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Spinor-type fields with linear, affine and general coordinate transformations

by

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ABSTRACT. — We demonstrate the existence of bivalued linear (infinite) spinorial representations of the Group of General Coordinate Transformations. We discuss the topology of the G. G. C. T. and its subgroups $GA(nR)$, $GL(n, R)$, $SL(nR)$ for $n = 2, 3, 4$, and the existence of a double covering.

We demonstrate the construction of the half-integer spin representations in terms of Harish-Chandra modules. We give D. W. Joseph's explicit matrices for $j_0 = \frac{1}{2}$, $c = 0$ in $SL(3R)$, which will act as little group in $GA(4R)$.

1. INTRODUCTION AND RESULTS

Einstein's Principle of General Covariance imposes two constraints on the equations of Physics in the presence of gravitational fields :

a) a smooth transition to the equations of Special Relativity ; note that we require a formulation of the Equivalence Principle in Field Theory [1]. Operationally, « locally, the properties of « special-relativistic » matter in a non-inertial frame of reference cannot be distinguished from the properties of the same matter in a corresponding gravitational field [2] ».

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b) under a general coordinate transformation $x^\mu \rightarrow \bar{x}^\mu$, the equations are general-covariant, i. e. form preserving.

This article relates to (b), i. e. it deals with representations of the Covariance Group \mathcal{C} , also known as the Group of General Coordinate Transformations (CGT) or in Mathematical language, the Group of Diffeomorphisms. This group has as a subgroup \mathcal{G} the General Linear Group $GL(4\mathbb{R})$; both are defined over a 4-dimensional real manifold L_4 . We prove that in addition to conventional tensors (namely tensorial representations of \mathcal{G} and \mathcal{C}) there exist bivalued linear spinorial representations of these groups, reducing to bivalued representations of the Poincaré group \mathcal{P} . These « new » representations are *infinite-dimensional and of discrete type*; e. g. for time-like momenta, they reduce to an infinite sum of « ordinary » \mathcal{P} spinors (e. g. $\frac{1}{2} \oplus \frac{5}{2} \oplus \frac{9}{2} \oplus \dots$) thus somewhat resembling a band of rotational excitations over a half-integer spin deformed nucleus. We shall therefore use the term band-spinor (or « bandor ») for these infinite spinor representations, so as to distinguish them from (finite) conventional spinors.

Historically, spinors were « fitted » into General Relativity [3] [4] after their incorporation into Special Relativity through the Dirac equation. It was noted that they behaved like (holonomic, or « world ») scalars under \mathcal{C} , their spinorial behavior corresponding only to the action of a physically distinct local Lorentz group \mathcal{L}_E , with

$$\mathcal{C} \cap \mathcal{L}_E = 0 \quad (1.1)$$

Both \mathcal{L}_E and conventional spinors thus required the introduction of a Bundle of Cotangent (or Tangent) Frames E, i. e. an orthonormal set of 1-forms (« vierbeins »; $a = 0, 1, \dots, 3$ the « anholonomic » indices)

$$\left. \begin{aligned} e^a &= e^a_\mu(x) dx^\mu \\ e_a &= \eta_{ab} e^b \\ \eta_{ab} &= \text{the Minkowski metric} \end{aligned} \right\} \quad (1.2)$$

with the L_4 (general affine) or U_4 (Riemann-Cartan) manifold metric given by

$$e^a_\mu \eta_{ab} e^b_\nu = g_{\mu\nu} \quad (1.3)$$

Band-spinors are « world » spinors, and thus do not require E for their definition. Contrary to what is stated in most texts on General Relativity, the introduction of E should indeed not be construed as resulting just from the world-scalar behavior of spinors. E represents a further geometrical construction corresponding to the physical constraints of a local gauge group of the Yang-Mills type, in which the gauged group is the isotropy group of the space-time base manifold. We can thus even introduce band-spinors in the vierbein system [5]: the isotropy group would then

have to be enlarged from \mathcal{L}_E to \mathcal{G}_E , i. e. the theory would have to realize a global General Affine (\mathcal{F}) symmetry as its starting point,

$$\mathcal{F} = \mathcal{G} \times \mathcal{I}, \quad \mathcal{F} \supset \mathcal{P} \tag{1.4}$$

where \mathcal{I} represents the translations. The quotient of \mathcal{F} by \mathcal{I} is \mathcal{G}_E (i. e. \mathcal{G} acting on the anholonomic indices).

Note that one source of the prevalent belief that there are no \mathcal{C} spinors (« world » spinors) stems from an unwarranted extrapolation from a theorem of E. Cartan [6]:

« It is impossible to introduce spinor fields, the term « spinor » being taken in the classical Riemannian connotations; i. e., given an arbitrary coordinate system x^μ , it is impossible to represent a spinor by any finite number N of components u_α , so that these should admit covariant derivatives of the form (α, β are spinor indices, $\mu\nu$ are vector indices)

$$D_\mu u_\alpha = \partial_\mu u_\alpha + \Gamma_{\mu\alpha}^\beta(x)u_\beta \tag{1.5}$$

with the $\Gamma_{\mu\alpha}^\beta$ as specific functions of x ».

As can be seen, Cartan was aware of the restriction of his proof to a finite number of spinor components. Our band-spinors $\Psi^{\bar{\alpha}}$ indeed do admit covariant derivatives as in (1.5), in the « world » (holonomic) system,

$$D_\mu \Psi^{\bar{\alpha}} = \partial_\mu \Psi^{\bar{\alpha}} + \Gamma_{\mu\nu}^{\tau} (G_{\tau\bar{\beta}}^{\nu}) \bar{\Psi}^{\bar{\beta}} \tag{1.6}$$

where $\bar{\alpha}$ runs over the sets $\alpha, \alpha \times \lambda, \alpha \times \lambda\rho\sigma, \dots$, i. e. spins

$$j = \frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \dots,$$

G_{τ}^{ν} is an infinite dimensional representation of \mathcal{G} , and $\Gamma_{\mu\nu}^{\tau}$ is the usual affine connection.

2. TOPOLOGICAL CONSIDERATIONS : THE COVERING GROUP OF $SL(nR)$ AND $GL(nR)$

We are studying the groups,

$$\mathcal{C} \supset \mathcal{F} \supset \mathcal{G} \supset \mathcal{S} \supset \mathcal{O} \tag{2.1}$$

$$\mathcal{C} \supset \mathcal{F} \supset \mathcal{P} \supset \mathcal{O} \tag{2.2}$$

where \mathcal{S} is the Unimodular Linear $SL(4R)$ and \mathcal{O} is the Special Orthogonal $SO(4)$. We do not enter into the further structure induced by the Minkowski metric at this stage. At various stages we shall also deal with the same groups over $n = 3$ and $n = 2$; we shall then use the notation $\mathcal{G}_3, \mathcal{S}_3$, etc.

Since our aim is to find unitary representations of $\mathcal{C}, \mathcal{F}, \mathcal{G}, \mathcal{S}$ which

reduce to bivalued unitary representations of \mathcal{P} and \mathcal{O} , we have *a priori* two candidate solutions :

$$\begin{aligned}
 (a) \quad & \mathcal{S} \supset \bar{\mathcal{O}}, \quad \mathcal{S}_3 \supset \bar{\mathcal{O}}_3, \quad \mathcal{F} \supset \bar{\mathcal{P}} \\
 (b) \quad & \mathcal{C} \supset \bar{\mathcal{F}} \supset \bar{\mathcal{G}} \supset \bar{\mathcal{P}} \supset \bar{\mathcal{O}} \\
 & \quad \quad \quad \bar{\mathcal{F}} \supset \bar{\mathcal{P}}
 \end{aligned}$$

where the bars denote double-covering of the relevant groups. In the first case, we would be dealing with single-valued representations of \mathcal{C} and its subgroups, and \mathcal{O} would be contained through its covering $\bar{\mathcal{O}}$. In the second case, all groups would display the same bivaluedness as \mathcal{O} , and we would have to go to their respective coverings to find a single-valued representation containing $\bar{\mathcal{O}}$. Since

$$\bar{\mathcal{O}}_3 = \text{SU}(2)$$

it is enough that we show that $\mathcal{S}_3 \not\supset \text{SU}(2)$ to cancel solution (a).

We introduce an Iwasawa decomposition [7] of \mathcal{S} . For a non compact real simple (all invariant subgroups are discrete and in the center) Lie group \mathcal{B} , it is always possible to find

$$\mathcal{B} = \mathcal{K} \cdot \mathcal{A} \cdot \mathcal{N} \tag{2.3}$$

where \mathcal{K} is the maximal compact subgroup, \mathcal{A} is a maximal Abelian subgroup homeomorphic to that of a vector space, \mathcal{N} is a nilpotent subgroup isomorphic to a group of triangular matrices with the identity in the diagonal and zeros everywhere below it. The decomposition is unique and holds globally

$$\mathcal{K} \cap \mathcal{A} = \mathcal{A} \cap \mathcal{N} = \mathcal{N} \cap \mathcal{K} = \{ 1 \}. \tag{2.4}$$

Applying (2.3) to \mathcal{S}_3 , \mathcal{K} is \mathcal{O}_3 . Since this is maximal and unique,

$$\mathcal{S}_3 \not\supset \bar{\mathcal{O}}_3$$

and we are left with solution – b) only. Applying (2.3) to \mathcal{S} ,

$$\mathcal{S} = \mathcal{O} \cdot \mathcal{A}_s \cdot \mathcal{N}_s \tag{2.5}$$

we also have

$$\bar{\mathcal{S}} = \bar{\mathcal{O}} \cdot \bar{\mathcal{A}}_s \cdot \bar{\mathcal{N}}_s \tag{2.6}$$

Now the groups \mathcal{A} and \mathcal{N} in an Iwasawa decomposition are simply connected, and $\bar{\mathcal{A}}\bar{\mathcal{N}} = \bar{\mathcal{A}}\bar{\mathcal{N}}$ is contractible to a point. Thus, *the topology of $\bar{\mathcal{S}}$ is that of $\bar{\mathcal{O}}$* . The same result has been shown to hold [8] for \mathcal{C} when the L_4 is Euclidean or Spherical and holds under some weak conditions for any L_4 .

By the same token, $\bar{\mathcal{G}}$ has the topology of $\overline{\mathcal{O}(n, \mathbb{R})}$, the double covering of the full Orthogonal (which includes the improper orthogonal matrices, with $\det = - 1$). \mathcal{G} and $\bar{\mathcal{G}}$ thus have two connected components.

For $n \geq 3$, \mathcal{S} is thus completely covered by $\bar{\mathcal{F}}$, the double-covering. However, $O(2)$ and $SL(2R)$ are infinitely connected.

$$\bar{\mathcal{F}}_2 < \bar{\mathcal{F}}_2 \tag{2.7}$$

where $\bar{\mathcal{F}}_2$ is the full covering.

Topologically, solution (b) is thus realizable. The single-valued unitary (and thus infinite-dimensional) irreducible representations of \mathcal{S} correspond to double-valued representations of \mathcal{S} and reduce to a sum of double valued representations of \mathcal{O} .

This being established, it is interesting to check a second source of confusion at the origin of the statements found in the literature of General Relativity and denying the existence of such double-valued representations. This is based upon an error in the statement of a theorem of E. Cartan [9]:

« The three linear unimodular groups of transformations over 2 variables ($SL(2C)$, $SU(2)$, $SL(2R)$) admit no linear many-valued representation ».

As can be seen from Cartan's proof of this theorem in ref. [9], it holds only for $SL(2C)$ and $SU(2)$. Moreover, Bargmann [10] has constructed the unitary representations of $SL(2R)$, since this is the double covering $Spin(3)_{(+--)}$ of the 3-Lorentz group $(1, -1, -1)$; and even though only single-valued representations of $SL(2R)$ are required for this role, he has also constructed (§ 7 d) multivalued linear representations of that group. The representations

$$C_q^h, h = \frac{1}{4}, q = \frac{1}{4} + s^2 \tag{2.8}$$

are bivalued representations of $Spin(3)_{(+--)} = SL(2R)$ as can be derived from Bargmann's formula

$$U(b) = \exp(4ilh\pi)U(a) \tag{2.9}$$

for two elements lying over the same element of $SL(2R)$. We take $l = 1$.

Note that in reducing $SL(4R)$ to $SL(2R)$, the generators are represented on the coordinates (holonomic variables) by,

$$\begin{aligned} \Sigma_1 &= \frac{i}{2}(x_1\partial_1 - x_2\partial_2), & \Sigma_2 &= \frac{i}{2}(x_1\partial_2 + x_2\partial_1), \\ & & \Sigma_3 &= -\frac{i}{2}(x_1\partial_2 - x_2\partial_1) \end{aligned} \tag{2.10}$$

with commutation relations

$$[\Sigma_1, \Sigma_2] = -i\Sigma_3, \quad [\Sigma_3, \Sigma_1] = i\Sigma_2, \quad [\Sigma_2, \Sigma_3] = i\Sigma_1 \tag{2.11}$$

with Σ_3 generating the compact subalgebra (eigenvalues m in ref. [10]). However, when using the same algebra as the double-covering [10] of $SO(1, 2)$, the identification in terms of the (completely different) $(1, -1, -1)$ space is given by,

$$\begin{aligned} \Sigma_1 &= i(x'_0\partial'_1 + x'_1\partial'_0), & \Sigma_2 &= -i(x'_2\partial'_0 + x'_0\partial'_2), \\ & & \Sigma_3 &= i(x'_1\partial'_2 - x'_2\partial'_1) \end{aligned} \tag{2.12}$$

with the same commutators and the same role for Σ_3 . We stress this correspondence because it has led to some additional confusion and arguments [11] against the existence of bivalued representations of \mathcal{S}_2 , and with it \mathcal{S}_4 .

3. THE $\overline{\text{SL}(3\mathbb{R})}$ BAND-SPINORS : EXISTENCE

The unitary infinite-dimensional representations of $\text{SL}(3\mathbb{R})$ were introduced [12] in the context of an algebraic description of hadron rotational excitations (« Regge trajectories [13] »). A construction was provided (« ladder representations) for the multiplicity-free $|\Delta j| = 2$ bands, where j is the \mathcal{O}_3 spin. Such representations are characterized by j_0 (the lowest j) and c , a real number,

$$\mathcal{D}(\mathcal{S}_3; j_0, c) \quad (3.1)$$

the ladder representations corresponding to $j_0 = 0$ and $j_0 = 1$.

We shall not dwell here upon the physical context of shear stresses in extended structures, connected with ref. [12], and we refer the reader to the first part of ref. [5], for that purpose. However, it was a result of this physical context that the author noted with D. W. Joseph the possible existence of similar bivalued representations, i. e. band-spinors. Joseph provided [14] a construction for $\mathcal{D}\left(\mathcal{P}_3; \frac{1}{2}, 0\right)$ and proved that together with the subsets $\mathcal{D}(\mathcal{S}_3; 1, c)$, $\mathcal{D}(\mathcal{S}_3; 0, c)$, $-\infty < c < \infty$, this formed the entire set of $|\Delta j| = 2$ multiplicity-free representations. The latter result was recently confirmed by Ogievetsky and Sokachev [15], after having been put in question [16].

We shall provide here a different construction, based upon the « subquotient » theorem for Harish-Chandra modules [17]. We return to the Iwasawa decomposition (2.6) for \mathcal{P}_3

$$\mathcal{P}_3 = \overline{\mathcal{O}}_3 \mathcal{A} \mathcal{N} \quad (3.2)$$

and define \mathcal{M}_3 , the Centralizer of \mathcal{A} in \mathcal{K} , i. e. in \mathcal{O}_3 . This is the set of all $\sigma \in \mathcal{O}_3$ such that

$$(\sigma \in \mathcal{M}_3 \mid \sigma a \sigma^{-1} = a) \quad (3.3)$$

for any $a \in \mathcal{A}$. The elements of \mathcal{A} span a 3-vector space, and \mathcal{M}_3 thus has to be in the diagonal. Since $\det(\mathcal{M}_3) = 1$, the elements of \mathcal{O}_3 belonging to \mathcal{M}_3 are the inversions in the 3 planes: $(+1, -1, -1)$, $(-1, +1, -1)$ and $(-1, -1, +1)$. Together with the identity element they form a group of order 4, with a multiplication table $m_1 m_2 = m_3$, $m_2 m_3 = m_1$, $m_3 m_1 = m_2$, $m_n^2 = 1$. It appears Abelian in this representation.

Returning now to \mathcal{P}_3 and $\overline{\mathcal{O}}_3$, we look for $\overline{\mathcal{M}}_3 \subset \overline{\mathcal{O}}_3$. The inversions are given in $\text{SU}(2)$ by $\exp(i\pi\sigma_n/2)$, which yields the Non-Abelian group

$$\overline{\mathcal{M}}_3 : (\pm i\sigma_n, \pm 1). \quad (3.4)$$

The subgroup $\bar{\mathcal{D}}_3 \subset \bar{\mathcal{P}}_3$

$$\bar{\mathcal{D}}_3 = \bar{\mathcal{M}}_3 \mathcal{A} \mathcal{N}$$

can now be used to induce the representations of $\bar{\mathcal{P}}_3$. Note that

$$\bar{\mathcal{P}}_3 / \bar{\mathcal{D}}_3 = \text{SU}(2) / \bar{\mathcal{M}}_3. \tag{3.5}$$

The representations $\rho(j_0, c)$ of $\bar{\mathcal{D}}_3$ are given by j_0 for a representation of the $\bar{\mathcal{M}}_3$ group of « plane inversions » in $\text{SU}(2)$, and λ for the characters of \mathcal{A} , since \mathcal{N} is represented trivially. The representations of $\bar{\mathcal{P}}_3$ will thus be labelled accordingly; from (3.5) we see that they will be spin-valued representations of $\bar{\mathcal{D}}_3$. Since $\bar{\mathcal{A}}\bar{\mathcal{N}} = \mathcal{A}\mathcal{N}$, univalence is guaranteed.

4. THE $\overline{\text{SL}(3\mathbb{R})}$ BAND-SPINORS : CONSTRUCTION OF $\mathcal{D} \left(\frac{1}{2}, 0 \right)$

Following our original introduction [12] of infinite-dimensional single-valued representations of $\text{SL}(3\mathbb{R})$, we now turn to our algebraic point of view. The five non-compact generators of \mathcal{S}_3 are isomorphic to a multiplication of the symmetric λ matrices of $\mathfrak{su}(3)$ by $\sqrt{-1}$, and behave like a $j = 2$ representation under the compact \mathcal{O}_3 (the antisymmetric λ matrices). They can thus mediate transitions between $|\Delta j| = 2, 1, 0$ levels of the compact subalgebra. In the following analysis we shall deal with a highly degenerate subset: the multiplicity-free $|\Delta j| = 2$ representations.

Although several treatments have appeared since [15] [16] we choose to reproduce the results of D. W. Joseph's unpublished 1970 work [14].

In Joseph's notation, the \mathcal{S}_3 generators are chosen to be :

$$\hat{h} := 2 \begin{vmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{vmatrix}, \quad \hat{e}_{\pm} := \begin{vmatrix} 0 & -i & 0 \\ i & 0 & \pm 1 \\ 0 & \mp 1 & 0 \end{vmatrix}, \quad \hat{f} := \sqrt{\frac{2}{3}} \begin{vmatrix} i & 0 & 0 \\ 0 & -2i & 0 \\ 0 & 0 & i \end{vmatrix} \tag{4.1}$$

$$\hat{f}_{\pm} := \begin{vmatrix} 0 & -1 & 0 \\ -1 & 0 & \pm i \\ 0 & \pm i & 0 \end{vmatrix}, \quad \hat{f}_{\pm\pm} := \begin{vmatrix} i & 0 & \pm 1 \\ 0 & 0 & 0 \\ \pm 1 & 0 & -i \end{vmatrix} \tag{4.1}$$

and using Capital letters for their Unitary Representations,

$$H^+ = H, \quad E^+ = E_-, \quad F^+ = F, \quad F^+_{\pm} = -F_-, \quad F^+_{\pm\pm} = F_{--} \tag{4.2}$$

the $\text{SL}(3\mathbb{R})$ matrices are produced by

$$\left. \begin{aligned} s &= e^{i\alpha} \\ \alpha &= \xi \hat{h} + \xi_+ e_+ + \xi_- e_- + \xi f + \xi_+ f_+ + \xi_- f_- + \xi_{++} f_{++} + \xi_{--} f_{--} \\ \xi^* &= \xi, \quad \xi^*_+ = \xi_-, \quad \xi^*_- = \xi, \quad \xi^*_{++} = -\xi_-, \quad \xi^*_{--} = \xi_{--} \end{aligned} \right\} \tag{4.3}$$

The commutation relations are,

$$\left. \begin{aligned}
 [\hat{h}, \hat{e}_{\pm}] &= \pm 2\hat{e}_{\pm}, \quad [\hat{e}_{+}, \hat{e}_{-}] = \hat{h} \\
 [\hat{h}, \hat{f}_{\pm\pm}] &= \pm 4\hat{f}_{\pm\pm}, \quad [\hat{h}, \hat{f}_{\pm}] = \pm 2\hat{f}_{\pm}, \quad [\hat{h}, \hat{f}] = 0 \\
 [\hat{e}_{\pm}, \hat{f}_{\pm\pm}] &= 0, \quad [\hat{e}_{\pm}, \hat{f}_{\pm}] = \sqrt{4}\hat{f}_{\pm\pm}, \quad [\hat{e}_{\pm}, \hat{f}] = \sqrt{6}\hat{f}_{\pm} \\
 [\hat{e}_{\pm}, \hat{f}_{\mp}] &= \sqrt{6}\hat{f}, \quad [\hat{e}_{\pm}, \hat{f}_{\mp\mp}] = \sqrt{4}\hat{f}_{\mp} \\
 [\hat{f}_{++}, \hat{f}_{--}] &= -2\hat{h}, \quad [\hat{f}_{+}, \hat{f}_{-}] = \hat{h}, \quad [\hat{f}_{\pm\pm}, \hat{f}_{\mp}] = 2\hat{e}_{\pm} \\
 [\hat{f}_{\pm\pm}, \hat{f}_{\pm}] &= 0, \quad [\hat{f}, \hat{f}_{\pm}] = \sqrt{6}\hat{e}_{\pm}, \quad [\hat{f}, \hat{f}_{\pm\pm}] = 0.
 \end{aligned} \right\} (4.4)$$

By imposing the $|\Delta j| = 2$ requirement upon the generator matrix elements and making use of (4.4), Joseph found a unique half-integer spins solution:

$$\left. \begin{aligned}
 E_{\pm} |j, m\rangle &= \sqrt{(j+m)(j \pm m + 1)} |j, m \pm 1\rangle \\
 H |j, m\rangle &= 2m |j, m\rangle \\
 \langle j+2, m+2 | F_{++} |j, m\rangle &= \sqrt{(j+m+4)(j+m+3)(j+m+2)(j+m+1)} t \\
 \langle j+2, m+1 | F_{+} |j, m\rangle &= 2\sqrt{(j+m+3)(j+m+2)(j+m+1)(j-m+1)} t \\
 \langle j+2, m | F |j, m\rangle &= \sqrt{6(j+m+2)(j+m+1)(j-m+2)(j-m+1)} t \\
 \langle j+2, m-1 | F_{-} |j, m\rangle &= 2\sqrt{(j+m+1)(j-m+3)(j-m+2)(j-m+1)} t \\
 \langle j+2, m-2 | F_{--} |j, m\rangle &= \sqrt{(j-m+4)(j-m+3)(j-m+2)(j-m+1)} t \\
 |t|^2 &= \frac{(2j+3)^2 + c^2}{4(2j+5)(2j+1)(2j+3)^2} \\
 j &= \frac{4n+1}{2}, \quad n = 0, 1, 2, \dots, \quad c^2 \rightarrow 0
 \end{aligned} \right\} (4.5)$$

All other matrix elements vanish.

This describes $\mathcal{D}\left(\frac{1}{2}, 0\right)$. The same method showed that the only other representations in that set were the previously derived [12] band-tensors

$$\left. \begin{aligned}
 \mathcal{D}(0, c) \\
 \mathcal{D}(1, c)
 \end{aligned} \right\} -\infty < c < \infty. \quad (4.6)$$

This degenerate set of representations corresponds to the $\overline{\text{SL}(3\mathbb{R})}$ case of a recently discovered class of representations of semi-simple Lie groups in connection with the study of the enveloping algebras and the A. Joseph ideal [18] [19].

5. $\overline{\text{GA}(4\mathbb{R})}$ AND $\overline{\text{GL}(4\mathbb{R})}$

Since the representations of $\overline{\mathcal{G}}$ are those of $\overline{\mathcal{F}}$, the physical states can be described by induced representations of $\overline{\mathcal{F}}$ over its stability subgroup and the translations. The stability subgroup is $\overline{\text{GL}(3\mathbb{R})}$, and we can thus use the product of our representations of \mathcal{S}_3 by the 2-element factor group $\text{O}(3)/\text{SO}(3)$, since \mathcal{G}_3 will have the topology of $\text{O}(3)$.

Further complications will arise as a result of the local Minkowskian metric η_{ab} of (1.2). The representations we developed fit the case of time-like momenta. We shall study the other possibilities in another publication.

For the construction of fields, we should use \mathcal{G}_4 . Our analysis in section 3 can be repeated for this group; the $\overline{\mathcal{M}}_4$ will correspond to a product of two sets $\pm(\sigma_i, 1)$ and $\overline{\mathcal{G}}_4/\overline{\mathcal{D}}_4$ is $\text{SU}(2) \times \text{SU}(2)/\overline{\mathcal{M}}_4$. Band-tensor representations of \mathcal{S}_4 were constructed in the second paper referred under [12], in connection with the states of a spinning top. For Band-spinors, we shall need $\mathcal{D}\left(j_0^{(1)} = \frac{1}{2}, j_0^{(2)} = 0\right) \oplus \mathcal{D}\left(j_0^{(1)} = 0, j_0^{(2)} = \frac{1}{2}\right)$, with $(\Delta j^{(1)} = 1, \Delta j^{(2)} = 1)$ non-compact action.

Each $(j^{(1)}, j^{(2)})$ level of a band spinor field satisfies a Bargmann-Wigner equation [20] for $j = |j^{(1)}| + |j^{(2)}|$.

The covariant derivative of a band-spinor field Ψ^α will be given by eq. (1.6). We shall deal with the field formalism in a future publication.

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