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Section A :
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Matter field space times admitting symmetry mappings satisfying vanishing contraction of the Lie deformation of the Ricci tensor

by

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ABSTRACT. — We examine some of the physical implications of symmetry mappings satisfying vanishing contraction of the Lie deformation of the Ricci tensor [*i. e.*, $g^{ij}\mathscr{L}_\xi R_{ij} = 0$ where $R_{ij} = \varkappa(T_{ij} - 1/2\, g_{ij}T)$] for matter field space-times. Also, the concomitant conservation law generator $\nabla_j[\sqrt{-g}\,(T_i^j - 1/2\,\delta_i^j T)\xi^i] = 0$ is investigated. Certain cases of these symmetry mappings representing symmetry properties are. developed as theorems with emphasis placed on cases where ξ^i is a timelike eigenvector of the given matter tensor. Several particular applications are given for perfect fluid and perfect magnetofluid space-times.

RÉSUMÉ. — Nous examinons quelques-unes des implications physiques des cartes de symétrie pour lesquelles la contraction de la déformation de Lie du tenseur de Ricci disparaît

$$[i.\,e.,\ g^{ij}\mathscr{L}_\xi R_{ij} = 0 \quad \text{où} \quad R_{ij} = \varkappa(T_{ij} - 1/2\, g_{ij}T)]$$

pour les espaces-temps des champs matériels. Nous recherchons aussi le générateur concomitant de la loi de conservation

$$\nabla_j[\sqrt{-g}\,(T_i^j - 1/2\delta_i^j T)\xi^i] = 0\,.$$

On développe certains cas de ces cartes de symétrie qui représentent des propriétés de symétrie avec insistance sur le cas où ξ^i est le vecteur propre orienté dans le temps du tenseur matériel donné. Nous donnons plusieurs applications particulières pour les fluides parfaits relativistes et pour la magnétohydrodynamique relativiste.

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1. INTRODUCTION

In this paper we wish to draw attention to the important role of certain symmetry mappings and the related conservation laws when they are admitted by matter field space-times [1] (MFS). In particular, we will be interested in investigating symmetry mappings which satisfy [2] $g^{ij}\mathcal{L}R_{ij} = 0$ (*i. e.*, members of the family of contracted Ricci collineations-FCRC) because they are directly related to the properties of the matter tensor and a particularly interesting and simple conservation law generator [3]

$$\nabla_j[\sqrt{-g}(T_i^j - 1/2\,\delta_i^j T)\xi^i] = 0\,. \tag{1.1}$$

It follows that this conservation law generator may be regarded as a generalization of the familiar Trautman [5] expression $\nabla_j(\sqrt{-g}\,T_i^j\xi^i) = 0$ which only holds, for general T_{ij}, if the ξ^i is a symmetry vector characterizing a group of motions (M) (isometries) admitted by the given space-time.

Some progress has already been made in the area of investigations of Riemannian space-times admitting Ricci collineations (RC) [2] [6]-[9], curvature collineations (CC) [6] [10]-[12] and other symmetry properties in the main symmetry chain of RC (see Symmetry Property Inclusion Diagram [4]). To date, these studies include results relating to conservation laws in particle mechanics in the form of m^{th}-order first integrals [13], the

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([1]) A matter field space-time (MFS) is defined to be a space-time with a matter tensor that has a unique unit timelike eigenvector $u^i(u^iu_i = 1$ with signature of metric-2) with positive eigenvalue ρ.

([2]) In accord with the notations and definitions used by J. A. Schouten [1], here and throughout this paper we use (*i*) ∇_k for the operation of covariant differentiation, (*ii*) $\mathcal{L}$ for the operation of Lie differentiation with respect to the vector ξ^i (unless otherwise noted) and (*iii*) round and square brackets on indices for the operations of symmetrization and antisymmetrization, respectively. Einstein's field equations $(R_{ij} - 1/2 g_{ij}R = \varkappa T_{ij})$ are assumed.

([3]) This family of symmetries was first introduced by Davis, Green and Norris [2] and further discussed by Oliver and Davis [3] and Green, Norris, Oliver and Davis [4]. More about the related conservation law generator and its history can be found in reference [2].

([4]) A more complete version of this diagram and other references relating to these symmetries can be found in the paper by Davis, Green and Norris [2].

extension of invariant classifications of geometries [*14*] and the development of expressions that are described as field conservation law generators ([5]).

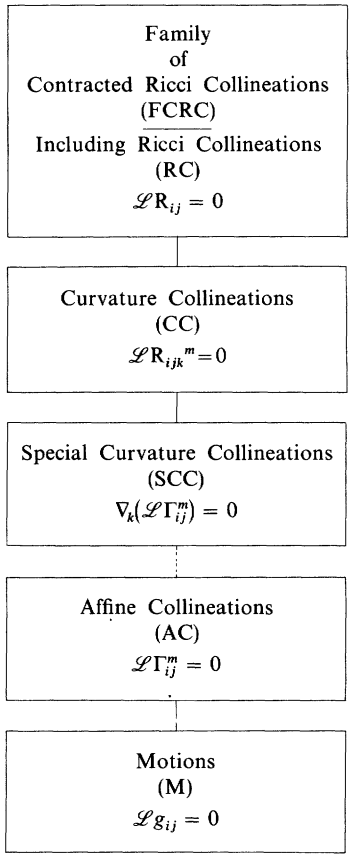

DIAGRAM 1. — Symmetry Property Inclusion Diagram.

([5]) By a conservation law generator we mean a local covariant conservation expression that involves the vanishing divergence of a vector density which is subject to a symmetry condition. Ordinarily this symmetry condition takes the form of a symmetry vector that satisfies certain symmetry conditions (*e. g.*, symmetry conditions given in the symmetry property inclusion diagram). It is sometimes useful to use « strong conservation law generator » to refer to a local « conservation expression » that holds as an identity independent of any symmetry condition.

Also in this connection it might be mentioned that among more general results Katzin-Levine-Davis [*6*] have shown for the case of Petrov type N space-times admitting members of FCRC which are CC that one can obtain conservation expressions of the form

$$\nabla_j(\sqrt{-g}\,\mu k^j) = 0 \qquad \text{with} \quad k_j k^j = 0\,.$$

These expressions are obtained by starting with a conservation law generator similar to (1.1) but based on the Bel-Robinson tensor rather than the matter tensor. For a general discussion of field and particle conservation laws see the survey article of W. R. Davis [*15*].

In this connection, it might be mentioned that Shaha [16] has obtained special results for magnetofluids that relate to groups of motions being admitted and also a very special result when an RC is admitted. In addition Glass [17] has obtained certain results relating to conservation laws for shear-free perfect fluids. These results have been extended and generalized by Oliver and Davis [3] using methods which utilize symmetry mappings.

As indicated earlier, we wish to investigate symmetry mappings satisfying $\mathscr{L}R_{ij} = H_{ij}(g^{ij}H_{ij} = 0)$ and the related field conservation expressions (1.1) in the context of Riemannian space-times corresponding to solutions of Einstein's field equations that can be characterized as MFS. Thus the matter tensor takes the form

$$T^{ij} = \rho u^i u^j - S^{ij} \tag{1.2}$$

where ρ, u^i, and S^{ij} $(S^{ij}u_j = 0)$ are the proper mass-energy density, unitary four-velocity and stress tensor of the fluid, respectively. The familiar « dynamical » and « conservation » equations follow from $\nabla_j T^{ij} = 0$ and take the form

$$\rho a_k = \gamma_{ik} \nabla_j S^{ij} \tag{1.3}$$

and

$$u^i \partial_i \rho + \{ \rho + (S/3) \} \theta + \sigma^{ij} S^T_{ij} = 0 \tag{1.4}$$

with $a_i = u^j \nabla_j u_i$ (acceleration), $\theta = \nabla_i u^i$ (expansion), $\gamma_{ij} = g_{ij} - u_i u_j$ (projection tensor), $S = \gamma_{ij} S^{ij}$ (trace of S^{ij}), $S^T_{ij} = S_{ij} - (1/3)\gamma_{ij} S$ (trace-free part of S^{ij}) and $\sigma_{ij} = \nabla_{(j} u_{i)} - a_{(i} u_{j)} - (1/3)\theta \gamma_{ij}$ (shear tensor). In Section 2 we proceed to investigate some general properties of FCRC members and the related conservation expressions. Special attention is given to the case where the symmetry vector is a timelike eigenvector of the matter tensor because of the intimate relation of this eigenvector to the source matter flow in the MFS. In Section 3 we examine several simple applications of FCRC symmetry properties and their particular physical consequences.

2. FCRC SYMMETRIES AND RELATED CONSERVATION EXPRESSIONS

In this section we consider symmetry mappings of the form

$$\mathscr{L}(T_{ij} - 1/2\, g_{ij} T) = H_{ij} \tag{2.1}$$

where H_{ij} is a symmetry trace-free tensor. First we investigate some of the properties of general FCRC symmetry mappings. For MFS, we give special consideration to cases where the symmetry vector is a timelike eigenvector of the matter tensor because of this direction's importance.

Finally, we will examine some of the FCRC related conservation expressions.

The specification of a particular FCRC symmetry property can be made by requiring H_{ij} in (2.1) to be sufficiently special to determine ξ^i to within constants. If ξ^i is only partially determined by H_{ij}, then the mapping could be referred to as a FCRC symmetry mapping or a FCRC quasi-symmetry property. In general FCRC symmetries will involve demands on the third derivatives of the metric which cannot be reduced to metrical demands of lower differential order. However, in certain cases it is possible to reduce or at least partially reduce the differential order of the demand. As an example of this we consider the case when $\xi^i = \lambda v^i$ is a timelike FCRC symmetry vector with the shear vanishing for the curve congruence associated with the vector field $v^i(x)$. Here, part of the FCRC symmetry demand can be re-expressed at the level of the first derivatives of the metric as $\mathscr{L} g_{ij} = 2\lambda[a_{(i}v_{j)} + (1/3)\theta\gamma_{ij}] + 2v_{(i}\partial_{j)}\lambda$. Other special FCRC symmetry mappings can also be specified totally at the level of the first derivatives of the metric or connection. Clearly motions and affine collineations (AC) are examples of such degenerate FCRC members.

For a MFS, FCRC symmetry and quasi-symmetry properties with symmetry vectors $\xi^i = \lambda u^i$ (u^i the unit timelike eigenvector of the matter tensor) are of particular interest because of their role in the determination of the timelike flow of the source matter. In order to examine these particular symmetry mappings in the case of MFS, we observe that $\mathscr{L}(T_{ij} - 1/2 g_{ij}T)$ with $\xi^i = \lambda u^i$ may be expressed in the form

$$(2.2)\quad \begin{aligned} \mathscr{L}(T_{ij} - 1/2 g_{ij}T) = {} & [A(u_i u_j - 1/4 g_{ij})] + [(\rho + S)u_{(i}\gamma^k_{j)}(\partial_k\lambda + \lambda a_k)] \\ & - \lambda[\mathscr{L}_u S^T_{ij} - (1/3)\gamma_{ij}(\gamma^{km}\mathscr{L}_u S^T_{km}) + \{\rho - (S/3)\}\sigma_{ij}] \\ & + 1/2\,[g_{ij}\nabla_k\{(\rho + S)\lambda u^k\}], \end{aligned}$$

with $A = (1/3)[\nabla_k\{(\rho + S)\lambda u^k\} + 2(\rho + S)(u^i\partial_i\lambda - \lambda\theta)]$. This particular decomposition has several interesting properties. First, each of the first three bracketed terms is trace-free and hence a possible simple choice for H_{ij} in (2.1). Also no sum of the bracketed terms can vanish without each individual term vanishing ([6]).

As an example of the use of this decomposition, we investigate the conditions for a RC to be admitted with symmetry vector $\xi^i = \lambda u^i$.

THEOREM 2.1. — *A* MFS *which satisfies* (i)

$$\mathscr{L}_u S^T_{ij} - (1/3)\gamma_{ij}(\gamma^{km}\mathscr{L}_u S^T_{km}) + \{\rho - (S/3)\}\sigma_{ij} = 0$$

and (ii) *either* $\rho + S = 0$ *or* $(\partial_i\lambda)/\lambda = -a_i + \theta u_i$ *and* $\nabla_k[(\rho + S)\lambda u^k] = 0$

([6]) This can be shown by letting A_{ij} be the sum of bracketed terms which is assumed to vanish. Then by taking the following contractions $g^{ij}A_{ij} = 0$, $u^i\gamma^j_k A_{ij} = 0$, $\gamma^i_k\gamma^j_l A_{ij} = 0$ and $u^i u^j A_{ij} = 0$ one can show that each individual bracketed term vanishes.

for some function λ *admits a* RC *with symmetry vector* $\xi^i = \lambda u^i$. *The converse of this theorem is also true.*

Proof. — The proof of this theorem follows simply from the decomposition (2.2) and the comments relating to footnote 6.

For the special case when the given MFS is a perfect fluid ($S^T_{ij} = 0$ and $S = 3p$), we obtain the following corollary.

COROLLARY 2.1. — *A perfect fluid admits a* RC *with symmetry vector* $\xi^i = \lambda u^i$ *if and only if* (i) *either* $\rho = p$ *or* $\sigma_{ij} = 0$ *and* (ii) *either* $\rho + 3p = 0$ *or* $(\partial_i \lambda)/\lambda = - a_i + \theta u_i$ *and* $\nabla_k \{ \rho + 3p) \lambda u^k \} = 0$.

The above corollary shows that vanishing shear, in general, underlies this particular perfect fluid symmetry. If the fluid is a dust, then Theorem 2.1 specializes to the following corollary.

COROLLARY 2.2. — *If a dust solution admits a* RC *with symmetry vector* $\xi^i = \lambda u^i$, *then this symmetry is a translation* (7).

Proof. — For a dust solution one has $S_{ij} = 0$. Using this in (1.3) and (1.4) one finds that $\nabla_k(\rho u^k) = 0$ and $a^i = 0$. For this case, Theorem 2.1 gives $\sigma_{ij} = 0$, $\partial_i \lambda = \theta \lambda u_i$ and $\nabla_k(\rho \lambda u^k) = 0$. Combining the above we get $\partial_i \lambda = 0$ and $\theta = 0$. Therefore since $\sigma_{ij} = \theta = a_i = 0$, $\mathscr{L} g_{ij} = 0$ with $\xi^i = \lambda u^i$ and $\xi^i \xi_i = \lambda^2$ is a constant. Thus ξ^i is a symmetry vector corresponding to a translation.

As noted in the introduction, the existence of a symmetry mapping satisfying $g^{ij} \mathscr{L} R_{ij} = 0$ implies the conservation expression (1.1). In addition, there exists another conservation expression related to FCRC symmetry mappings.

THEOREM 2.2. — *For a* FCRC *symmetry mapping with symmetry vector* ξ^i, *the following conservation expression holds*

$$\nabla_j [\sqrt{-g}(1/2\, \mathscr{L} g^{ij} + g^{ij} \nabla_m \xi^m)] = 0 \,. \tag{2.3}$$

Proof. — The proof follows from the identity

$$\nabla_i \nabla^{[j} \xi^{i]} - \nabla_i (1/2\, \mathscr{L} g^{ij} + g^{ij} \nabla_m \xi^m) \equiv R^j_m \xi^m \,,$$

Einstein's equations and (1.1).

Although each symmetry mapping $\mathscr{L}(T_{ij} - 1/2\, g_{ij} T) = H_{ij}$ with $g^{ij} H_{ij} = 0$ leads to the same formal conservation expressions (1.1) and (2.3), the particular choice of H_{ij} will make (1.1) and (2.3) take a different form for each distinct symmetry. For MFS, the conservation expression (1.1) is of special interest when $\xi^i = \lambda u^i$. Then (1.1) simplifies to

$$\nabla_i(\sqrt{-g}\, n u^i) = 0 \tag{2.4}$$

(7) A translation [18] is a motion ($\mathscr{L} g_{ij} = 0$) with $\xi^i \xi_i$ a constant.

with $n = \lambda(\rho - 1/2\ T)$. This expression can be interpreted in terms of « particle number » conservation of the source matter in the MFS (8) (9) [2] [4].

3. APPLICATIONS

In this section we present some simple examples which illustrate the properties of particular FCRC members. We will look at some of the physical properties of several different types of space-times which could admit a RC with symmetry vector $\xi^i = \lambda u^i$. We begin by showing the connection between S^T_{ij} and σ_{ij} when such a RC is admitted by a MFS. Next we examine the rotation when the MFS degenerates to a perfect fluid. Finally we look at a perfect magnetofluid and investigate the shear of the fluid.

The trace-free part of the stress tensor (S^T_{ij}) and the shear tensor (σ_{ij}) are somewhat related in all MFS in that they enter equation (1.4) in the form $\sigma^{ij}S^T_{ij}$. This connection is clearer if one rewrites $\sigma^{ij}S^T_{ij} = 1/2\ g^{ij}\mathscr{L}_u S^T_{ij}$ which shows that if the shear vanishes then $g^{ij}\mathscr{L}_u S^T_{ij}$ must also vanish. For a MFS admitting a RC with symmetry vector $\xi^i = \lambda u^i$, the relation between shear and the trace-free part of the stress tensor is even stronger.

THEOREM 3.1. — *For a* MFS *admitting a* RC *with symmetry vector* $\xi^i = \lambda u^i$, $\mathscr{L}_u S^T_{ij} = 0$ *if and only if either* (i) $\sigma_{ij} = 0$ *or* (ii) $\rho = S/3$.

Proof. — Assume $\mathscr{L}_u S^T_{ij} = 0$. Then (*i*) of Theorem 2.1 shows that either $\sigma_{ij} = 0$ or $\rho = S/3$. Now rewrite (*i*) of Theorem 2.1 as

$$\mathscr{L}_u S^T_{ij} + \{ \rho - (S/3) \} \sigma_{ij} - (2/3)\gamma_{ij}\sigma^{kl}S^T_{kl} = 0 . \tag{3.1}$$

If $\sigma_{ij} = 0$, then $\mathscr{L}_u S^T_{ij} = 0$ by (3.1). Now assume $\rho = S/3$. By combining both parts of (*ii*) of Theorem 2.1, we get $u^i\partial_i\rho + 2\rho\theta = 0$. Putting this into (1.4) we get $\sigma^{kl}S^T_{kl} = 0$. Using these results in (3.1) we find that $\mathscr{L}_u S^T_{ij} = 0$.

If the MFS admitting a RC with symmetry vector $\xi^i = \lambda u^i$ is also a perfect fluid, then we can obtain the following result which relates to the rotation of the fluid.

(8) For example see Green, Norris, Oliver and Davis [4] where FCRC symmetry mappings are examined in the context of Robertson-Walker models. These models have first fundamental form $\Phi = (dx^0)^2 - \mathscr{F}^2(x^0)\{ dr^2/(1 - kr^2) + r^2[d\theta^2 + \sin^2\theta d\varphi^2] \}$. It is shown that in this case (2.4) takes the form $\nabla_i[\sqrt{-g}\,(f/\mathscr{F}^3)u^i] = 0$ where f is an arbitrary function of r, θ and φ. This expression has been interpreted as particle number conservation.

(9) The conservation expression given in footnote 8 has been obtained from other considerations in Weinberg [19] where it is indicated that baryon number conservation takes this form.

THEOREM 3.2. — *If a perfect fluid* ($\rho \neq p$ *and* $\rho + 3p \neq 0$) *admits a* RC *with symmetry vector* $\xi^i = \lambda u^i$, *then* $u^i\partial_i\omega^2 = (2/3)\theta\omega^2$ *where* $\omega_{ij} = \nabla_{[j}u_{i]} - a_{[i}u_{j]}$ *and* $2\omega^2 = \omega_{ij}\omega^{ij}$.

Proof. — It can be shown that [10]

$$u^i\partial_i\omega^2 + (4/3)\theta\omega^2 - 2\omega_{ij}\sigma_k{}^j\omega^{ki} = \nabla_j a_i\omega^{ij}. \tag{3.2}$$

Using Corollary 2.1 we see that $\sigma_{ij} = 0$ and $\omega^{ij}\nabla_j a_i = 2\theta\omega^2$. Therefore (3.2) becomes $u^i\partial_i\omega^2 = (2/3)\theta\omega^2$.

COROLLARY 3.1. — *If a* MFS *admits a motion with symmetry vector* $\xi^i = \lambda u^i$, *then* $u^i\partial_i\omega^2 = 0$.

Proof. — If $\mathcal{L} g_{ij} = 0$ for $\xi^i = \lambda u^i$, then $\sigma_{ij} = \theta = 0$ and $(\partial_i\lambda)/\lambda = -a_i$. Using these results in (3.2), one finds that $u^i\partial_i\omega^2 = 0$.

Finally we examine the case of a perfect magnetofluid admitting a RC with symmetry vector $\xi^i = \lambda u^i$. For a perfect magnetofluid [11] the matter tensor is $T_{ij} = (\rho + 1/2\ \mu h^2)u_iu_j - (p + 1/2\ \mu h^2)\gamma_{ij} - \mu h_ih_j$ and Maxwell's equations are $\nabla_j(u^ih^j - u^jh^i) = 0$.

THEOREM 3.3. — *If a perfect magnetofluid* ($\rho - p - \mu h^2 \neq 0$) *admits a* RC *with symmetry vector* $\xi^i = \lambda u^i$, *then* h^i *is an eigenvector of* σ_{ij}.

Proof. — By rewriting (*i*) of Theorem 2.1 for perfect magnetofluids one finds that $\mathcal{L}_u S^T_{ij} + [\rho - p + (1/3)\mu h^2)]\sigma_{ij} - (2/3)\mu\gamma_{ij}\sigma_{kl}h^kh^l = 0$. Using Maxwell's equations and the form of S^T_{ij} one can rewrite this as

$$2h^k(\sigma_{ik}h_j + \sigma_{jk}h_i) - (2/3)\theta h_ih_j - (1/3)\gamma_{ij}\{(2/3)\theta h^2 + 4\sigma_{kl}h^kh^l\} + \frac{1}{\mu}(\rho - p + \mu h^2)\sigma_{ij} = 0.$$

Contracting this equation with h^i gives

$$\frac{1}{\mu}(\rho - p - \mu h^2)\sigma_{ij}h^i = -\{(2/3)\sigma_{kl}h^kh^l + (4/9)\theta h^2\}h_j.$$

Therefore h^i is an eigenvector of σ_{ij} if $\rho - p - \mu h^2 \neq 0$.

The examples given in this section serve to illustrate some of the types of physical information that follow when it is known that a given space-time admits a particular FCRC member. However, we have in no way attempted to systematically examine in depth all the consequences of each type of FCRC symmetry that has been mentioned. Further extensions of this work could be to examine the particular examples given in Section 3

(10) See, for example, Greenberg [*20*]. The difference in the two expressions arises from a difference in signature and in the definition of the expansion.

(11) See, for example, Lichnerowicz [*21*].

more completely as well as to look at other FCRC members. Also, one could proceed to a systematic invariant classification of particular FCRC members along the lines of the work of Davis, Green and Norris [2] for RC. These results could also be exploited for matter tensors in flat space-time ([12]) since $g^{ij}\mathscr{L}(T_{ij} - 1/2\ g_{ij}T) = 0$ and $\nabla_j T^{ij} = 0$ imply

$$\nabla_j[\sqrt{-g}(T_i^j - 1/2\ \delta_i^j T)\xi^i] = 0\,.$$

This would be of particular interest in certain studies relating to special relativistic hydrodynamics and plasma physics.

([12]) Here (without Einstein's field equations) the relevant symmetries are defined by the matter tensor itself which has no specific given relation to the symmetries that exist in flat space-time. Clearly, the symmetries based on the curvature tensor (e. g., CC and RC) are no longer defined. Nonetheless, beyond SCC$\left(\nabla_k\mathscr{L}\begin{Bmatrix} m \\ ij \end{Bmatrix} = 0\right)$ in curved or flat space-time there are indefinitely many other symmetries that could be of interest

$$[\text{e. g., } \nabla_{k_1} \ldots \nabla_{k_m}(\mathscr{L} g_{ij}) = H_{ijk_1 \ldots k_m} \quad \text{where} \quad H_{ijk_1 \ldots k_m}$$

are special tensors that do not satisfy these relations by definition]. The general symmetry vector η^i representing an SCC in flat space-time (N = 4) involves 60 parameters (see, G. H. Katzin and J. Levine [22]); however, the SCC does not form a 60 parameter group. The SCC admitted by flat space-time includes conformal collineations (24-parameters) and projective collineations (24-parameters) which, of course, includes the Poincaré group. In connection with the above considerations in flat space-time, see reference [2].

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