Annales de l'I. H. P., section A

L. Trlifaj

Exceptional and orthosymplectic graded Lie algebras

Annales de l'I. H. P., section A, tome 29, nº 1 (1978), p. 111-122

<http://www.numdam.org/item?id=AIHPA_1978__29_1_111_0>

Ann. Inst. Henri Poincaré, Section A :
Vol. XXIX, n° 1, 1978, p. 111-122. *Physique théorique.*

Exceptional and orthosymplectic graded Lie algebras

by

L. TRLIFAJ

Institute of Physics of the Czechoslovak Acad. Sci., Prague

ABSTRACT. — The connection between the exceptional and the associated orthosymplectic graded Lie algebras is shown by redefining the anticommutator of the odd generators. Possible modifications of the anticommutator as well as admissible Lie algebras of the even generators are restricted by the two Jacobi identities, so that the well-known three cases of the exceptional graded Lie algebras are obtained. The concept of non-standard matrix representation based on such a modification can be introduced.

1. INTRODUCTION

Physical application of the graded Lie algebras (GLAs) in the problems of supersymmetry was reviewed in many papers (for example see [*1-3*]), the questions of their classification were solved in a series of papers [*4-9*]. The aim of this note is, to show the connection between the exceptional and the orthosymplectic GLAs, which was not mentioned in them. They belong to a class of the simple GLAs [*6*], which can be characterized, essentially, by some regular metric matrices in the used approach.

In Section 2, the orthosymplectic GLAs are constructed on the base of the correlated odd generators by using two isomorphic graded vector spaces with the bilinear form (in the bra and ket formalism). Some of their main properties are also reviewed. The exceptional GLAs are related to the orthosymplectic ones by redefining the anticommutator of the odd generators in Section 3. Such a modification must comply with the two Jacobi identities which specify it considerably. The first of them restricts the symplectic even part of the GLA to the sp(2, R) Lie algebra (LA) and requires some symmetry condition to be fulfilled by the quantity which determines the modification of the anticommutator. According to the second identity the embedding of the allowed subalgebra into the remaining even (orthogonal) part of the GLA is performed. The detailed analysis has shown that there exist three known [8] subalgebras of this kind, hence using the techniques of the projection operators introduced in Section 3 we briefly demonstrate their properties from the point of view of the applied approach only in Section 4. This approach allows also to introduce the lowest non-standard matrix representations beside the standard (next lowest adjoint) representations of the exceptional GLAs. In the latter the commutators and the anticommutators of two matrices A and B are represented by the conventional expressions $[A, B]_{\pm} = AB \pm BA$, in the former the redefined form of the anticommutator, which is a linear combination of the conventional anticommutators, is used.

2. PRELIMINARIES

The general GLA can be constructed [6, 8] by means of a (graded) vector space $V = V_0 \oplus V_1$ and the isomorphic vector space $V' = V_0' \oplus V_1'$ whose even subspaces V_0 and V_0' and odd subspaces V_1 and V_1' are spanned by the associated basis vectors $a\rangle$, $a'\rangle$ $(a = 1, \ldots, m)$ and $\alpha\rangle, \alpha'\rangle$ $(\alpha = 1, \ldots, n)$ respectively. The bilinear form is assumed to be even

$$(1) \qquad \langle V_0' \mid V_1 \rangle = 0 = \langle V_1' \mid V_0 \rangle$$

the associated matrices with the elements $\langle a' \mid b \rangle$ and $\langle \alpha' \mid \beta \rangle$ being regular.

To beginn with we define the 2mn independent odd generators of the GLA

$$(2) \qquad G_{a\alpha} = h_1 a\rangle\langle \alpha' + h_2 \alpha\rangle\langle a', \qquad F_{a\alpha} = h_1 a\rangle\langle \alpha' - h_2 \alpha\rangle\langle a',$$

where h_1 and h_2 are fixed numbers. The purely algebraic anticommutator of generators $G_{a\alpha}$ and $G_{b\beta}$ is expressed in terms of the $(m^2 + n^2)$ independent even generators of the GLA and their « unit » element as follows

$$(3) \quad [G_{a\alpha}, G_{b\beta}]_+ = G_{a\alpha}G_{b\beta} + G_{b\beta}G_{a\alpha} = h_1 h_2 \{ s_{\alpha\beta}S_{ab} + t_{\alpha\beta}O_{ab} + s_{ab}S_{\alpha\beta} + t_{ab}O_{\alpha\beta} + [\langle a' \mid b \rangle\langle \beta' \mid \alpha \rangle + \langle b' \mid a \rangle\langle \alpha' \mid \beta \rangle]E \}.$$

Using the inverse metric matrices $\langle ' \mid \rangle^{-1}$ $(\langle ' \mid \rangle^{-1}\langle ' \mid \rangle = \mathbb{1})$ associated to the bilinear form (1) in each of the subspaces we give the explicit expressions for the even and traceless generators S_{ab} and $O_{\alpha\beta}$ only

$$(4) \quad S_{ab} = a\rangle\langle b' + b\rangle\langle a' - \frac{1}{m}(\langle b' \mid a\rangle + \langle a' \mid b\rangle)\sum_{cd}(\langle ' \mid \rangle^{-1})_{cd}\, c\rangle\langle d' = d'$$

$$(5) \quad O_{\alpha\beta} = \alpha\rangle\langle \beta' - \beta\rangle\langle \alpha' - \frac{1}{n}(\langle \beta' \mid \alpha\rangle - \langle \alpha' \mid \beta\rangle)\sum_{\gamma\delta}(\langle ' \mid \rangle^{-1})_{\gamma\delta}\, \gamma\rangle\langle \delta'$$

as the generators O_{ab} and $S_{\alpha\beta}$ are defined in the similar way. When the trace of the unit element E

$$(6a)\qquad E = \frac{1}{m}\sum_{cd}(\langle ' | \rangle^{-1})_{cd}c\rangle\langle d' + \frac{1}{n}\sum_{\gamma\delta}(\langle ' | \rangle^{-1})_{\gamma\delta}\gamma\rangle\langle\delta'$$

is taken, the matrix γ_A of the grading automorphism [7] must be applied

$$(6b)\qquad \mathrm{tr}\,\gamma_A E = \frac{1}{m}\sum_{cd}(\langle ' | \rangle^{-1})_{cd}\langle d' | c\rangle - \frac{1}{n}\sum_{\gamma\delta}(\langle ' | \rangle^{-1})_{\gamma\delta}\langle\delta' | \gamma\rangle = 1 - 1 = 0.$$

In what follows new metric matrices with the elements

$$(7)\qquad s_{ab} = \frac{1}{2}(\langle a' | b\rangle + \langle b' | a\rangle), \qquad s_{\alpha\beta} = \frac{1}{2}(\langle\alpha' | \beta\rangle + \langle\beta' | \alpha\rangle)$$

$$t_{ab} = \frac{1}{2}(\langle a' | b\rangle - \langle b' | a\rangle), \qquad t_{\alpha\beta} = \frac{1}{2}(\langle\alpha' | \beta\rangle - \langle\beta' | \alpha\rangle)$$

prove useful. They are symmetric and skew-symmetric respectively as well as the generators S_{ab} and $O_{\alpha\beta}$ in the indices ab and $\alpha\beta$. We could close the GLA by calculating the remaining anticommutators for generators G and F and the commutators for odd and/or even generators. Such GLA is classified as a real form $(SU(m', m'') \times SU(n', n'') \times SU(1); m\otimes\bar{n}\oplus\bar{m}\otimes n)$ with (variable) signatures $m = m' + m''$ and $n = n' + n''$ according to the notation of ref. [9] provided the isomorphism of vector spaces V and V' along with the associated bilinear form is suitably chosen (Hermitian).

Since we are interested in the subalgebras of these GLAs we remark only that

$$(8)\qquad [E, G_{c\gamma}]_- = EG_{c\gamma} - G_{c\gamma}E = \frac{n-m}{m}F_{c\gamma}.$$

Hence, the preservation of a half ($G_{c\gamma}$) of the odd basis (2) only results in the necessary elimination of the unit element, which means according to relation (3), that

$$(9a)\qquad \langle a' | b\rangle\langle\beta' | \alpha\rangle + \langle b' | a\rangle\langle\alpha' | \beta\rangle = 0.$$

The generators $F_{c\gamma}$ must be eliminated also in the commutators $[S_{ab}, G_{c\gamma}]_-$ and $[O_{\alpha\beta}, G_{c\gamma}]_-$, where they occur. This is accomplished, if

$$(9b)\qquad \langle a' | b\rangle = -\langle b' | a\rangle, \qquad \langle\alpha' | \beta\rangle = \langle\beta' | \alpha\rangle.$$

But then the generators O_{ab} and $S_{\alpha\beta}$ disappear in the anticommutator (3) too and they need not be considered at all. The multiplication of the basis vectors $\rangle$ by some arbitrary constants C induces the multiplication of the associated vectors $\langle'$ within the generator (2) by the complex conjugate constants C* or by C depending, whether the corresponding bilinear form is Hermitian or real. The correlation in eq. (2) is, therefore, not changed, only if bilinear forms $\langle' | \rangle$ and accordingly constants $g = h_1 h_2$ and the forthcoming LAs are taken real.

Introducing the notation

$$(10)\quad \frac{1}{2}(t_{ac}\delta_{bd} + t_{bc}\delta_{ad}) = \left\{ {ab \atop dc} \right\},$$
$$2\Psi_0(ab, cd, fg) = \left\{ {ab \atop fc} \right\} \delta_{dg} + \left\{ {ab \atop gc} \right\} \delta_{df} + \left\{ {ab \atop gd} \right\} \delta_{cf} + \left\{ {ab \atop fd} \right\} \delta_{cg},$$

$$(11)\quad \frac{1}{2}(- s_{\alpha\gamma}\delta_{\beta\delta} + s_{\beta\gamma}\delta_{\alpha\delta}) = \left[{\alpha\beta \atop \delta\gamma} \right],$$
$$2\Phi_0(\alpha\beta, \gamma\delta, \rho\omega) = \left[{\alpha\beta \atop \rho\gamma} \right]\delta_{\delta\omega} - \left[{\alpha\beta \atop \omega\gamma} \right]\delta_{\delta\rho} + \left[{\alpha\beta \atop \omega\delta} \right]\delta_{\gamma\rho} - \left[{\alpha\beta \atop \rho\delta} \right]\delta_{\gamma\omega},$$

we can write the only non-zero structure relations of the investigated subalgebra in a lucide form

$$(12\,a)\quad [G_{a\alpha}, G_{b\beta}]_+ = g s_{\alpha\beta} S_{ab} + g t_{ab} O_{\alpha\beta}, \quad g = h_1 h_2,$$
$$(12\,b)\quad [S_{ab}, G_{c\gamma}]_- = 2\left\{ {ab \atop dc} \right\} G_{d\gamma}, \quad [O_{\alpha\beta}, G_{c\gamma}]_- = 2\left[{\alpha\beta \atop \delta\gamma} \right] G_{c\delta},$$
$$(12\,c)\quad [S_{ab}, S_{cd}]_- = 2\Psi_0(ab, cd, fg) S_{fg}, \quad [O_{\alpha\beta}, O_{\gamma\delta}]_- = 2\Phi_0(\alpha\beta, \gamma\delta, \rho\omega) O_{\rho\omega}.$$

Here and in what follows the sum over two repeated indices is implied. The right-hand sides of relations (12 *b*) give us the representations of the even generators within the independent and closed subspaces of the odd generators. These representations are parts of the usual adjoint representation [*10*] of the given GLA, in which the representation of the brackets $[\,,\,]_\pm$ is conventional and determined by the Jacobi identities [*5*]. The constructive way of building up the whole subalgebra ensures us their automatical fulfilment.

Inside each of the LA (12 *c*) the Cartan subalgebra can be introduced. The usual analysis [*11*] then shows that the whole GLA is orthosymplectic [*6*], the orthogonal part allowing, in general, an indefinite metric $s_{\alpha\beta}$ in the used real forms. Using again the notation of ref. [*9*] we classify it as $(\mathrm{Sp}(2p) \times \mathrm{O}(n', n'')\,;\, 2p \times n)$, where $2p = m$ and the (variable) signature $n = n' + n''$. After the complexification it is converted to $(\mathrm{Sp}(2p) \times \mathrm{O}(n)\,;\, 2p \times n)$ over the complex numbers. The byproduct of this analysis is the more precise form [*8*] of the so-called root-weight theorem [*5*].

The relations (12) do not essentially change under the replacement of the first set of the odd generators $G_{a\alpha}$ of eq. (2) by the second one $F_{a\alpha}$, which induces the sign change in the right-hand side of eq. (12 *a*) only. This corresponds to the special types of representations (grade star representations [*12*]) of the orthosymplectic GLAs.

Relation (8) indicates that cases $n = m$ can lead to the special subalgebras in construction of which the possible isometric isomorphism between the subspaces V_0 and V_1 of equal dimensions is employed beside correlations within the odd basis of the GLA. These cases were investigated in refs [*7*, *8*].

3. EXCEPTIONAL ALGEBRAS

Apart from the derivation the structure relations (12) represent an abstract GLA. The involved LAs as well as their possible substructures are fixed by the relations (12 *c*) and (12 *b*). A realization of such substructures, which could be completed to (new) GLAs, is consequently subjected, in general, to a change of the relation (12 *a*).

Hence, we redefine it by applying a (homomorphic) mapping $\{,\} = (\mathbb{1} + D)[\,,\,]_+$

$$(13)\quad \{ G_{a\alpha}, G_{b\beta} \} = [G_{a\alpha}, G_{b\beta}]_+ + D_{\alpha\beta;\gamma\delta}[G_{a\gamma}, G_{b\delta}]_+ = g s_{\alpha\beta} S_{ab} + g t_{ab} A_{\alpha\beta;\gamma\delta} O_{\gamma\delta},$$

where the quantities D and A

$$(13\,a)\qquad A_{\alpha\beta;\gamma\delta} = D_{\alpha\beta;\gamma\delta} + \binom{\alpha\beta}{\gamma\delta}, \qquad \binom{\alpha\beta}{\gamma\delta} = \frac{1}{2}\delta_{\alpha\gamma}\delta_{\beta\delta} - \frac{1}{2}\delta_{\alpha\delta}\delta_{\beta\gamma}$$

are chosen skew-symmetric in the indices $\alpha\beta$ and $\gamma\delta$ respectively due to the inherent symmetry properties of the anticommutators.

The new anticommutator does not fulfil the Jacobi identities automatically, but being related to the old basis it can be tested by those two identities which experience change (13). The first of them involving three odd generators $G_{d\delta}$ in the cyclic permutations of the couples of indices $d\delta$

$$(14)\qquad \sum_{\text{cycle}\binom{abc}{\alpha\beta\gamma}} [\{ G_{a\alpha}, G_{b\beta} \}, G_{c\gamma}]_- = 0$$

confines the dimensions of the vector subspaces V_0 and V_0' to $m = 2$ according to eqs. (12 *a*, *b*). At the same time the following symmetry conditions

$$(15)\quad s_{\gamma\rho}D_{\alpha\beta;\rho\delta} + s_{\beta\rho}D_{\alpha\gamma;\rho\delta} = 0 \quad \text{or} \quad s_{\gamma\rho}A_{\alpha\beta;\rho\delta} + s_{\beta\rho}A_{\alpha\gamma;\rho\delta} = s_{\gamma\rho}\binom{\alpha\beta}{\rho\delta} + s_{\beta\rho}\binom{\alpha\gamma}{\rho\delta}$$

must be fulfilled. The obtained real form sp(2, R) is isomorphic to the real form sl(2, R) (and su(1, 1)) [*13*]. The restraint $m = 2$ is typical [*8*] of the exceptional GLAs, so that the exceptionality can be tied to the modification (13).

Only now the anticipated form of special choice (13) may be made clearer. One can namely show in a more detailed analysis that, from the more general forms of the anticommutator modification which include, for example, the symplectic deformation $D_{ab;cd}$ too, the expression (13) only leads to non-trivial conclusions. The symmetry conditions imposed by the Jacobi identity (14) and the requirement of the general independence of the LAs (12 *c*) are very strong, so that the chosen form (13) of the anti-commutator modification is sufficiently general.

The next restraint on tensor A comes from the possible reduction or rearrangement of the basis of the remaining orthogonal LA. Let us decompose the « unit » operator of relation (13 *a*) into two projection operators

$$(16)\quad \binom{\alpha\beta}{\gamma\delta} = P_{\alpha\beta;\gamma\delta} + Q_{\alpha\beta;\gamma\delta}, \qquad P^2 = P, \quad Q^2 = Q, \quad PQ = QP = 0,$$

where the abbreviated notation $P^2 \rightarrow P_{\alpha\beta;\rho\omega}P_{\rho\omega;\gamma\delta}$ is used. If the redundant generators $O_{\alpha\beta}$ of the orthogonal LA are eliminated by means of the projection operator Q, then the equality $Q\{\ ,\ \} = 0$ must hold, hence, according to relation (13)

$$(17\,a)\qquad A_{\alpha\beta;\gamma\delta} = KP_{\alpha\beta;\gamma\delta},$$

where K is some constant. The specification of the operator Q (and P) which is, as mentioned, identical with the determination of the admissible subalgebra of the orthogonal LA, is a problem by itself. The subalgebra, whose generators $U_{\alpha\beta}$ consist of the linear combinations of generators $O_{\alpha\beta}$, is not only embedded into the orthogonal LA, but its generators must also obey the second Jacobi identity of the new GLA

$$(18)\quad [U_{\gamma\delta}, \{G_{a\alpha}, G_{b\beta}\}]_- = \{[U_{\gamma\delta}, G_{a\alpha}]_-, G_{b\beta}\} + \{[U_{\gamma\delta}, G_{b\beta}]_-, G_{a\alpha}\}.$$

The choice of the subalgebra is, therefore, not arbitrary, it must be consistent with relations (18) as well as (15).

If the orthogonal LA is not reduced and decays only into the direct sum of two subalgebras, then the tensor A splits into two projection operators of type (16)

$$(17\,b)\qquad A_{\alpha\beta;\gamma\delta} = K_1P_{\alpha\beta;\gamma\delta} + K_2Q_{\alpha\beta;\gamma\delta}$$

where K_1 and K_2 are some constants.

Using the results of ref. [*8*] we can, now, avoid the lengthy way of considerations which would lead to the specification of operators (16) and review only the known exceptional GLA from the point of view of the given approach.

4. SPECIAL CASES

The three-dimensional real sp(2, R) LA will be described by means of the known C matrix which is related to the Pauli matrix τ_2

$$(19)\qquad t_{ab} = C_{ab} = (i\tau_2)_{ab} = -C_{ba}.$$

We can easily transform its structure relations into those of the (complex) su(2) LA by means of Pauli matrices performing its complexification at the same time. The complexification must be then extended to the whole

GLA. This procedure is brought about straight-forward by the admissible embedding into the orthogonal LA in the second case investigated 4.2. It does not contradict the allowed values of constant $g = h_1 h_2$ in eq. (13), which can be chosen complex, in general. The final procedure of complexification will be always understood in the all investigated cases.

4.1. $sp(2) \times G_2$

The G_2 algebra is embedded into the o(7) LA, hence, we choose $s_{\alpha\beta} = \delta_{\alpha\beta}$ ($\alpha, \beta = 1, \ldots, 7$), so that quantities (11) are as follows

(20)
$$\left[\begin{smallmatrix}\alpha\beta\\ \gamma\delta\end{smallmatrix}\right] = \binom{\alpha\beta}{\gamma\delta}, \qquad \sum_{\text{cycle}(\alpha\beta\gamma)} \binom{\alpha\beta}{\gamma\delta} = 0,$$
$$\Phi_0(\alpha\beta, \gamma\delta, \rho\omega) = \binom{\alpha\beta}{\rho\tau}\binom{\gamma\delta}{\tau\omega} - \binom{\alpha\beta}{\tau\omega}\binom{\gamma\delta}{\rho\tau}.$$

It is completely characterized by a totally skew-symmetric tensor $\xi_{\alpha\beta\gamma}$, whose absolute value $|\xi_{\alpha\beta\gamma}| = 1$ for seven triples of indices only (details see refs [*8*, *14*]). Projection operators corresponding to the elimination of seven independent generators $U_\alpha = \xi_{\alpha\beta\gamma} O_{\beta\gamma}$ ($\alpha = 1, \ldots, 7$) of the o(7) LA are

(21)
$$Q_{\alpha\beta;\gamma\delta} = \frac{1}{6}\xi_{\alpha\beta\tau}\xi_{\gamma\delta\tau}(Q_{\alpha\beta;\alpha\beta} = 1),$$
$$D_{\alpha\beta\gamma\delta} = \frac{1}{2}\binom{\alpha\beta}{\gamma\delta} - \frac{1}{4}\xi_{\alpha\beta\tau}\xi_{\gamma\delta\tau},$$
$$A_{\alpha\beta;\gamma\delta} = \frac{3}{2}P_{\alpha\beta;\gamma\delta} = \frac{3}{2}\left(\binom{\alpha\beta}{\gamma\delta} - \frac{1}{6}\xi_{\alpha\beta\tau}\xi_{\gamma\delta\tau}\right),$$

where tensor $D_{\alpha\beta\gamma\delta}$ must be, according to eq. (15), and also is totally skew-symmetric being non-zero for seven quadruples of indices only [*8*, *14*]. If the dependent (14 of them independent) generators $U_{\alpha\beta}$ ($\alpha, \beta = 1, \ldots, 7$)

(22)
$$U_{\alpha\beta} = A_{\alpha\beta;\gamma\delta} O_{\gamma\delta} = -U_{\beta\alpha}$$

are introduced, then, according to the second eq. (12 *b*), matrices $2(A_{\alpha\beta})_{\gamma\delta} = 2A_{\alpha\beta;\gamma\delta}$ constitute their adjoint representation within the odd-generator subspace, so that the function

(23)
$$4\Phi(\alpha\beta, \gamma\delta, \rho\omega) = 4A_{\alpha\beta;\mu\nu}A_{\gamma\delta;\lambda\varkappa}\Phi_0(\mu\nu, \lambda\varkappa, \rho\omega) = 2C^{\xi\eta}_{\alpha\beta,\gamma\delta}A_{\xi\eta;\rho\omega}$$

must be expressible by means of the structure constants $C^{\rho\omega}_{\alpha\beta,\gamma\delta}$ of the G_2 LA. Henceforward the conventional representation of the matrix commutator $[A, B]_- = AB - BA$ is used. One can show making use of the identity [*8*]

(24)
$$\sum_{\text{cycle}(\alpha\beta\gamma)} \xi_{\alpha\beta\delta} P_{\delta\gamma;\rho\omega} = 0$$

and of the expression for D in eqs. (21), that the relation (23) is, indeed, fulfilled, if $C^{\rho\omega}_{\alpha\beta,\gamma\delta}$ are the structure constants of the G_2 LA [8]

$$C^{\rho\omega}_{\alpha\beta,\gamma\delta} = 3\Phi_0(\alpha\beta, \gamma\delta, \rho\omega) + \frac{1}{2}\xi_{\alpha\beta\mu}\xi_{\gamma\delta\nu}\binom{\mu\nu}{\rho\omega}. \tag{25}$$

The Jacobi identity (18) requires function $\Phi(\alpha\beta, \gamma\delta, \rho\omega)$ defined by eq. (23) to be totally skew-symmetric in the couples of indices $(\alpha\beta)$, $(\gamma\delta)$ and $(\rho\omega)$. However,

$$\Phi(\alpha\beta, \gamma\delta, \rho\omega) = \frac{2}{3}A_{\alpha\beta;\lambda\varkappa}A_{\gamma\delta;\sigma\tau}A_{\rho\omega;\xi\eta}\Phi_0(\lambda\varkappa, \sigma\tau, \xi\eta) + \Phi(\alpha\beta, \gamma\delta, \sigma\tau)Q_{\rho\omega;\sigma\tau} \tag{25 a}$$

according to eqs. (23), (16) and (21). The last term in eq. (25 *a*) being zero due to eq. (23) and to the orthogonality of projectors P and Q, function $\Phi(\alpha\beta, \gamma\delta, \rho\omega)$ is totally skew-symmetric along with function $\Phi_0(\alpha\beta, \gamma\delta, \rho\omega)$.

Since the Jacobi identities (14) and (18) are satisfied and the anticommutator $\{\ ,\ \}$ has the needed symmetry properties, we can now forget the explicit definitions of the anticommutator in eq. (13) and of the generators $U_{\alpha\beta}$ (22) and consider them as given in an abstract wav. Summing up the G_2 part of the investigated GLA we have then

$$\begin{gathered} \{ G_{a\alpha}, G_{b\beta} \} = g\delta_{\alpha\beta}S_{ab} + gC_{ab}U_{\alpha\beta}, \\ [U_{\alpha\beta}, G_{c\gamma}]_- = 2A_{\alpha\beta;\delta\gamma}G_{c\delta}, \qquad [U_{\alpha\beta}, U_{\gamma\delta}]_- = C^{\rho\omega}_{\alpha\beta,\gamma\delta}U_{\rho\omega}, \end{gathered} \tag{26}$$

where the structure constants are given by eqs. (20) and (25). The sp(2) part of the GLA remains unchanged as given by the first equations in eqs. (12 *b*) and (12 *c*). The obtained structure relations conform with those of ref. [8], where some other details are also mentioned.

The relevant feature of the used approach is a possibility of introducing the lowest irreducible $(m + n) \times (m + n)$ $(= 9 \times 9)$ matrix representation, which can be called non-standard. If namely the anticommutator for the odd generators is taken explicitly in form (13) and its interior anticommutator of two representation matrices is represented in a conventional way $[A, B]_+ = AB + BA$, then the representation matrices are as follows

$$G_{a\alpha} = \begin{pmatrix} 0 & h_1 Z_{a\alpha} \\ h_2 \tilde{Z}_{a\alpha} C & 0 \end{pmatrix}, \quad (Z_{a\alpha})_{c\gamma} = \delta_{ca}\delta_{\gamma\alpha}, \quad (\tilde{Z}_{a\alpha}C)_{\gamma c} = -\delta_{\gamma\alpha}C_{ca}, \tag{27 a}$$

$$h_1 h_2 = g, \quad U_{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 0 & \underline{U}_{\alpha\beta} \end{pmatrix}, \quad (\underline{U}_{\alpha\beta})_{\gamma\delta} = 3\binom{\alpha\beta}{\gamma\delta} - \frac{1}{2}\xi_{\alpha\beta\tau}\xi_{\gamma\delta\tau}, \tag{27 b}$$

$$S_{ab} = \begin{pmatrix} \underline{S}_{ab} & 0 \\ 0 & 0 \end{pmatrix}, \quad (\underline{S}_{ab})_{cd} = \delta_{ac}C_{bd} + \delta_{bc}C_{ad}. \tag{27 c}$$

If the anticommutator $\{ A, B \}$ is taken direct in the conventional way, there is no such possibility. Such a form of the matrix anticommutator is related to the standard 9×9 representation of the associated osp(2, 7) GLA (notation see ref. [*12*]).

4.2. sp(2) × o(7)

Unlike the GLA of this type which has been reviewed in Section 2 the « new » (complex) o(7) LA is, now, assumed to be embedded into the o(8) LA, whose real form used is o(4, 4). The adequate description of this spinor embedding was developped in ref. [*15*] by means of Hermitian 8×8 matrices $\Gamma_a = \Gamma_a^+$ $(a = 1, \ldots, 7)$ of the Dirac type

(28 *a*) $$\Gamma_{ab} = \Gamma_a\Gamma_b \qquad \Gamma_{ab} + \Gamma_{ba} = 2\delta_{ab} \qquad (a, b = 1, \ldots, 7).$$

There exists a charge conjugation matrix Γ^c defining an invariant bilinear form with respect to the full 7-dimensional rotation group [*15*] with properties

(28 *b*) $$\tilde{\Gamma}^c = \Gamma^c = \Gamma^{c+}, \qquad \tilde{\Gamma}^c\Gamma^c = 1, \qquad \tilde{\Gamma}_a\Gamma^c + \Gamma^c\Gamma_a = 0, \qquad \mathrm{tr}\,\Gamma^c = 0,$$

where symbol $\sim$ over matrices denotes their transposition. Due to the first two properties (28 *b*) we identify the metric matrix $s_{\alpha\beta}$ in eqs. (12) with matrix $(\Gamma^c)_{\alpha\beta} = s_{\alpha\beta}$. Since the basis $O_{\alpha\beta}$ of the o(4, 4) LA is reduced to that U_{ab} of o(7) LA by the transformation

(29) $$U_{ab} = \frac{1}{4}(\Gamma_{ab}\tilde{\Gamma}^c)_{\alpha\beta}O_{\alpha\beta} = -\,U_{ba} \qquad (a \neq b)$$

and

(30 *c*) $$[U_{ab}, U_{cd}]_- = 2\Phi_0(ab, cd, fg)U_{fg}(s_{ab} = \delta_{ab}, \left[{}^{ab}_{cd}\right] = \left({}^{ab}_{cd}\right), \ldots)$$

according to eqs. (12 *c*) and (28), the projection operator (17 *a*), owing to $\mathrm{tr}\,(\Gamma_{ab}\Gamma_{cd}) = 8\delta_{ab}\delta_{cd} - 16\left({}^{ab}_{cd}\right)$, is

(31) $$A_{\alpha\beta;\gamma\delta} = \frac{4}{3}P_{\alpha\beta;\gamma\delta} = \frac{1}{12}(\Gamma^c\Gamma_{ab})_{\alpha\beta}\left({}^{ab}_{cd}\right)(\Gamma_{cd}\Gamma^c)_{\gamma\delta},$$

where symbol $\left({}^{ab}_{cd}\right)$ is given by the second eq. (13 *a*).

The projection operator A fulfils eq. (15), which can be interpreted as the generalized form of the well-known Fierz identity. The second Jacobi identity (18) is also fulfilled, for matrix $\frac{1}{2}\Gamma_{ab}$ is a representation matrix of o(7) LA as follows direct from relations (29) and (12 *b*). This is explicitly evident from the remaining structure relations of the o(7) part of the investigated GLA

(30 *a*) $$\{ G_{a\alpha}, G_{b\beta} \} = g(\Gamma^c)_{\alpha\beta}S_{ab} + \frac{1}{3}gC_{ab}(\Gamma^c\Gamma_{pq})_{\alpha\beta}U_{pq},$$

(30 *b*) $$[U_{ab}, G_{c\gamma}]_- = \frac{1}{2}(\Gamma_{ab})_{\beta\gamma}G_{c\beta},$$

which again hold independently of the way, they were derived. Formulae (30) agree with those of ref. [*8*].

In consequence of the introduction of the modified anticommutator (13), one can again construct the non-standard 10×10 matrix representation of the given GLA. We write down the odd and o(7) parts only

$$(32\,a) \quad G_{a\alpha} = \begin{pmatrix} 0 & h_1 Z_{a\alpha}\Gamma^c \\ h_2 \tilde{Z}_{a\alpha} & 0 \end{pmatrix}, \quad (Z_{a\alpha}\Gamma^c)_{c\gamma} = \delta_{ca}\Gamma^c_{\alpha\gamma}, \quad (\tilde{Z}_{a\alpha}C)_{\gamma c} = -\delta_{\gamma\alpha}C_{ca}$$

$$(32\,b) \quad h_1 h_2 = g, \quad U_{ab} = \begin{pmatrix} 0 & 0 \\ 0 & \underline{U}_{ab} \end{pmatrix}, \quad (\underline{U}_{ab})_{\alpha\beta} = \frac{1}{2}(\Gamma_{ab})_{\alpha\beta}$$

as the sp(2) part is represented by expressions (27 *c*).

4.3. sp(2) × sp(2) × sp(2)

We state beforehand the well-known fact [*13*] that the o(2, 2) LA is isomorphic to the direct sum of two sp(2, R) LAs. The isomorphism is easily realized by taking the vector spaces V_1 and V_1' as direct products of two two-dimensional vector spaces on which the sp(2, R) LA can be built up. As for the abstract o(2, 2) LA this means that the single indices $\alpha, \beta, \ldots$ of generators $O_{\alpha\beta}$ are replaced by the couples of spinor indices (a_1, a_2), (b_1, b_2), $\ldots$ $(a_i, b_i, \ldots = 1, 2 ; i = 1, 2)$ and that the (indefinite) metric matrix $s_{\alpha\beta}$ (7) is defined by means of matrices (19)

$$(33) \qquad s_{\alpha\beta} = C_{a_1 b_1} C_{a_2 b_2} .$$

The contracted generators

$$(34) \qquad \begin{aligned} S_{a_1 b_1} &= - C_{a_2 b_2} O_{(a_1, a_2)(b_1, b_2)} = S_{b_1 a_1} \\ S_{a_2 b_2} &= - C_{a_1 b_1} O_{(a_1, a_2)(b_1, b_2)} = S_{b_2 a_2} \end{aligned}$$

fulfil the structure relations $(i, j = 1, 2, 3)$

$$(35\,a) \quad [S_{a_i b_i}, S_{c_j d_j}] = \delta_{ij}(C_{b_i c_i} S_{a_i d_i} + C_{a_i d_i} S_{b_i c_i} + C_{a_i c_i} S_{b_i d_i} + C_{b_i d_i} S_{a_i c_i})$$

according to eqs. (12 *c*). Here the generatorss S_{ab} of the first eq. (12 *c*) are included in a symmetric way by denoting them $S_{a_3 b_3}$ and their indices $a, b, \ldots$, in general, $a_3, b_3, \ldots$.

The projection operators P and Q of eq. (17 *b*) which correspond to the decomposition (34) are

$$(36\,a) \quad P_{\alpha\beta;\gamma\delta} = \frac{1}{2}\binom{a_1 b_1}{c_1 d_1}_+ C_{a_2 b_2} C_{c_2 d_2}, \qquad Q_{\alpha\beta;\gamma\delta} = \frac{1}{2}\binom{a_2 b_2}{c_2 d_2}_+ C_{a_1 b_1} C_{c_1 d_1},$$

where now $\binom{ab}{cd}_+ = \frac{1}{2}\delta_{ac}\delta_{bd} + \frac{1}{2}\delta_{ad}\delta_{bc}$ and the tensor A itself is

$$(36\,b) \qquad g_3 A_{\alpha\beta;\gamma\delta} = - 2 g_1 P_{\alpha\beta;\gamma\delta} - 2 g_2 Q_{\alpha\beta;\gamma\delta} .$$

Here new constants $g_3 = g, g_1$ and g_2 are introduced instead of g, K_1 and K_2. If the odd generators $G_{a\alpha}$ are denoted as $G_{a_1a_2a_3}$ their modified anticommutator (13) is expressed in a symmetric form too

$$(35\,b) \qquad \{ G_{a_1a_2a_3}, G_{b_1b_2b_3} \} = \sum_{\text{cycle}(123)} g_1 C_{a_2b_2} C_{a_3b_3} S_{a_1b_1}$$

by using eqs. (19), (33), (34) and (36). Similarly we rewrite eqs. (12 *b*) as

$$(35\,c) \qquad [S_{a_ib_i}, G_{\cdot c_i \cdot}]_- = C_{a_ic_i} G_{\cdot b_i \cdot} + C_{b_ic_i} G_{\cdot a_i \cdot}$$

using eqs. (19), (33) and (34). The structure relations (35) are equivalent to those of ref. [8].

It is still necessary to prove, that identities (15) and (18) are satisfied. After some calculations one can show, that the former applies if

$$g_1 + g_2 + g_3 = 0$$

and

$$(37) \qquad \sum_{\text{cycle}(dab)} (\delta_{ac} C_{bd} - \delta_{bc} C_{ad}) = 0 = \sum_{\text{cycle}(cab)} (\delta_{ac} C_{bd} - \delta_{bc} C_{ad}) .$$

Being totally skew-symmetric in the triple of indices (*dab*) and (*cab*) respectively the last sums in eq. (37) are always zero, if indices *a*, *b*, *c*, *d* may assume two values only. But this is the case of the investigated LA and the other reasoning for the restriction of the possible orthogonal LAs. The Jacobi identity (18) is fulfilled automatically.

The non-standard 8 × 8 matrix representation of the given GLA

$$(38\,a) \qquad G_{a_1a_2a_3} = \begin{pmatrix} 0 & h_1 Z_{a_3a_1a_2} S \\ h_2 \tilde{Z}_{a_3a_1a_2} C & 0 \end{pmatrix}, \qquad h_1 h_2 = g_3 ,$$

$$(Z_{a_3a_1a_2} S)_{c_3c_1c_2} = \delta_{c_3a_3} C_{a_1c_1} C_{a_2c_2}, \qquad (\tilde{Z}_{a_3a_1a_2} C)_{c_1c_2c_3} = - \delta_{c_1a_1} \delta_{c_2a_2} C_{c_3a_3} ,$$

$$(38\,b) \quad S_{a_jb_j} = \begin{pmatrix} 0 & 0 \\ 0 & S_{a_jb_j} \otimes \mathbb{1}_{j'} \end{pmatrix}, \qquad (S_{a_jb_j})_{c_1c_2,d_1d_2}$$

$$= (\delta_{a_jc_j} C_{b_jd_j} + \delta_{b_jc_j} C_{a_jd_j}) \delta_{c_{j'}d_{j'}} \qquad (j = 1, 2 \text{ and } j' = 2, 1),$$

where the matrix $S_{a_3b_3}$ given by eq. (27 *c*) with $a = a_3$, $b = b_3, \ldots$, is obtained in an analogous way to representations (27) and (32).

Beside the non-standard representations (27), (32) and (38) there exist the next lowest matrix representations which are given by the usual adjoint representations. Since the composition law of the adjoint representation follows from the Jacobi identities (14), (18) and the other two ones, the commutators as well as the anticommutators of two representation matrices are given by the conventional expressions. Such representations differ from the non-standard ones not only in the definition of the anticommutator of the odd elements, but in other respects too. The Killing form