omega, [-l, l])$$

The existence of this limit was shown in [1].

The aim of this paper is to prove that for states

$$\omega_A(\delta_\psi) = \omega_F(\delta_\psi) \exp - \frac{1}{2} \langle \psi \mid a(-\Delta)\psi \rangle \qquad (*)$$

$$s(\omega_A) = \frac{1}{2\pi} \int_{\mathbb{R}} dx\, e(a(x^2))$$

where

$$e(x) = (1 + x)\ln(1 + x) - x \ln x\,, \qquad x \geqslant 0$$

3. COMPUTATION OF $s(\omega_A)$ FOR A RESTRICTED CLASS OF STATES

Let A be a non-negative self-adjoint operator on an Hilbert space $\mathscr{K}$. We put

$$S(A) = \operatorname{Tr}_{\mathscr{K}} e(A).$$

THEOREM 3.1. — Let ω_B be a gauge-invariant, quasi-free state on $\mathscr{A}$ and $\Lambda \subset \mathbb{R}$ a compact set. Then:

$$S(\omega_B, \Lambda) = S(E_\Lambda B E_\Lambda)$$

where E_Λ is the orthogonal projection from $\mathscr{L}^2(\mathbb{R}, dx)$ on $\mathscr{L}^2(\Lambda, dx)$

Proof. — The proof of this theorem is well-known and consists in a straightforward computation of $S(\omega_B, \Lambda)$.

LEMMA 3.2. — The function $A \to S(A)$ is concave.

Proof. — Let A, B be self-adjoint non-negative operators on $\mathscr{K}$ and $0 < \alpha < 1$. Put $C = \alpha A + (1 - \alpha)B$. We have to show that:

$$\alpha S(A) + (1 - \alpha)S(B) \leqslant S(C)$$

Suppose that $S(A) = +\infty$, then also $S(\alpha A) = +\infty$ and as $C \geqslant \alpha A$ and $S(\cdot)$ is an increasing function (see Lemma 4.1) $S(C) = +\infty$.

Suppose next that S(A) and S(B) are finite. Applying Klein's inequality [9] to the function $x \to e(x)$ one gets

$$\operatorname{Tr} (1 + A)\ln (1 + A) - A \ln A \leqslant \operatorname{Tr} (1 + A)\ln (1 + C) - A \ln C$$

and

$$\operatorname{Tr} (1 + B)\ln (1 + B) - B \ln B \leqslant \operatorname{Tr} (1 + B)\ln (1 + C) - A \ln C$$

and so:

$$\alpha S(A) + (1 - \alpha)S(B) \leqslant S(C)$$

Let $x \in \mathbb{R} \to a(x^2) \in \mathbb{R}$ be a measurable, essentially non-negative function. We define $s'(a)$ as:

$$s'(a) = \frac{1}{2\pi} \int_{\mathbb{R}} dx\, e(a(x^2))$$

THEOREM 3.3. — Let $x \in \mathbb{R} \to a(x^2) \in \mathbb{R}$ be a non-negative, continuous function with compact support and ω_A the state on $\mathscr{A}$ corresponding to $a(\cdot)$ (see (*)) then:

$$s(\omega_A) = s'(a)$$

Proof. — 1) Let $l > 0$ and define a group $\{ U^l(s), s \in \mathbb{R} \}$ of unitaries on $\mathscr{L}^2([-l, l], dx)$ by:

$$(U^l(s)\phi)(x) = \phi((x + s) \bmod 2l).$$

Denote by E^l the orthogonal projection from $\mathscr{L}^2(\mathbb{R}, dx)$ on $\mathscr{L}^2([-l, l], dx)$ and let $A^l = E^l A E^l$. Define now the operator A^l_c on $\mathscr{L}^2([-l, l], dx)$ by:

$$A^l_c = \frac{1}{2l} s - \int_0^{2l} ds\, U^l(s) A^l U^{l*}(s) \qquad (a)$$

Then:

i) $0 \leqslant A^l_c$ as $A^l \geqslant 0$

ii) $\frac{1}{2l} \operatorname{Tr} A^l_c = \frac{1}{2l} \operatorname{Tr} A^l = \frac{1}{2\pi} \int_{\mathbb{R}} dx\, a(x^2) < \infty$

iii) $[A^l_c, U^l(s)] = 0, \; s \in \mathbb{R}$.

As the group $\{ U^l(s) \mid s \in \mathbb{R} \}$ has simple discrete spectrum:

$$U^l(x) \frac{1}{\sqrt{2l}} e^{\frac{i\pi n x}{l}} = e^{\frac{i\pi n s}{l}} \frac{1}{\sqrt{2l}} e^{\frac{i\pi n x}{l}} \qquad n \in \mathbb{Z},\ x \in [-l, l]$$

iii) implies that

$$A_c^l \frac{1}{\sqrt{2l}} e^{\frac{i\pi n x}{l}} = \alpha_c^l(n) \frac{1}{\sqrt{2l}} e^{\frac{i\pi n x}{l}}$$

and a simple calculation shows that

$$\alpha_c^l(n) = \frac{1}{\pi} \int_{\mathbb{R}} dx \frac{a(x^2) \sin^2 lx}{l\left(\frac{n\pi}{l} + x\right)^2} \qquad n \in \mathbb{Z}$$

Using (*a*), Theorem 3.1 and Lemma 3.2 one gets:

$$S(\omega_A, [-l, l]) \leqslant S(A_c^l) = \sum_{n \in \mathbb{Z}} e(\alpha_c^l(n))$$

Dividing both sides by $2l$, taking the limit $l \to \infty$ and using Lemma 5.1 this becomes:

$$s(\omega_A) \leqslant s'(a)$$

2) Consider next for $l > 0$, the gauge-invariant, locally-normal, quasi-free state ω_A^l with period $2l$ constructed on:

$$\delta_\psi \to \omega_F(\delta_\psi) \exp - \frac{1}{2} \langle \psi \mid A_c^l \psi \rangle \qquad \psi \in \mathscr{A}_{[-l,l]}$$

An easy computation shows that

$$w^* - \lim \omega_A^l = \omega_A ,$$

and, as the entropy density is w^*-upper-semi-continuous [8] on the periodic, locally normal states, one gets

$$s(\omega_A) \geqslant \lim_{l \to \infty} s(\omega_A^l) = \lim_{l \to \infty} \frac{1}{2l} S(A_c^l) = s'(a)$$

using as above Lemma 5.1.

4. EXTENSION OF THE ENTROPY DENSITY FORMULA TO GENERAL LOCALLY-NORMAL GAUGE AND EUCLIDIAN INVARIANT, QUASI-FREE STATES

LEMMA 4.1. — Let $0 \leqslant A \leqslant B$ and $S(B) < \infty$, then:

$$0 \leqslant S(B) - S(A) \leqslant S(B - A)$$

Proof. — 1) One has

$$e(\mathrm{A}) = \int_0^1 \{ \mathbb{1} + \ln (s\mathbb{1} + \mathrm{A}) \} \, dx ,$$

and as $\ln(\cdot)$ is a monotone operator increasing function [2] $e(\mathrm{A}) \leqslant e(\mathrm{B})$ and so $\mathrm{S}(\mathrm{A}) \leqslant \mathrm{S}(\mathrm{B})$.

2) Define for $0 < \lambda \leqslant 1$

$$f(\lambda) = \mathrm{S}(\mathrm{B} - \mathrm{A} + \lambda\mathrm{A}) - \mathrm{S}(\lambda\mathrm{A})$$

Then

$$f'(\lambda) = \mathrm{Tr}\, \mathrm{A} \{ e'(\mathrm{B} - \mathrm{A} + \lambda\mathrm{A}) - e'(\lambda\mathrm{A}) \}$$

As

$$e'(\mathrm{A}) = \int_0^1 \frac{ds}{s\mathbb{1} + \mathrm{A}}$$

$e'(\cdot)$ is monotone operator decreasing and so $f'(\lambda) \leqslant 0$. Then:

$$\begin{aligned} \mathrm{S}(\mathrm{B}) - \mathrm{S}(\mathrm{A}) = f(1) &\leqslant f(\lambda) \qquad 0 < \lambda \leqslant 1 \\ &\leqslant \lim_{\lambda \to 0} f(\lambda) = \mathrm{S}(\mathrm{B} - \mathrm{A}) \end{aligned}$$

LEMMA 4.2. — Suppose that

$x \in \mathbb{R} \to a(x^2)$ is non-negative and measurable

$x \in \mathbb{R} \to a_n(x^2)$, $n \in \mathbb{N}$ is a sequence of non-negative measurable functions, a. e. bounded by $a(\cdot)$ and such that

$$\lim_{n \to \infty} a_n(x^2) = a(x^2) \qquad \text{a. e.}$$

Let ω_{A} and ω_{A_n} be the states corresponding to $a(\cdot)$ and $a_n(\cdot)$ and suppose $s(\omega_{\mathrm{A}}) < \infty$

Then:

$$\lim_{n \to \infty} s(\omega_{\mathrm{A}_n}) = s(\omega_{\mathrm{A}})$$

Proof. — Using the same notation as in the proof of Theorem 3.3 one has for any interval $[-l, l]$, $l > 0$:

$$0 \leqslant \mathrm{A}_n^l \leqslant \mathrm{A}^l$$

and as $\mathrm{S}(\mathrm{A}^l) < \infty$ [1] one has by Lemma 1:

$$0 \leqslant \mathrm{S}(\mathrm{A}^l) - \mathrm{S}(\mathrm{A}_n^l) \leqslant \mathrm{S}((\mathrm{A} - \mathrm{A}_n)^l)$$

and therefore

$$0 \leqslant s(\omega_{\mathrm{A}}) - s(\omega_{\mathrm{A}_n}) \leqslant s(\omega_{\mathrm{A} - \mathrm{A}_n})$$

Choose an $l > 0$. From the subadditivity of the entropy [1] one gets

$$s(\omega_{\mathrm{A} - \mathrm{A}_n}) \leqslant \frac{1}{2l} \mathrm{S}(\mathrm{A} - \mathrm{A}_n)^l) .$$

So it is sufficient to show that

$$\lim_{n \to \infty} \mathrm{S}(\mathrm{A} - \mathrm{A}_n)^l) = 0 .$$

Under the conditions of the lemma one has:

$$(A - A_n)^l \xrightarrow[n\to\infty]{s} 0$$

$$0 \leqslant (A - A_n)^l \leqslant A^l$$

$$S(A^l) < \infty$$

but this implies

$$\lim_{n\to\infty} S((A - A_n)^l) = 0$$

THEOREM 4.3. — Let $x \in \mathbb{R} \to a(x^2)$ be a non-negative bounded, measurable function and ω_A the state corresponding to $a(\cdot)$, then:

$$s(\omega_A) = s'(a)$$

Proof. — 1) Suppose that

$$\int_{\mathbb{R}} dx\, a(x^2) = +\infty$$

Then $s'(a) = +\infty$ and ω_A is not locally normal, but this implies $s(\omega_A) = +\infty$

2) Suppose that

$$\int_{\mathbb{R}} dx\, a(x^2) < +\infty$$

but $s'(a) = +\infty$. This case implies by an immediate application of the second part of Theorem 3.3 that $s(\omega_A) = +\infty$

3) Suppose that

$$\int_{\mathbb{R}} dx\, a(x^2) < +\infty \quad \text{and} \quad s'(a) < +\infty$$

Choose then a sequence $x \in \mathbb{R} \to a_n(x^2)$, $n \in \mathbb{N}$, satisfying:

i) $0 \leqslant a_n(x^2) \leqslant a(x^2)$ a. e.

ii) $x \to a_n(x^2)$ continuous with compact support

iii) $\lim_{n\to\infty} a_n(x^2) = a(x^2)$ a. e.

and let ω_{A_n} be the state corresponding to $a_n(\cdot)$.

Applying Theorem 3.3 and Lemma 4.2 one finds:

$$s(\omega_A) = \lim_{n\to\infty} s(\omega_{A_n}) = \lim_{n\to\infty} s'(a_n) = s'(a)$$

where the last equality follows from the Lebesgue dominated convergence theorem.

APPENDIX

LEMMA 5.1. — Suppose that

i) $x \in \mathbb{R} \to a(x) \in \mathbb{R}$ is continuous with compact support.

ii) $x \in \mathbb{R} \to g(l, x) \in \mathbb{R}$, $l > 0$ is a family of measurable functions satisfying:

— $\exists A > 0$ such that

$$\int_{\mathbb{R}} dx \mid g(l, x) \mid \leqslant A \qquad l > 0$$

— $\lim_{l \to \infty} \int_{\mathbb{R}} dx \; g(l, x) = 1$

— $\exists \alpha > 1$, $\beta > 0$ and $B > 0$ such that

$$\mid g(l, x) < B \quad \mid x \mid^{-\alpha} l^{-\beta} \; l > 0, \qquad x \in \mathbb{R}$$

iii) $x \in \mathbb{R} \to f(x) \in \mathbb{R}$ is continuous and $\exists \lambda > 0$ and $C > 0$ such that

$$\mid f(x) \mid \leqslant C \mid x \mid^{\gamma} \qquad x \in \mathbb{R}$$

Assume furthermore that $\gamma(\alpha - 1) < 1$ and that $\gamma\alpha > 1$, then:

$$\lim_{l \to \infty} \frac{1}{l} \sum_{n=-\infty}^{+\infty} f\left\{ \int_{\mathbb{R}} dx \, a\left(x + \frac{n}{l}\right) g(l, x) \right\} = \int_{\mathbb{R}} dx \, f \circ a(x) .$$

Proof. — 1) Choose $x_0 > 0$ such that $a(x) = 0$ for $\mid x \mid \geqslant x_0$

We show that

$$\lim_{l \to \infty} \frac{1}{l} \sum_{n=lx_0+1}^{\infty} f\left\{ \int_{\mathbb{R}} dx \, a\left(x + \frac{n}{l}\right) g(l, x) \right\} = 0$$

$$\left| \frac{1}{l} \sum_{n=lx_0+1}^{\infty} f\left\{ \int_{\mathbb{R}} dx \, a\left(x + \frac{n}{l}\right) g(l, x) \right\} \right|$$

$$\leqslant \frac{1}{l} \sum_{n=lx_0+1}^{\infty} \left| f\left\{ \int_{\mathbb{R}} dx \, a\left(x + \frac{n}{l}\right) g(l, x) \right\} \right|$$

$$\leqslant \frac{1}{l} \sum_{n=lx_0+1}^{\infty} C \left| \int_{\mathbb{R}} dx \, a\left(x + \frac{n}{l}\right) g(l, x) \right|^{\gamma}$$

$$\leqslant \frac{1}{l} \sum_{n=lx_0+1}^{\infty} C \, || a ||_{\infty}^{\gamma} \left| \int_{-x_0 - \frac{n}{l}}^{x_0 - \frac{n}{l}} dx \mid g(l, x) \right|^{\gamma}$$

$$\leqslant \frac{1}{l} \sum_{n=lx_0+1}^{\infty} C \, || a ||_{\infty}^{\gamma} B^{\gamma} l^{-\beta\gamma} \left| \int_{-x_0 - \frac{n}{l}}^{x_0 - \frac{n}{l}} dx \mid x \mid^{\alpha} \right|^{\gamma}$$

$$\leqslant \frac{1}{l} \sum_{n=lx_0+1}^{\gamma} \frac{C \, || a ||_{\infty}^{\gamma} B^{\gamma} l^{-\beta\gamma}}{(\alpha - 1)^{\gamma}} \left\{ \frac{1}{\left(\frac{n}{l} - x_0\right)^{\alpha - 1}} - \frac{1}{\left(\frac{n}{l} + x_0\right)^{\alpha - 1}} \right\}^{\gamma}$$

$$\leqslant \frac{1}{l} \sum_{n=lx_0+1}^{\infty} \frac{C \| a \|_\infty^\gamma B^\gamma l^{-\beta\gamma}}{(\alpha -)^\gamma} \left(\frac{n}{l}\right)^{-\gamma(\alpha-1)} \frac{x_0^\gamma \{ 2^{\alpha-1} + 2(\alpha-1) \}^\gamma}{\left(\frac{n}{l} - x_0\right)^{\gamma(\alpha-1)}} \left(\frac{n}{l}\right)^{-\gamma} \left(\frac{n}{l}\right)^{\gamma(\alpha-1)}$$

$$\leqslant \frac{C \| a \|_\infty^\gamma B^\gamma x_0^\gamma \{ 2^{\alpha-1} + 2(\alpha-1) \}^\gamma}{(\alpha-1)^\gamma} l^{-1-\beta\gamma} \sum_{n=lx_0+1}^{\infty} \left(\frac{n}{l}\right)^{-\gamma} \left(\frac{n}{l} - x_0\right)^{-\gamma(\alpha-1)}$$

$$\leqslant \frac{C \| a \|_\infty^\gamma B^\gamma x_0^\gamma \{ 2^{\alpha-1} + 2(\alpha-1) \}^\gamma}{(\alpha-1)^\gamma} l^{-1-\beta\gamma} \int_{lx_0}^{\infty} dx \left(\frac{x}{l}\right)^{-\gamma} \left(\frac{x}{l} - x_0\right)^{-\gamma(\alpha-1)}$$

$$= \frac{C \| a \|_\infty^\gamma B^\gamma x_0^\gamma \{ 2^{\alpha-1} + 2(\alpha-1) \}^\gamma}{(\alpha-1)^\gamma} l^{-\beta\gamma} \int_{x_0}^{\infty} dx\, x^{-\gamma}(x - x_0)^{-\gamma(\alpha-1)}$$

and this last expression tends to 0 als $l \to \infty$

2) One shows in the same way that

$$\lim_{l\to\infty} \frac{1}{l} \sum_{n=-\infty}^{-lx_0-1} f\left\{ \int_{\mathbb{R}} dx\, a\left(x + \frac{n}{l}\right) g(l, x) \right\} = 0$$

3) Computation of

$$\lim_{l\to\infty} \frac{1}{l} \sum_{n=-lx_0}^{lx_0} f\left\{ \int_{\mathbb{R}} dx\, a\left(x + \frac{n}{l}\right) g(l, x) \right\}$$

Choose $\varepsilon > 0$ then there is a $\delta > 0$ such that:

$$\left.\begin{array}{l} | x - y | < \delta \\ x, y \in \text{Range } a(\cdot) \end{array}\right\} \to | f(x) - f(y) | < \varepsilon$$

Choose now a $\mu > 0$ such that

$$| x - y | < \mu \to | a(x) - a(y) | < \frac{1}{2A} \delta$$

and divide $[-x_0, x_0]$ in intervals $I_k = \left[\frac{k\mu}{2}, (k+1)\frac{\mu}{2} \right]$

$$k = \frac{-2x_0}{\mu}, \ldots, \frac{2x_0}{\mu} - 1$$

For $\frac{n}{l}$, $y \in I_k$ one gets:

$$\left| \int_{\mathbb{R}} dx\, a\left(x + \frac{n}{l}\right) g(l, x) - a(y) \right| \leqslant$$

$$\left| \int_{-\mu/2}^{\mu/2} dx\, a\left(x + \frac{n}{l}\right) g(l, x) - a(y) \int_{-\mu/2}^{\mu/2} dx\, g(l, x) \right|$$

$$+ \left| \int_{\mathbb{R} - [\frac{-\mu}{2}, \frac{\mu}{2}]} dx\, a\left(x + \frac{n}{l}\right) g(l, x) \right| + \left| a(y) \int_{\mathbb{R} - [\frac{-\mu}{2}, \frac{\mu}{2}]} dx\, g(l, x) \right|$$

$$+ \left| a(y) \left\{ 1 - \int_{\mathbb{R}} dx\, g(l, x) \right\} \right|$$

$$\leqslant \frac{1}{2}\delta + \frac{2 \| a \|_\infty l^{-\beta} B 2^{2-\alpha}}{(1-\alpha)\mu^{1-\alpha}} + \| a \|_\infty \left| 1 - \int_{\mathbb{R}} dx\, g(l, x) \right| \leqslant \delta$$

for $l \geqslant l_0$

Then also:

$$\left| \int_{\mathbb{R}} dx\, f \circ a(x) - \frac{1}{l} \sum_{n=-lx_0}^{lx_0} f\left\{ \int_{\mathbb{R}} dx\, a\left(x + \frac{n}{l}\right) g(l, x) \right\} \right| \leqslant$$

$$\sum_{k=-\frac{2x_0}{\mu}}^{\frac{2x_0}{\mu}-1} \left| \int_{I_k} dx\, f \circ a(x) - \frac{1}{l} \sum_{\frac{n}{l} \in I_k} f\left\{ \int_{\mathbb{R}} dx\, a\left(x + \frac{n}{l}\right) g(l, x) \right\} \right| \leqslant \varepsilon$$

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